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ON CHARACTERISTIC FUNCTIONS AND MOMENTS

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SUMMARY. With every probability distribution function F on the real line is associated a $\lambda_F \geqslant 0$ such that F has absolute moments of all orders $< \lambda_F$ (if $\lambda_F > 0$) but not of any order $> \lambda_F$ (the moment of order λ_F itself may or may not exist). This paper first obtains certain results concerning the connection between the existence of moments and the behaviour (in the neighbourhood of the origin) of the characteristic function f of F, for arbitrary F; in particular, a necessary and sufficient condition on f for F to have the absolute moment of any given order (Theorem 5), and the procedure and formula for obtaining λ_F in terms of f (Theorems 6 and 7 and the remarks thereon). These results are then specialized to infinitely divisible (i.d.) laws (Theorems 8 and 9). Theorem 8 shows that an i.d. law has the absolute moment of a specified order λ if and only if $\int_{(-\infty,-1)} |u|^\lambda \, dM(u) + \int_{(1,\infty)} u^\lambda \, dN(u) < \infty$, where M and N are the Levy functions in the Levy representation for the law: for $\lambda \geqslant 2$, this takes the pleasant form that the law has the moment of order λ if and only if the Levy functions M and N also have the moment of that order. Theorem 9 identifies λ_F for an i.d. law F in terms of its Levy functions.

1. INTRODUCTION

A well-known result asserts that a probability distribution function (d.f.) F on the real line has the moment of a specified even-integer order iff (= if and only if) f, its characteristic function (c.f.), has the derivative of that order (at the origin, equivalently everywhere). Informative as this result is, it does not go very far in the study of the existence of (absolute) moments of d.f.'s through their c.f.'s (throughout what follows, the phrase "absolute moment" will be abbreviated as "moment"): for example, taking the non-Normal stable laws, every such law has an "exponent" α such that $0 < \alpha < 2$ and possesses moments of all orders $< \alpha$ but not of any order $\geqslant \alpha$; these facts have been established by obtaining estimates for the density functions of these laws *via* the inversion formula and by other *ad hoc* methods. It was obviously desirable to obtain criteria based directly on the c.f., for the existence of moments of other than even-integer orders. In what follows, we establish several results exhibiting the connection between the existence of moments and the behaviour of the c.f. (in the neighbourhood of the origin). Theorem 1 is taken from Ramachandran and Rao (1968), to provide us with an auxiliary result. Theorems 2 and 3 concern finite series expansions for the c.f. and their connection with the existence of moments. Theorems 4 and 5 provide a necessary, and a necessary and sufficient, condition respectively on f for F to have the moment of any specified order. Theorems 6 and 7 and the remarks thereon provide the formula and the procedure for obtaining the "critical order" $\lambda_F = \sup \{\lambda : \int |x|^\lambda \, dF(x) < \infty\}$ for arbitrary F, in terms of f. (Ramachandran, 1962, gives λ_F in terms of F.) Theorems 8 and 9 concern infinitely divisible (i.d.) laws. Theorem 8 shows that an i.d. law has the moment of order λ iff $\int_{(-\infty,-1)} |u|^\lambda \, dM(u) + \int_{(1,\infty)} u^\lambda \, dN(u) < \infty$ where M and N are the Levy functions in the Levy representation for $\log f$; for $\lambda \geqslant 2$, this condition can be expressed as follows: an i.d. law has the moment of a specified order iff its Levy functions have the same. Theorem 9 gives the formula for λ_F, if F is an i.d. law, in terms of the Levy functions M and N.

Theorem 5 is essentially due to R. P. Boas (see Boas, 1967) and the present author is grateful to the referee for drawing his attention to that paper. This further made it possible for us to establish Theorem 8; only a weaker version, namely Theorem 9 (a), had been given in the original version of this paper.

2. Results concerning the existence of absolute moments for arbitrary d.f.'s in terms of their c.f.'s

In Ramachandran and Rao (1968), investigating the properties of a class of probability laws which include the semi-stable laws of Paul Levy, the following auxiliary result was established. The reader is referred to that paper for the proof.

Theorem 1 : (a) *If, for some $\lambda < 2$, $\log |f(t)|/|t|^\lambda$ is bounded away from zero at some sequence $\{t_n\}$ of points tending to zero, then F does not have moments of any order $> \lambda$; in particular, this is true if $\log |f(t)|/|t|^\lambda$ is not bounded in the (deleted) neighbourhood of the origin.*

(b) *If $\log |f(t)|/|t|^\lambda$ is bounded in the (deleted) neighbourhood of the origin, then F has moments of all orders $< \lambda$; more generally, if $\log |f(t)|/|t|^\lambda$ is bounded at some sequence $\{t_n\}$ of points tending to zero such that (i) $\Sigma |t_n|^\varepsilon < \infty$ for every $\varepsilon > 0$, and (ii) $\{t_{n-1}/t_n\}$ is a bounded sequence (example : $t_n = r^n$, $0 < r < 1$), then F has moments of all orders $< \lambda$.*

(c) *If $\log |f(t)|/t^2$ is bounded at some sequence of points tending to zero, then F has the second moment. (The converse is true for all such sequences, i.e., as $t \to 0$).*

(d) *If $\log |f(t)|/t^2 \to 0$ as $t \to 0$ through some sequence of values, then F is degenerate. (The converse is trivially true for all such sequences).*

Remarks : (1) We prove below (Theorem 4) a stronger version of Theorem 1(a), namely, that under those conditions, *the moment of order λ itself does not exist.*

(2) If a is real and $1/2 \leqslant a \leqslant 1$, we have from the logarithmic series expansion that $1-a \leqslant -\log a \leqslant 2(1-a)$, so that in an interval around the origin where $|f(t)|^2 \geqslant 1/2$, we have

$$1-|f(t)|^2 \leqslant -2\log |f(t)| \leqslant 2[1-|f(t)|^2].$$

Hence, in all the statements (a)-(d) of Theorem 1, we my replace $\log|f(t)|$ by $1-|f(t)|^2$.

(3) Proceeding as in the proof of Theorem 1(b), we can establish the following sufficient condition for the existence of the moment of order $\lambda < 2$: for some sequence

$\{a_n\} \to 0$ such that $\{a_{n-1}/a_n\}$ is bounded, we must have $\Sigma |a_n|^{-\lambda}[1-|f(a_n)|] < \infty$; this of course has its extension to moments of orders $2n+\lambda$ ($n \geqslant 0$ integer, $0 < \lambda < 2$). See also our comments following Theorem 5.

We may remark at this stage that, in investigating the existence of moments, it suffices to consider only symmetric d.f.'s, since F has the moment of any given order iff $F^* = F * \tilde{F}$ has, where $*$ (on the RHS) denotes the convolution operation and $\tilde{F}$ is the d.f. conjugate to F (defined for all real x by $\tilde{F}(x) = 1-F(-x-0)$), and consequently F^* is symmetric, i.e., $\tilde{F}^* = F^*$. If f be the c.f. of F, then that of F^* is $|f|^2$, as is well-known.

We begin by considering two results concerning finite series expansions for a c.f. (in the neighbourhood of the origin). Both were used in an earlier version of this paper to prove Theorem 9(a). Theorem 2 is used in the proof of Theorem 7; a proof thereof may be found for instance in Loeve (1963, pp. 199-200). Theorem 3, though no longer used elsewhere in this paper, is of some independent interest because of its relationship to Theorem 2.

Theorem 2 : *Let F be a symmetric d.f. having the moment of order $2n+\lambda$, $n \geqslant 0$ integer and $0 \leqslant \lambda \leqslant 2$. Then f admits the following expansion in the neighbourhood of the origin :*

$$f(t) = 1+\frac{t^2}{2!}f''(0)+\ldots+\frac{t^{2n}}{2n!}f^{(2n)}(0)+O(|t|^{2n+\lambda}). \qquad \ldots \quad (*)$$

The result below is in the nature of a converse to Theorem 2.

Theorem 3. *Let F be a symmetric d.f. having the moment of even-integer order $2n (n \geqslant 0)$ and suppose that f admits an expansion of the above form, with $0 < \lambda < 2$, in the neighbourhood of the origin. Then F has moments of all orders $< 2n+\lambda$.*

Remark : The moment of order $2n+\lambda$ itself cannot, in general, be asserted to exist, as illustrated by any non-Normal symmetric stable law, whose c.f. is of the form $\exp(-c|t|^\alpha)$, $c > 0$ and $0 < \alpha < 2$; it does not possess the moment of order α. Thus the above is a best possible result.

Proof : For $n = 0$, the assertion of the theorem follows from Theorem 1(b) (or from Theorem 5 below). We consider below the cases $n \geqslant 1$. F being symmetric, f is real and given by $f(t) = \int \cos tx\, dF(x)$ for all real t. Hence, Re z denoting the real part of the complex number z, we have for sufficiently small $|t|$ and $|h|$ that

$$f(t+2nh)-\binom{2n}{1}f[t+2(n-1)h]+\ \ldots\ +f(t-2nh)$$
$$= \text{Re}\{\int e^{itx}(e^{ihx}-e^{-ihx})^{2n}dF(x)\} = (-1)^n \int \cos\ tx(2\sin hx)^{2n}dF(x).$$

The same differencing operation applied to the right side of relation $(*)$ gives : $(2h)^{2n}f^{(2n)}(0)+A(t,h)$, where $A(h,h) = O(|h|^{2n+\lambda})$. Hence, setting $t = h$ in the above, we have (for all sufficiently small $|h|$)

$$\int \cos\ hx(2\sin hx)^{2n}dF(x) = (2h)^{2n}\int x^{2n}dF(x)+O(|h|^{2n+\lambda}).$$

or,

$$\int \left[1-\left(\frac{\sin hx}{hx}\right)^{2n}\right] x^{2n}dF(x) \leqslant \int \left[1-\cos hx\left(\frac{\sin hx}{hx}\right)^{2n}\right]x^{2n}dF(x) = O(|h|^{\lambda}).$$

Setting $G(x) = \int\limits_{(-\infty,x]} u^{2n}dF(u)$, so that $G(+\infty) < \infty$, we have for small $h > 0$,

$$(1-2^{-2n}) \int\limits_{[hx\geqslant 2]} dG(x) \leqslant \int\limits_{[hx\geqslant 2]} \left[1-\left(\frac{\sin hx}{hx}\right)^{2n}\right] dG(x) = O(h^{\lambda}),$$

whence $G(+\infty)-G(y) \leqslant O(y^{-\lambda})$ as $y\to\infty$. Hence G has moments of all orders $< \lambda$, i.e., F has moments of all orders $< 2n+\lambda$.

Theorem 4 : *Let F be a symmetric d.f. having the moment of order $2n (n \geqslant 0$ integer). A necessary condition for F to have the moment of order $2n+\lambda$, where $0 < \lambda < 2$, is that*

$$\lim_{t\to 0} |t|^{-\lambda}. \log [f^{(2n)}(t)/f^{(2n)}(0)] \text{ (exist and be)} = 0.$$

Proof : We need only consider the case $n = 0$, the general case following from the fact that, if F has the moment of order $2n$, then $f^{(2n)}(t)/f^{(2n)}(0)$ is a c.f., the d.f. corresponding to which has the moment of order λ iff F has the moment of order $2n+\lambda$. We begin by remarking that, for the moment of order λ to exist for F, it is necessary that $x^{\lambda}[1-F(x)] \leqslant \int\limits_{(x,\infty)} u^{\lambda}dF(u)\to 0$ as $x\to\infty$, so that, if a sequence $s_k\to\infty$ as $k\to\infty$ exists such that $s_k^{\lambda}[1-F(s_k)]$ is bounded away from zero, then the moment of order λ does not exist.

Suppose then that $|t|^{-\lambda}. \log f(t)$ does not tend to zero as $t\to 0$. Let c_1, c_2 and c_3 below denote suitable positive constants. Then there exist c_1 and a sequence $\{t_n\} \downarrow 0$ such that $-t_n^{\lambda}. \log f(t_n) \geqslant c_1$. Since, for real x such that $1 \geqslant x \geqslant 1/2$, as we have already noted, $2(1-x) \geqslant -\log x (\geqslant 1-x)$, it follows that $[1-f(t_n)] . t_n^{-\lambda} \geqslant c_2$ for all (sufficiently large) n, i.e., $\int\limits_{(0,\infty)} (1-\cos t_n x)dF(x) \geqslant c_3 t_n^{\lambda}$ (note that the integrand vanishes at the origin so that $\int (1-\cos t_n x)dF(x) = 2 \int\limits_{(0,\infty)} (1-\cos t_n x)dF(x)$). On integrating by parts, we have

$$\lim_{X\to\infty} \int\limits_{(0,X)} \sin t_n x . [1-F(x)]dx \geqslant c_3 t_n^{\lambda-1}.$$

Since $1-F(x)$ is a non-increasing function of x; $(0, X] = (0, \pi/t_n] \bigcup (\pi/t_n, 2\pi/t_n] \bigcup \ldots$; and $\sin t_n(x+\pi/t_n) = -\sin t_n x$; we easily see that the LHS of the above relation is $\leqslant \int\limits_{(0,\pi/t_n)} \sin t_n x . [1-F(x)]dx$, so that setting $h(x, t) = [1-F(x/t)].(x/t)^{\lambda}$, we have

$$\int\limits_0^{\pi} h(x, t_n) . x^{-\lambda} \sin x \, dx \geqslant c_3.$$

Now, either (A) $h(x, t_n)$ is uniformly bounded for all x in $(0, \pi)$ and all n, or (B) there exist sequences $\{x_k\}$ and $\{t_{n_k}\}$ such that $h(x_k, t_{n_k})\to\infty$. In case (B), setting $s_k = x_k/t_{n_k}$, we see that, $1-F(s_k)$ being bounded, $s_k\to\infty$ as $k\to\infty$, and then $[1-F(s_k)]s_k^{\lambda}\to\infty$ implies (by our reasoning earlier) that the moment of order λ does not exist. In case

(A), $x^{-\lambda} \sin x$ being integrable over $(0, \pi)$ since $\lambda < 2$, and $h(x, t_n)$ being uniformly bounded, we have by Fatou's lemma that

$$\int_0^\pi [\limsup_{n\to\infty} h(x, t_n)] x^{-\lambda} \sin x \, dx \geqslant c_3$$

so that $\limsup_{n\to\infty} h(x, t_n) > 0$ on a set of positive Lebesgue measure. All that we need is that this should be true for some $x > 0$, and it follows that F does not have the moment of order λ.

Corollary : *If, at some sequence of points* $\{t_k\} \to 0$, $|t_k|^{-\lambda}. \log |f^{(2n)}(t_k)/f^{(2n)}(0)|$ *is bounded away from zero, where* $0 < \lambda < 2$, *then F does not have the moment of order* $2n+\lambda$. *In particular, a non-Normal stable law with exponent* α *does not have the moment of order* α *(and so of any higher order).*

Remark : As an example due to A. Wintner shows (Lukacs, 1960, p. 32), the condition of the theorem is not sufficient.

We pass on to a NASC (necessary and sufficient condition) on the c.f. for a d.f. to have the absolute moment of any prescribed order. This is essentially contained in (the proof of) Theorem 3 of Boas (1967), where, however, it is obscured by the fact that the concern of that theorem is with the integrability properties of c.f.'s and not with moments *per se*.

Theorem 5 : (a) *A NASC for F to have the moment of order* λ, *where* $0 < \lambda < 2$ *is that, for some* $c > 0$,

$$\int_0^c t^{-\lambda-1}. \log |f(t)| \, dt \text{ exist finitely}$$

or, equivalently,

$$\int_0^c t^{-\lambda-1}. [1 - |f(t)|^2] dt < \infty.$$

(b) *A NASC for F, having the moment of order* $2n$, $n \geqslant 0$ *integer, to have the moment of order* $2n+\lambda$ $(0 < \lambda < 2)$ *is that, for some* $c > 0$,

$$\int_0^c t^{-\lambda-1}. [\log |f^{(2n)}(t)/f^{(2n)}(0)|] dt \text{ exist finitely,}$$

or equivalently,

$$\int_0^c t^{-\lambda-1}. [1 - |f^{(2n)}(t)/f^{(2n)}(0)|^2] dt < \infty.$$

Proof : We need prove only (a). (b) follows from (a) by the usual argument. The fact that F has the moment of an even-integer order iff f has the derivative of that order (at the origin) and Theorem 5 together give us criteria for the existence of the moment of any prescribed order.

Since (cf. our Remark 2 following the statement of Theorem 1).

$$2[1-|f(t)|^2] \geqslant -\log [|f(t)|^2] \geqslant 1-|f(t)|^2 \text{ for } |t| \leqslant \delta,$$

if $\delta > 0$ be chosen and fixed such that $|f(t)|^2 \geqslant 1/2$ for $|t| \leqslant \delta$, we need only establish the "equivalent" condition in (a). It follows that we need only show that for a symmetric d.f. F to have the moment of order λ, $0 < \lambda < 2$, it is necessary and sufficient that $\int_0^\delta t^{-\lambda-1}[1-f(t)]dt < \infty$, where $\delta > 0$ is chosen and fixed such that $f(t) \geqslant 1/2$ for $|t| \leqslant \delta$: in fact, in *this* form for the NASC, $\delta > 0$ can be arbitrary, as is easily seen.

Supose then that $\int |x|^\lambda dF(x) < \infty$ for a symmetric d.f. F. Then

$$\int_0^\delta t^{-\lambda-1}[1-f(t)]dt = \int_0^\delta t^{-\lambda-1}[\int(1-\cos tx)dF(x)]dt$$

$$= \int\Big[\int_0^\delta t^{-\lambda-1}(1-\cos tx)dt\Big]dF(x)$$

$$= \int |x|^\lambda \Big[\int_0^{\delta|x|} v^{-\lambda-1}\,(1-\cos v)dv\Big]dF(x).$$

Now, since $0 < \lambda < 2$, the inner integral is bounded $\int_0^\infty v^{-\lambda-1}(1-\cos v)dv < \infty$ and since $\int |x|^\lambda dF(x) < \infty$. the condition stated is necessary. Conversely, suppose

$$\int_0^\delta [\int(1-\cos tx)dF(x)]t^{-\lambda-1}\,dt < \infty.$$

Then, as above, we see that

$$\int |x|^\lambda \Big[\int_0^{\delta|x|} v^{-\lambda-1}\,(1-\cos v)dv\Big]\,dF(x) < \infty,$$

so that

$$\int_1^\infty x^\lambda \Big[\int_0^{\delta x} v^{-\lambda-1}\,(1-\cos v)dv\Big]dF(x) < \infty.$$

But the inner integral is bounded away from zero, being $\geqslant \int_0^\delta v^{-\lambda-1}(1-\cos v)dv > 0$ since $x \geqslant 1$, so that $\int_1^\infty x^\lambda\,dF(x) < \infty$. F being symmetric, the same is true of $\int_{-\infty}^{-1} |x|^\lambda dF(x)$, and the sufficiency part of the theorem follows.

Remarks : (1) It is interesting to note that, as a consequence of Theorems 4 and 5, it follows that if $\int_0^c [\log|f(t)|]t^{-\lambda-1}dt$ exists finitely for some λ, $0 < \lambda < 2$, and some $c > 0$, then $\log|f(t)|/|t|^\lambda \to 0$ as $t \to 0$. The latter fact is, however, not a simple and direct consequence of the former in that, if g is a non-negative continuous function on $(0, c)$ such that $\int_0^c [g(t)/t]dt$ exists finitely, it does not necessarily follow that $g(t) \to 0$ as $t \to 0$. Theorem 4 is thus not a simple corollary of Theorem 5.

(2) Though the first assertion of Theorem 1(b) follows from Theorem 5, the "more generally" part of Theorem 1(b) is, again, not a simple consequence of

Theorem 5. Similarly for the sufficient condition given under Remark (3) following the statement of Theorem 1. While Theorem 5 is in a sense the last word on the subject, giving as it does a condition which is both necessary and sufficient, we are sometimes faced with situations where it is not applicable, whereas Theorems 1(b) and 4 provide a complete answer regarding the existence of moments : as an example, we may cite the proof of Theorem 3.1 in Ramachandran and Rao (1968), where, from the fact that a certain c.f. f has the property : $\log |f(t_n)|/|t_n|^{\lambda} = \log|f(1)| \neq 0$ for all n, where $\{t_n\}$ is a certain sequence tending to zero and $0 < \lambda < 2$, it is inferred that F has moments of all orders $< \lambda$ but not of any order $\geqslant \lambda$.

(3) Theorems 6 and 7 below again provide instances where Theorems 1, 2 and 4 enable us to draw conclusions which are apparently not directly derivable from Theorem 5.

(4) The NASC of Theorem 5 is an integrability condition on the c.f. Other such NASC's can be given—for example, the condition that the function $t^{-\lambda-2} \int_0^t \log f(h)dh$ be integrable over $(0, c)$ for some $c > 0$ is necessary and sufficient for the symmetric d.f. F to have the moment of order $\lambda (0 < \lambda < 2)$, and of course extends easily to cover moments of order $2n+\lambda$. The proof runs along lines similar to those of the proof of Theorem 5.

We now take up the question of identifying the "critical order" λ_F in terms of f. Theorem 6 is an immediate consequence of Theorem 1 : Part (b) thereof also follows from Theorem 5. We consider below only *symmetric d.f.'s which are further non-degenerate* so that $\log f$ does not vanish identically in a neighbourhood of the origin. Also the phrase "*bounded*" is used below for brevity in place of the complete phrase "*bounded in a deleted neighbourhood of the origin*".

Theorem 6 : *Suppose F has the moment of order $2n (n \geqslant 0$ integer) and let $\lambda = \sup\{\delta : |t|^{-\delta}. \log [f^{(2n)}(t)/f^{(2n)}(0)]$ is bounded\}. Then*

(a) $\lambda \leqslant 2$;

(b) *F has moments of all orders $< 2n+\lambda$;*

(c) *if $\lambda < 2$, then F has no moments of order $> 2n+\lambda$; and*

(d) *if ($\lambda = 2$ and further) $t^{-2}. \log [f^{(2n)}(t)/f^{(2n)}(0)]$ is bounded, then F has the moment of order $2n+2$, and conversely. (The boundedness of this function at some sequence of points tending to the origin is sufficient for the existence of the moment of order $2n+2$).*

Proof : Assertion (a) is a consequence of Theorem 1(d) and our assumption that F is non-degenerate. The other assertions follow from the corresponding assertions of Theorem 1. Assertion (b) above also follows from Theorem 5.

Theorem 7 : *For a d.f. F with moment of order $2n (n \geqslant 0$ integer),*

$$\sup\{\delta : 0 \leqslant \delta \leqslant 2, |t|^{-\delta} \log [f^{(2n)}(t)/f^{(2n)}(0)] \text{ is bounded}\}$$
$$= \sup\{\delta : 0 \leqslant \delta \leqslant 2, \int |x|^{2n+\delta} dF(x) < \infty\}.$$

Proof : We begin by remarking that, apparently, Theorem 5 only permits the deduction that the LHS $\leqslant$ the RHS; to obtain the reverse inequality, some supplementary argument (essentially an appeal to Theorem 2 as in the proof below) appears necessary.

If $\psi_\delta(t) = |t|^{-\delta}. \log [f^{(2n)}(t)/f^{(2n)}(0)]$ is bounded, then, by Theorem (3 or 5 or) 6, moments of all orders $< 2n+\delta$ exist. Conversely, if moments of all orders $< 2n+\delta$ exist, then $f^{(2n)}(t)/f^{(2n)}(0)$ is the c.f. of a (symmetric) d.f. with moments of all orders $< \delta$, so that, by Theorem 2, $f^{(2n)}(t) = f^{(2n)}(0)+O(|t|^\epsilon)$ for every $\epsilon < \delta$, whence it follows that $\psi_\epsilon(t)$ is bounded for all $\epsilon < \delta$. Hence the theorem.

A criterion in terms of the c.f. for the existence of even-integer-order moments being available, Theorems 6 and 7 point the way for identifying the "critical order" λ_F for arbitrary F. Theorem 5 or Theorems 1 and 4 may then be used to test whether the moment of the critical order itself exists or not.

3. Application to infinitely divisible laws

We proceed to apply the results of Section 2 to infinitely divisible (i.d.) laws. If f is an i.d. c.f., suppose $\phi = \log f$ has the Levy representation $L(a, \gamma, M, N)$, i.e.,

$$\phi(t) = iat-\gamma t^2+\int_{(-\infty,\,0)} h(t, u)dM(u)+\int_{(0,\,\infty)} h(t, u)dN(u),$$

where $h(t, u) = e^{itu}-1-[itu/(1+u^2)]$, a is real, $\gamma \geqslant 0$, M and N are respectively non-decreasing on $(-\infty, 0)$ and $(0, \infty)$ respectively, with $M(-\infty) = N(+\infty) = 0$ and $\int_{[-1,\,0)} u^2dM(u)+\int_{(0,\,1]} u^2dN(u) < \infty$. M and N will be called the Levy functions of f or F. The Normal component with $\exp(iat-\gamma t^2)$ as its c.f. has moments of all orders. Also, if $\phi_1(t) = \int_{(0,\,1]} h(t, u)\, dN(u)$, then the component with $\exp(\phi_1)$ as its c.f. also has moments of all orders : for, $\int_0^1 u^2dN(u) < \infty$ implies the existence of ϕ_1' and ϕ_1''; in particular, $\phi_1''(t) = \int_0^1 (iu)^2e^{itu}dN(u)$; then, trivially, $\phi_1^{(n)}(t)$ exists for all t and $= \int_0^1 (iu)^n e^{itu}dN(u)$ for every $n \geqslant 2$; hence $\exp(\phi_1)$ also has derivatives of all orders, and the above assertion follows. Similarly, the component with $\exp\left\{\int_{[-1,\,0)} h(t, u)dM(u)\right\}$ as its c.f. also has moments of all orders. Thus we need only investigate the existence of moments of d.f.'s whose c.f.'s are of one or the other of the forms : $\exp\left[\int_{(1,\,\infty)} h(t, u)dN(u)\right]$ and $\exp\left[\int_{(-\infty,\,-1)} h(t, u)dM(u)\right]$ — indeed it suffices to consider only the former class in view of an obvious duality that exists between the two classes; yet again, it suffices to consider d.f.'s with c.f.'s of the form $\exp\left[\int_{(1,\,\infty)} (\cos tu-1)dN(u)\right]$ by the usual "symmetrization" argument.

We require a couple of auxiliary results.

Lemma 1 : *Let f be a real-valued and even function defined and non-vanishing on the compact interval $I : [-c, c]$, with $f(0) = 1$. If $f^{(2n)}$, the derivative of order $2n$, exists and is continuous on I, and ϕ denotes the continuous version of the logarithm of f with $\phi(0) = 0$, then, for $0 < \lambda < 2$, the two statements :*

$$\int_0^c |f^{(2n)}(t)-f^{(2n)}(0)|\, t^{-1-\lambda}dt < \infty \quad \textit{and} \quad \int_0^c |\phi^{(2n)}(t)-\phi^{(2n)}(0)|\, t^{-1-\lambda}dt < \infty$$

are equivalent. In particular, this is true if f is the c.f. of a symmetric d.f. having the moment of order $2n$.

Proof : Since $f(0) = 1$ and f is continuous (our assumptions imply that f and all its derivatives upto and including order $2n$ are continuous on I), there exists a $\delta > 0$ such that $|f(t)-1| \leqslant 1/2$ for $|t| \leqslant \delta$, so that $\frac{1}{2}\ |f(t)-1| \leqslant |\phi(t)| \leqslant 2\ |f(t)-1|$ for such t. (Note that $1-a \leqslant -\log a \leqslant 2(1-a)$ if $\frac{1}{2} \leqslant a \leqslant 1$, and $\frac{1}{2}(a-1) \leqslant \log a \leqslant a-1$ if $1 \leqslant a \leqslant 2$). For $n = 0$, these relations immediately yield the assertion of the lemma. We discuss below the cases $n \geqslant 1$. Let $g_k(t) = f^{(k)}(t)/f(t)$ for $k = 1, 2, \ldots, 2n$. Then every g_k is defined on I and it is well-known and easy to check that if $\phi = \log f$, then $\phi^{(2n)} = g_{2n}+P(g_{2n-1}, g_{2n-2}, \ldots, g_1)$, where P is a polynomial in the functions indicated, with the property that if a typical term be $=$ constant. $\prod_{j=1}^{2n-1} g_j^{\alpha_j}$, then $\sum_{j=1}^{2n-1} j.\, \alpha_j = 2n$. This property implies that the terms in the polynomial $P(g_{2n-1}, \ldots, g_1)$ can only be one of two kinds : (A) terms in which no g with an odd index occurs as a factor, and (B) terms in which at least one g with an odd index occurs as a factor and consequently *at least two such g's* occur as factors (the same g may be repeated). Since (1) a function continuous on a compact interval is bounded, (2) f is even and (3) $\int_0^c t^{-\lambda+1}\, dt < \infty$ since $(0 <)\lambda < 2$, we have :

(i) $f^{(k)}(t)/f(t)$ is bounded on I for all $k \leqslant 2n$, by (1).

(ii) Since $[f^{(2k)}(t)-f^{(2k)}(0)]/t^2 \to \frac{1}{2} f^{(2k+2)}(0)$ by (2), and so is bounded, for $0 \leqslant k \leqslant n-1$, and hence, in particular, $[f(t)-1]/t^2$ is bounded as well, it follows from (3) that

$$\int_0^c \left| \frac{f^{(2k)}(t)}{f(t)} - f^{(2k)}(0) \right| t^{-1-\lambda}dt < \infty.$$

(iii) $f^{(2m-1)}(0) = 0$ for $m \leqslant n$, by (2); and $f^{(2p-1)}(t)\, .\, f^{(2q-1)}(t)/t^2$ is therefore bounded, since it converges to $f^{(2p)}(0)\ \ f^{(2q)}(0)$, as $t \to 0$; hence it follows from (3) that

$$\int_0^c \left| \frac{f^{(2p-1)}(t) f^{(2q-1)}(t)}{[f(t)]^2} \right| t^{-1-\lambda}dt < \infty.$$

Invoking facts (i) and (ii) above in the case of terms of type (A) above, and all the three facts in the case of terms of type (B), we see that if $P(g_{2n-1}, \ldots, g_1)$ be denoted by Q, then $\int_0^c |Q(t)-Q(0)| \, t^{-1-\lambda}dt$ exists finitely. Hence it follows that (for $0 < \lambda < 2$)

$$\int_0^c |\phi^{(2n)}(t)-\phi^{(2n)}(0)|\, t^{-1-\lambda}dt \quad \text{and} \quad \int_0^c |g_{2n}(t)-g_{2n}(0)| \, . \, t^{-1-\lambda}dt$$

are both of them finite or both of them infinite. But the latter is finite iff

$$\int_0^c |f^{(2n)}(t)-f^{(2n)}(0)| \, . \, t^{-1-\lambda}dt < \infty,$$

since $\int_0^c [1-f(t)]t^{-1-\lambda}dt < \infty$. Hence the lemma.

Lemma 2 : *If* $\phi(t) = \int_{(1,\infty)} (\cos tu-1)dN(u)$, *then the d.f. with* ϕ *as its* $\log f$ *has the moment of an even-integer order* $2n$ *iff* $\int_{(1,\infty)} u^{2n} \, dN(u) < \infty$.

Proof : If F has the moment of order $2n$, then f and hence $\phi = \log f$ has the derivative of order $2n$, so that

$$(-1)^n\phi^{(2n)}(0) = (-1)^n \lim_{h\to 0} \left\{ \left[\phi(nh)-\binom{2n}{1} \phi((n-1)h)+\ldots+\phi(-nh)\right] \Big/ h^{2n}\right\}$$

$$= (-1)^n \lim_{h\to 0} \{h^{-2n} \text{ Re} \int_{(1,\infty)} (e^{ihu/2}-e^{-ihu/2})^{2n}dN(u)\}$$

$$= \lim_{h\to 0} \int_{(1,\infty)} \left(\frac{\sin (hu/2)}{hu/2}\right)^{2n} . \, u^{2n}dN(u) \geqslant \int_{(1,\infty)} u^{2n}dN(u)$$

by Fatou's lemma. Conversely, if $\int_{(1,\infty)} u^{2n} \, dN(u) < \infty$, then $\phi(t) = \int_{(1,\infty)} (\cos tu-1)dN(u)$ can be differentiated (under the integral sign) $2n$ times, so that the same is true of f, and consequently F has the moment of order $2n$.

Theorem 8 : *Let* f *be an i.d.c.f. with the Levy representation* $L(u, \gamma, M, N)$. *Then* F *has the moment of order* λ *iff* $\int_{(-\infty,-1)} |u|^\lambda dM(u)+ \int_{(1,\infty)} u^\lambda dN(u) < \infty$. (*If* $\lambda \geqslant 2$, *this can be more succinctly stated thus : F has the moment of order λ iff both M and N have the same*).

Proof : In view of the remarks earlier, it suffices to prove the assertion in the case where $\log f = \phi$ has the form : $\phi(t) = \int_{(1,\infty)} (\cos tu-1)dN(u)$. From Lemma 2 above, it follows that F has the moment of order $2n(n \geqslant 0$ integer) iff $\int_{(1,\infty)} u^{2n}dN(u) < \infty$. Suppose then that F has the moment of order $2n$ and let us investigate conditions for the existence of the moment of order $2n+\lambda$, where $0 < \lambda < 2$. By Theorem 5, this moment exists iff $\int_0^c t^{-1-\lambda} |f^{(2n)}(0)-f^{(2n)}(t)| dt < \infty$ for some $c > 0$. By Lemma 1, this is equivalent to

$$\int_0^c t^{-1-\lambda} |\phi^{(2n)}(t)-\phi^{(2n)}(0)| dt < \infty,$$

i.e.,
$$\int_0^c t^{-1-\lambda} \left[\int_{(1,\infty)} u^{2n}(1-\cos tu) dN(u) \right] dt < \infty,$$

i.e.,
$$\int_{(1,\infty)} u^{2n} \left[\int_0^c t^{-1-\lambda}(1-\cos tu) dt \right] dN(u) < \infty,$$

i.e.,
$$\int_{(1,\infty)} u^{2n+\lambda} \left[\int_0^{cu} v^{-1-\lambda}(1-\cos v) dv \right] dN(u) < \infty.$$

As in the proof of Theorem 5, this is equivalent to $\int_{(1,\infty)} u^{2n+\lambda} dN(u) < \infty$. Hence the theorem.

Theorem 9 : *Let F be an i.d. law with the Levy representation* $L(a, \gamma, M, N)$ *for* $\log f$. *Then*

(a) $\lambda_F = sup \; \{\delta : \int_{(-\infty,-1)} |u|^\delta dM(u) + \int_{(1,\infty)} u^\delta dN(u) < \infty\}$;

(b) $\lambda_F =$ *the smaller of the two quantities below :*

$$\lim_{u\to-\infty} inf \left[\frac{-log\; M(u)}{log\; |u|} \right] \quad and \quad \lim_{u\to\infty} inf \left[\frac{-log\; |N(u)|}{log\; u} \right].$$

$\left(\text{In terms of } F \text{ itself, } \lambda_F = \liminf_{x\to\infty} \left\{ \frac{-\log [1-F(x)+F(-x)]}{\log x} \right\}. \right)$

Note : These three quantities are to be taken as $+\infty$ if respectively $M(-u)$, $N(u)$, $1-F(u)+F(-u)$ vanishes for (some $u > 0$ and so for) all large $u > 0$.

Proof : Assertion (a) follows at once from Theorem 8. Assertion (b) follows from (a) and Theorem 3.1 of Ramachandran (1962)— also see pp. 19-20 of Ramachandran (1967) : the statement in parentheses above gives the statement of that theorem.

4. Concluding remarks

It is instructive to examine the foregoing results in relation to the stable laws. A non-Normal stable law with 'exponent' α ($0 < \alpha < 2$) has its $|f(t)|$ of the form $\exp(-c|t|^\alpha)$, where $c > 0$; it is infinitely divisible, with the Levy functions M and N given by the formulas : $M(u) = c_1|u|^{-\alpha}$ and $N(u) = -c_2 u^{-\alpha}$ (for $u < 0$ and $u > 0$ respectively), where $c_1 \geqslant 0$, $c_2 \geqslant 0$, and $c_1+c_2 > 0$. Theorems 1(a) and 1(b) as well as Theorem 9 confirm that such a law has moments of all orders $< \alpha$ (this is also confirmed by Theorem 3) but not of any order $> \alpha$, and Theorem 4 confirms that the moment of the critical order α does not exist. All these conclusions are simultaneously yielded by Theorem 5 or by Theorem 8. As already stated, Theorems 1(b) and 4 enable us to arrive at similar conclusions in respect of the semi-stable laws and of a certain class of 'generalized stable laws' considered by Ramachandran and Rao (1968, Section 3). In all these cases, the critical order λ_F coincides with the 'exponent' of the law concerned.

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