SOME RESULTS ON IDEMPOTENT MATRICES AND A MATRIX EQUATION CONNECTED WITH THE DISTRIBUTION OF QUADRATIC FORMS

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SUMMARY. In this paper general solutions (X) are obtained for matrix equations (i) $XBX = 0$, (ii) $XBXB = XB$ and (iii) $XBXBX = XBX$.

1. Introduction and preliminaries

In an earlier paper one of the authors, Mitra (1968) obtained the most general solution (X) of matrix* equations $XBX = X$ and $BXBXB = BXB$. In the present paper we solve some other matrix equations of related interest.

The following notations are used. Matrices and vectors are denoted by bold face letters such as B, C, D, X, a, b etc. matrices by capital letters and column vectors by lower case letters. 0 indicates a null matrix and 0 a null vector. A matrix B of order $m\times n$ will sometimes be denoted by $B(m\times n)$. For a matrix $B(m\times n)$,

$R(B)$ denotes its rank.

B^- denotes a generalised inverse (see Rao, 1967).

B_r^- denotes a reflexive g-inverse.

$B^\perp$ denotes a matrix of rank $m-R(B)$ such that $B'B^\perp = 0$.

2. Some results on idempotent matrices

Lemma 2.1 : *The most general form of an idempotent matrix of order n is given by*

$$H = C^-C$$

where m as well as $C(m\times n)$ are arbitrary.

Proof: Observe that by Theorem 2a of Rao (1967), $H = C^-C$ is idempotent. Conversely, if $H(n\times n)$ is idempotent, H is a g-inverse of itself. Choosing $m = n$, $C=C^-=H$ we have $C^-C = H^2 = H$. Q.E.D.

Lemma 2.2 : *For a matrix $B(m\times n)$, XB is idempotent i.e. $XBXB = XB$ if and only if $X(n\times m)$ is of the form*

$$X = (CB)^-C+E(B^\perp)'$$

where $p, q \geq m-R(B)$, $C(p\times m)$, $E(n\times q)$ and $B^\perp(m\times q)$ are otherwise arbitrary.

Proof: $[(CB)^-C+E(B^\perp)']B = (CB)^-CB$ is obviously idempotent. Conversely, if XB is idempotent XB is a g-inverse of itself. Choosing $p = n$, $C = X$, $(CB)^- = XB$ we have $(CB)^-C = XBX$. Check that $[X-(CB)^-C]B = [X-XBX]B = 0$. Q.E.D.

*In this paper we consider matrices over the field of real numbers. The minor modifications necessary for the complex case are obvious.

Lemma 2.3 : *If X and B are symmetric matrices of order m then XB is idempotent if and only if X is of the form*

$$C'(CBC')_r^- C+B^{\perp}D(B^{\perp})'$$

where $p, q \geqslant m-R(B)$, $C(p\times m)$, $B^{\perp}(m\times q)$ are arbitrary, $D(q\times q)$ is an arbitrary diagonal matrix and $(CBC')_r^- = GCBC'G'$, G being an arbitrary g-inverse of CBC'.

Proof : Observe that

$$[C'(CBC')_r^- C+B^{\perp}D(B^{\perp})']B = C'(CBC')_r^- CB$$

is idempotent. Conversely, if XB is idempotent, BXB is a symmetric g-inverse of XBX. Choosing $C = X$,

$$G = BXB, \ (CBC')_r^- = BXBXBXBXB = BXB$$

we have

$$C'(CBC')_r^- C = XBXBX = XBX.$$

Rest of the proof follows as in Lemma 2.2. Q.E.D.

3. Nonnegative definite generalised inverse

In this section we state a few lemmas which are easy to establish.

Lemma 3.1 : *A symmetric matrix has a nonnegative definite (n.n.d.) g-inverse if and only if the matrix itself is n.n.d*

Lemma 3.2 : (The most general form of a n.n.d. g-inverse) : *Let $B = MM'$ be a n.n.d. matrix of order m where $M(m\times r)$ is a matrix of rank r. Then G is a n.n.d. g-inverse of B if and only if G can be expressed as $K'K$ where*

$$K = LM^- + U(I-MM^-)$$

where

n is arbitrary positive integer,

$L(n\times r)$ is an arbitrary semiorthogonal matrix i.e. $L'L = I_r$,

$U(n\times m)$ is arbitrary,

and M^- is any g-inverse of M.

Lemma 3.3 : *Let B be a n.n.d. matrix of order m and $X(s\times m)$ of rank s, then XBX' is idempotent if and only if X can be expressed as $X = YC$, where*

$$Y = L(N'N)^{-1}N' + U[I-N(N'N)^{-1}N'],$$

$C(p\times m)$ is an arbitrary matrix of rank p, the matrix

$N(p\times t)$ satisfies the equation $CBC' = NN'$, $t = R(N) = R(CBC')$,

$L(s\times t)$ is an arbitrary semiorthogonal matrix, i.e. $L'L = I_t$,

$U(s\times p)$ is arbitrary otherwise except that

$$R[U(I-N(N'N)^{-1}N')] = s-t.$$

4. Solution of equations $XBX = 0$ and $XBXBX = XBX$

Lemma 4.1 : *For a matrix* $B(m \times n)$, $XBX = 0$ *if and only if* $X(n \times m)$ *is of the form*

$$X = YC$$

where p *as well as* $C(p \times m)$ *are arbitrary and* Y *is an arbitrary solution of the equation* $CBY = 0$.

Proof: The "if" part is trivial. To prove the "only if" part, let X be a solution of the equation $XBX = 0$ of rank $R(X) = p$ and let $X = DC$ be a rank factorisation of X. Check $DCBDC = 0 \Longrightarrow CBD = 0$. Q.E.D.

Lemma 4.2 : *For matrices* $B(m \times n)$, $W(q \times m)$,

$$XBX = 0 \text{ and } WBX = 0$$

if and only if $X(n \times m)$ *is of the form*

$$X = YC$$

where p *as well as* $C(p \times m)$ *are arbitrary and* $Y(n \times p)$ *is an arbitrary solution of the equation*

$$\begin{pmatrix} C \\ W \end{pmatrix} BY = 0.$$

Proof: The proof is similar to that of Lemma 4.1.

Lemma 4.3 : *For a matrix* $B(m \times n)$, $XBXBX = XBX$ *if and only if* $X = Z + W$ *where* W *is a solution of the equation* $WBW = W$ *and* Z *satisfies the equations*

$$ZBZ = 0 \text{ and } WBZ = 0.$$

Proof: If Z and W are determined as in Lemma 4.3 we have

$$\begin{aligned}(Z+W)B(Z+W) &= ZBZ + WBZ + ZBW + WBW \\ &= ZBW + W\end{aligned}$$

and

$$\begin{aligned}(Z+W)B(Z+W)B(Z+W) &= (ZBW+W)B(Z+W) \\ &= ZBWBZ + ZBWBW + WBZ + WBW \\ &= ZBW + W.\end{aligned}$$

Hence $X = Z + W$ satisfies the equation $XBXBX = XBX$. Conversely, let X be a solution of the equation $XBXBX = XBX$. Observe that $W = XBX$ satisfies the equation $WBW = W$ and

$$\begin{aligned}(X - XBX)B(X - XBX) &= XBX - XBXBX - XBXBX + XBXBXBX \\ &= 0.\end{aligned}$$

Also, $$WB(X - XBX) = XBXBX - XBXBXBX = 0.$$

Q.E.D.

It was shown in Mitra (1968) that the most general solution of the equation $WBW = W$ is given by

$$W = Q(PBQ)_r^- P$$

where P and Q are arbitrary. Using Lemma 4.2 we observe therefore that the most general solution of the equation $XBXBX = XBX$ is given by

$$X = Z+W$$

where $$W = Q(PBQ)_r^- P$$

and $$Z = YC$$

C, P and Q being arbitrary and Y an arbitrary solution of the equation

$$\begin{pmatrix} C \\ W \end{pmatrix} BY = 0.$$

5. Concluding remarks

If the vector-valued random variable x follows a $N_m(0, \Sigma)$ distribution, the quadratic form $x'Bx$ has a chi-square distribution if and only if

$\Sigma B \Sigma B \Sigma = \Sigma B \Sigma$ (Ogasawara and Takahashi, 1951 ; Khatri, 1963 ; Rao, 1965).

Given a symmetric matrix B, Lemmas 3.3 and 4.3 are useful in determining the class of n.n.d. matrices Σ for which the chi-square distribution holds for $x'Bx$.

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Paper received : November, 1969.