A NOTE ON SOME INADMISSIBLE ESTIMATORS

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SUMMARY. Hanurav (1966) has conjectured that any non-zero function of a minimal sufficient statistic is an admissible estimator of its expectation. In this note we give examples to show that the conjecture is false both in the survey-sampling set-up and in the conventional set-up.

1. Introduction

Let $(U_1, \ldots, U_N)$ be a population of N identifiable units. Let X_i be the value of a real-valued characteristic on the unit U_i. The vector $\theta = (X_1, \ldots, X_N)$ then becomes an unknown parameter and R^N becomes the parameter space. For any mode of sampling, let s denote the set of distinct units in the sample and let Y_s denote the set of the X-values corresponding to the units in s. If each element of Y_s is measured without error, then the pair $T = (s, Y_s)$ is a minimal sufficient statistic. *In this note we consider only unbiased estimation with squared error as loss.*

Hanurav (1966, page 197) has conjectured that any non-zero function of T is an admissible estimator of its expectation. As stated, this conjecture is obviously false. For example, the zero parametric function admits the zero estimator as a uniformly-minimum-variance unbiased estimator and thus any non-zero unbiased estimator of the zero function is inadmissible. For this reason we propose the following modified conjecture, which makes sense even in the conventional set-up of statistical estimation.

Modified conjecture : Let $h(\theta)$ be a parametric function for which no uniformly-minimum-variance unbiased estimator exists. Let t be any function of a minimal sufficient statistic T which is an unbiased estimator of $h(\theta)$. Then t is admissible.

The purpose of this note is to show that even this modified conjecture is false. Counter-examples are given in Section 2 for survey-sampling situations and in Section 3 for conventional situations.

2. Counter examples in survey-sampling

Our counterexamples are based on the following simple

Lemma : *Let t_1 be an unbiased estimator of $h(\theta)$. Let t_2 be a non-zero unbiased estimator of zero. If t_1 and t_2 are uncorrelated for all θ, then (t_1+t_2) is inadmissible.*

Proof : The assumptions imply that $\text{var}_\theta(t_1+t_2) = \text{var}_\theta(t_1)+\text{var}_\theta(t_2)$. Thus t_1 dominates (t_1+t_2).

For the remainder of this section we assume that $N = 3$ and that we take a simple random sample of 2 units without replacement. We consider unbiased estimation of $\bar{X} = (X_1+X_2+X_3)/3$. Define two estimators t_1 and t_2 as follows :

s	Y_s	value of t_1	Value of t_2
(U_1, U_2)	(X_1, X_2)	$\alpha_1X_1+(1-\alpha_2)X_2$	$a_1X_1-a_2X_2$
(U_2, U_3)	(X_2, X_3)	$\alpha_2X_2+(1-\alpha_3)X_3$	$a_2X_2-a_3X_3$
(U_3, U_1)	(X_3, X_1)	$\alpha_3X_3+(1-\alpha_1)X_1$	$a_3X_3-a_1X_1$

Observe that t_1 is a general homogeneous linear unbiased estimator of $\bar{X}$ and that t_2 is a general homogeneous unbiased estimator of zero. Easy computations show that t_1 and t_2 are uncorrelated for all values of θ if, and only if, the following conditions hold.

$$a_1\alpha_1 = a_1(1-\alpha_1),\ a_2\alpha_2 = a_2(1-\alpha_2),\ a_3\alpha_3 = a_3(1-\alpha_3); \qquad \ldots \quad (1)$$

and

$$a_2\alpha_1 = a_1(1-\alpha_2),\ a_3\alpha_2 = a_2(1-\alpha_3),\ a_1\alpha_3 = a_3(1-\alpha_1). \qquad \ldots \quad (2)$$

Assume now that t_2 is a non-zero estimator. This means that at least one of the a's is non-zero. The situations in which t_1 and t_2 are uncorrelated for all θ can be classified as follows.

Class A : $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2}$ and $a_1 = a_2 = a_3 \neq 0$.

Class B : Exactly two of the α's equal $\frac{1}{2}$. For the sake of definiteness let $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $\alpha_3 \neq \frac{1}{2}$. Then (1) implies that $a_3 = 0$. Now (2) shows that $a_1 = a_2 \neq 0$ and yields the contradiction $\alpha_3 = 1$ and $\alpha_3 = 0$. Thus this class is empty.

Class C : Exactly one of the α's equals $\frac{1}{2}$ For the sake of definiteness, let $\alpha_1 = \frac{1}{2}$, $\alpha_2 \neq \frac{1}{2}$ and $\alpha_3 \neq \frac{1}{2}$. Then (1) shows that $a_2 = a_3 = 0$ and $a_1 \neq 0$. Now (2) implies that $\alpha_2 = 1$ and $\alpha_3 = 0$.

Class D : None of the α's equals $\frac{1}{2}$. Here (1) shows that $a_1 = a_2 = a_3 = 0$. Thus this class is empty.

The lemma now shows that the values of the vector $(\alpha_1, \alpha_2, \alpha_3, a_1, a_2, a_3)$ which make (t_1+t_2) inadmissible are

$$(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, a, a, a),\quad (\tfrac{1}{2}, 1, 0, a, 0, 0),$$

$$(0, \tfrac{1}{2}, 1, 0, a, 0) \text{ and } (1, 0, \tfrac{1}{2}, 0, 0, a),$$

where a is any non-zero real number. Since $\bar{X}$ does not possess a uniformly-minimum-variance unbiased estimator and since (t_1+t_2) is a function of T, the modified conjecture of Section 1 stands disproved.

3. A counterexample in the conventional set-up

We use a standard example of an incomplete family of probability distributions (cf. Lehmann, 1959, page 152). Let $0 < \theta < 1$ and let T be a random variable such that

$$P_\theta[T = -1] = \theta \text{ and } P_\theta[T = n] = (1-\theta)^2\theta^n \quad \text{for } n = 0, 1, 2, \ldots$$

Then it is easy to verify that the only unbiased estimators of zero are of the form cT where c is an arbitrary real number. It is also easy to check that $E_\theta(T^2) = 2\theta/(1-\theta)$. Here T is clearly a minimal sufficient statistic. An unbiased estimator of θ is t_0 defined by

$$t_0 = \begin{cases} 1 & \text{if } T = -1 \\ 0 & \text{if } T = 0, 1, 2, \ldots \end{cases}$$

Therefore any unbiased estimator of θ is of the form $t = t_0+cT$, where $c\epsilon R$.

Theorem : *For t to be admissible it is necessary and sufficient that $0 \leqslant c \leqslant \frac{1}{2}$.*

Proof: Observe first that $\text{var}_\theta\,(t_0) = \theta(1-\theta)$ and $\text{cov}_\theta\,(t_0, T) = -\theta$. Therefore

$$\text{var}_\theta\,(t_c) = \theta(1-\theta)-2c\theta+\frac{2c^2\theta}{(1-\theta)}$$

$$= \frac{\theta}{2(1-\theta)}[(1-\theta)^2+f_\theta(c)],$$

where $f_\theta(c) = (1-\theta-2c)^2$. Now $0 < \theta < 1$ and $c > \frac{1}{2}$ imply that $f_\theta(c) > f_\theta\left(\frac{1}{2}\right)$. Thus, for $c > \frac{1}{2}$, $t_{1/2}$ dominates t_c. Similarly, for $c < 0$, t_0 dominates t_c. Thus t_c is inadmisible whenever $c < 0$ or $c > \frac{1}{2}$. Next, if $0 < c < \frac{1}{2}$ and $c' \neq c$, then $f_{1-2c}(c) = 0 < f_{1-2c}(c')$. Thus t_c is admissible whenever $0 < c < \frac{1}{2}$. Finally the admissibility of t_0 and $t_{1/2}$ follows from the observations that, if $c \neq 0$, then

$$\lim_{\theta \to 1} \frac{\text{var}\,(t_0)}{\text{var}\,(t_c)} = 0,$$

and, if $c \neq \frac{1}{2}$, then

$$\lim_{\theta \to 0} \frac{\text{var}_\theta(t_{1/2})}{\text{var}_\theta(t_c)} < 1.$$

The proof of the theorem is thus complete.

Since there are at least two admissible estimators, there is no uniformly-minimum-variance unbiased estimator of θ. The theorem thus shows that the modified conjecture of Section 1 is false in the conventional set-up also.

REFERENCES

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