SANKHYA

THE INDIAN JOURNAL OF STATISTICS

Editors: P. C. MAHALANOBIS, C. R. RAO

SERIES A, VOL. 32 MARCH 1970 PART 1

SOLUTIONS OF FUNCTIONAL EQUATIONS ARISING IN SOME REGRESSION PROBLEMS AND A CHARACTERIZATION OF THE CAUCHY LAW

By B. RAMACHANDRAN

and

C. RADHAKRISHNA RAO

Indian Statistical Institute

SUMMARY: The paper examines in some detail the nature of the probability distribution of the independent and identically distributed random variables (i.i.d.r.v.'s) $X_1, X_2, \ldots$, which possess the property $E(a_1 X_1+a_2X_2+\ldots|b_1X_1+b_2X_2,+\ldots) = 0$ a. s. Both the cases of finite and infinite number of variables are considered. The distribution depends on the nature of the coefficients a_i, b_i, and their relationships.

1. INTRODUCTION

In Ramachandran and Rao (1968)—abbreviated hereafter as R-R (1968)—solutions of the regression equation

$$E(a_1X_1+\ldots+a_mX_m|b_1X_1+\ldots+b_mX_m) = 0 \text{ a.s., } m \geqslant 2 \qquad \ldots \quad (1)$$

where the X_j are independent and identically distributed random variables (i.i.d.r.v.'s) with $EX_1 = 0$, were considered under various conditions on the coefficients a_j, b_j (by a solution, we mean an identification or description of the distribution of X_1). Under certain conditions on these coefficients, the characteristic function (c.f.) of X_1 itself or the c.f. of X_1-X_2 (i.e., the squared modulus of the former) was found to be non-vanishing throughout the real line and to satisfy an equation of the form

$$f(t) = \prod_1^p [f(\beta_j t)]^{\gamma_j} \cdot \prod_{p+1}^n [f(-\beta_j t)]^{\gamma_j} \quad \text{for all } t \qquad \ldots \quad (2)$$

where $n \geqslant 1$, $0 \leqslant p \leqslant n$, $0 < \beta_j < 1$ and $\gamma_j > 0$ for all j. All c.f.'s f satisfying (2) —not merely those corresponding to d.f.'s with finite first moment —were studied in our earlier work (R-R, 1968, Theorems 3.1 and 3.2) and it was established that f is an infinitely divisible (i.d.) c.f., and ignoring the trivial case of degenerate laws as solutions, if λ be the unique real number such that $\sum_1^n \gamma_j \beta_j^\lambda = 1$, then we must have $0 < \lambda \leqslant 2$; further (i) f is normal iff $\lambda = 2$; (ii) if $0 < \lambda < 2$, then f corresponds to a distribution which is absolutely continuous, and has absolute moments of all orders $<\lambda$

but not of order λ. Since every stable law is a solution of (2) under suitable conditions on the β_j and γ_j (cf. Section 1 and the statements of Theorems 3 and 6 below) and since the above properties are strongly reminiscent of the stable laws, the class of c.f.'s satisfying (2) was called in our earlier work a class of 'generalized stable laws', and λ the *exponent* of such a law. We also merely noted there the fact that if $L(\mu, \sigma, M, N)$ be the Levy representation for $\log f$, where M and N are respectively the negative and the positive Poisson spectral functions, which we shall without loss of generality take to be respectively left-and right-continuous, then M and N satisfy the relations

$$N(u) = \sum_1^p \gamma_j N(u/\beta_j) - \sum_{p+1}^n \gamma_j M(-u/\beta_j) \quad \text{for } u > 0$$

$$M(u) = \sum_1^p \gamma_j M(u/\beta_j) - \sum_{p+1}^n \gamma_j N(-u/\beta_j) \quad \text{for } u < 0. \qquad \ldots \quad (3)$$

If we write $g(u) = -N(e^u)$ and $h(u) = M(-e^u)$, we have for all real u the relations $(B_j = -\log \beta_j)$

$$g(u) = \sum_1^p \gamma_j g(u+B_j) + \sum_{p+1}^n \gamma_j h(u+B_j)$$

$$h(u) = \sum_1^p \gamma_j h(u+B_j) + \sum_{p+1}^n \gamma_j g(u+B_j) \qquad \ldots \quad (4)$$

where g and h are both non-negative, non-increasing and right-continuous functions on the real line with $g(+\infty) = h(+\infty) = 0$.

Studying, *per se* and independently of such considerations as the above, c.f.'s f satisfying the equation

$$f(t) = \prod_1^p f(\beta_j t) \prod_{p+1}^n f(-\beta_j t) \quad \text{for all } t \qquad \ldots \quad (5)$$

where $0 < \beta_j < 1$ for all j—in which case the non-vanishing nature of f *follows* from (5), and the infinite divisibility of f is easily established—R. Shimizu (1968) obtained the explicit forms of the Levy functions M and N in the representation for $\log f$ in the various possible cases. His analysis generalizes readily to the more general equation (2). We shall however present below a much simpler and transparent proof which also has the advantage that it carries over to cases where an infinite number of factors are present in the R.H.S. of (2), i.e., where we have a relation of the form

$$f(t) = \prod_1^\infty [f(\beta_j t)]^{\gamma_j} \quad \text{for all } t \qquad \ldots \quad (6)$$

with $|\beta_j| < 1$ and $\gamma_j > 0$ for all j.

In Section 1, we briefly consider the case $n = 1$ and obtain a characterization of the Cauchy law. In Section 2, we discuss the complete solution of (2) for $n > 1$. In Section 3, we discuss solutions of (6). In Section 4, we deal with applications of the results of the earlier sections to regression problems, and finally in Section 5 with characterizations of certain stochastic processes.

It is convenient to use the term 'non-trivial c.f.' to denote a c.f. not pertaining to a degenerate distribution.

2. The case $n = 1$, and a characterization of the Cauchy law

In this case, (2) takes one of two forms :

$$f(t) = [f(\beta t)]^{\gamma} \quad \ldots \quad (7a)$$

$$f(t) = [f(-\beta t)]^{\gamma} \quad \ldots \quad (7b)$$

where $0 < \beta < 1$, $\gamma > 0$. The infinite divisibility of f in either case is easy to establish—in fact, both are 'semi-stable laws' in the sense of Paul Levy—and, f being assumed non-trivial, if λ be the unique real number such that $\gamma\beta^{\lambda} = 1$, then $0 < \lambda \leqslant 2$, and f is a normal c.f. iff $\lambda = 2$. For $0 < \lambda < 2$, $\sigma = 0$ in the Levy representation $L(\mu, \sigma, M, N)$, and the Levy functions M and N satisfy the relations

$$M(u) = \gamma M(u/\beta),\ N(u) = \gamma N(u/\beta) \qquad \text{if (7a) holds}$$

and

$$M(u) = \gamma N(-u/\beta),\ N(u) = -\gamma M(-u/\beta) \qquad \text{if (7b) holds.}$$

It is immediate that the solutions in these two cases are respectively

$$\text{(a)} \qquad M(u) = \frac{\xi(\log|u|)}{|u|^{\lambda}} \quad \text{and} \quad N(u) = \frac{-\eta(\log u)}{u^{\lambda}} \quad \ldots \quad (8a)$$

where ξ and η are non-negative right-continuous functions on the real line with period $B = -\log\beta$; and

$$\text{(b)} \qquad M(u) = \frac{\xi(\log|u|)}{|u|^{\lambda}} \quad \text{and} \quad N(u) = \frac{-\xi(\log u+B)}{u^{\lambda}} \quad \ldots \quad (8b)$$

where ξ is a non-negative, right-continuous function on R_1 with period $2B$.

Of course, in either case [(7a) or (7b)], f satisfies the relation $f(t) = [f(\beta^2 t)]^{\gamma^2}$ and thus is a 'semi-stable law'. It is known from R-R (1968) that the d.f. of f is absolutely continuous and has absolute moments of all orders $< \lambda$ but not of order λ.

Some of the above observations imply the following characterization of the Cauchy law. It is well-known that if $X_1, \ldots, X_n$ are i.i.d.r.v.'s with a Cauchy distribution, then X_1 and $\bar{X}_{(n)}$, the arithmetic mean of the n r.v.'s, have the same distribution. We have the following strong converse of this proposition. A similar theorem, under the extra assumption that the r.v.'s are symmetric, appears in Eaton (1966).

Theorem 1 : *Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s. If X_1 and $\bar{X}_{(n)}$ have the same distribution for two values n_1 and n_2 of n such that $\log n_1/\log n_2$ is irrational, then X_1 follows a Cauchy law.*

Proof : If f be the c.f. of X_1, then

$$f(t) = [f(t/n)]^{n} \quad \text{for all } t. \quad \ldots \quad (9)$$

If this is true for $n = n_1$ and $n = n_2$, then (8a) holds with $\lambda = 1$ in either case, with ξ and η having two periods $\rho_1 = \log n_1$ and $\rho_2 = \log n_2$. If ρ_1/ρ_2 is irrational, then the set

$$\{m_1\rho_1+m_2\rho_2 : m_1 \text{ and } m_2 \text{ integers}\}$$

is everywhere dense on the real line, and $\xi(m_1\rho_1+m_2\rho_2) = \xi(0)$ and the right-continuity of ξ then imply that $\xi \equiv \xi(0)$ and similarly $\eta \equiv \eta(0)$. Thus

$$M(u) = c_1/|u| \text{ and } N(u) = -c_2/u,$$

where $c_1 \geqslant 0$, $c_2 \geqslant 0$ and $c_1+c_2 > 0$. Hence (cf. Lukacs, 1960, p. 102), for some $c > 0$, and b real ($|b| \leqslant 1$),

$$\log f(t) = i\mu t - c|t| \left\{1+(2/\pi)ib \frac{t}{|t|} \log|t|\right\} \text{ for } t \neq 0$$

and on substituting in (0) with $n = n_1$ or n_2, we get $b = 0$ and f is therefore a Cauchy c.f.

3. The case $n > 1$

As stated in the Introduction, our results and proofs in this section are respectively suitably modified (to cover the more general case we are dealing with), and considerably simpler, versions of Shimizu (1968). We have invoked results from that paper as well as from the pioneering work of Yu. V. Linnik (1953) wherever necessary; we have however presented the proofs of this section in some detail for two reasons: there are many points of difference between Shimizu (1968) and the present section, which make cross-references to the former inconvenient; more importantly, since our approach here is also applied in Section 3 to the 'infinite case', the details are given here and reduced to a minimum there.

We introduce some notation (cf. Shimizu, 1968): Let A_n be the set of all vectors $B = (B_1, ..., B_n)$ with all elements positive, and consider the following subsets of A_n, where $0 \leqslant p \leqslant n$:

$A_n(0)$: at least one pair of the B_j are mutually incommensurable.

$A_n(\rho)$: the B_j are mutually commensurable, and $\rho > 0$ is such that $m_j = B_j/\rho$, $j = 1, 2, ..., n$, are positive integers with their greatest common factor $= 1$. (The m_j are described by oversight as 'mutually prime' in Shimizu, 1968).

$B_n^p(\rho)$: the subset of $A_n(\rho)$ such that at least one of $m_1, ..., m_p$ is odd and/or at least one of $m_{p+1}, ..., m_n$ is even.

$C_n^p(\rho)$: the subset of $A_n(\rho)$ such that $m_1, ..., m_p$ are all even and $m_{p+1}, ..., m_n$ are all odd.

Note that $B_n^n(\rho) = A_n(\rho)$ and $C_n^n(\rho)$ is empty, by the definition of ρ, and that any member of A_n must belong either to $A_n(0)$, or, for some $\rho > 0$, to $B_n^p(\rho)$ or to $C_n^p(\rho)$.

The following result is the extended version, applicable to our situation, of the basic auxiliary result of Shimizu (1968).

Theorem 2 : *Let g and h be non-negative, non-increasing and right-continuous functions defined on the real line, with $g(+\infty) = h(+\infty) = 0$ and satisfying the relations (4) for some γ and B in A_n. Then*

(i) $g \equiv h \equiv 0$ *if* $\gamma_1 + \ldots + \gamma_n \leqslant 1$.

(ii) *If* $\gamma_1 + \ldots + \gamma_n > 1$, *let* $\lambda > 0$ *be the unique solution of* $\sum_1^n \gamma_j e^{-B_j \lambda} = 1$. *Then we have*

(a) *if* $B \in A_n(0)$, *then*

$$g(u) = \xi\, e^{-\lambda u} \text{ and } h(u) = \eta\, e^{-\lambda u} \qquad \ldots \quad (10a)$$

where ξ and η are non-negative real constants (with $\xi + \eta > 0$), and $\xi = \eta$ if $p < n$;

(b) *if* $B \in B_n^p(\rho)$, *then*

$$g(u) = \xi(u) e^{-\lambda u} \text{ and } H(u) = \eta(u)\, e^{-\lambda u} \qquad \ldots \quad (10b)$$

where ξ and η are non-negative, right-continuous and periodic with period ρ; further, $\xi = \eta$ if $p < n$;

(c) *if* $B \in C_n^p(\rho)$, *so that* $p < n$, *then*

$$g(u) = [\xi(u) + \eta(u)] e^{-\lambda u} \text{ and } h(u) = [\xi(u) - \eta(u)] e^{-\lambda u} \qquad \ldots \quad (10c)$$

where ξ and η are right-continuous, ξ is periodic with period ρ and η has the property : $\eta(u+\rho) = -\eta(u)$ for all u.

Proof: Let $B_* = \min B_j$, $B^* = \max B_j$, and $k = g + h$, so that $k(u) = \sum_1^n \gamma_j k(u + B_j)$.

(i) If $\sum_1^n \gamma_j \leqslant 1$, we have from the above that $k(u) \leqslant k(u + B_*)$ since k is non-increasing, and, for the same reason, the reverse inequality holds as well. Hence $k(u) = k(u + B_*)$ for all u, so that $k(u) = k(+\infty) = 0$. Hence $0 \leqslant g,\ h \leqslant k$ implies that $g \equiv h \equiv 0$.

(ii) The basic idea of our proof in this case is simple. We first prove that the Laplace transform of g (or of h) is defined and analytic in $\operatorname{Re} z > -\lambda$ and coincides there with a function analytic everywhere except possibly for simple poles at a lattice of points lying on the vertical line $\operatorname{Re} z = -\lambda$. We then use the standard technique of Linnik (1953) and Shimizu (1968), of applying the complex inversion formula for the Laplace transform and the theorem of residues to obtain the form of g.

Let then $\gamma_1 + \ldots + \gamma_n = 1 + \delta$, $\delta > 0$. We have

$$k(0) \geqslant (1+\delta)\, k(B^*) \geqslant (1+\delta)^2\, k(2B^*) \geqslant \ldots$$

which is easily seen to imply, since k is non-increasing, that

$$k(u) \leqslant c_1 \exp(-c_2 u) \text{ for all } u \geqslant 0$$

where $c_1 > 0$ and $c_2 = [\log(1+\delta)]/B^* > 0$. This shows that $\int_u^\infty k(t)dt$ exists for all u, and the same is obviously true of g or h in place of k. Let then

$$g^*(u) = \int_u^\infty g(t)dt \text{ and } h^*(u) = \int_u^\infty h(t)dt. \qquad \ldots \quad (11)$$

g^* and h^* are a pair of non-negative, non-increasing functions with $g^*(+\infty) = h^*(+\infty) = 0$, and satisfying the relations (4); in addition to these properties which they have in common with the pair (g, h), they are also *continuous*. Now it is immediate that if the pair (g^*, h^*) is shown to have one of the forms (10a)-(10c), then the pair (g, h) also has correspondingly the same form. Thus we need and shall prove our theorem only under the further assumption that g and h (and hence k) are continuous. We then have

Lemma 1 : *$\int_0^\infty e^{xu} k(u)du < \infty$ for $x < \lambda$, so that $X_g(z) = \int_0^\infty e^{-zu}\ g(u)du$ and $X_h(z) = \int_0^\infty e^{-zu}\ h(u)du$ are defined and analytic for $Re\, z > -\lambda$.*

Proof : Let $k(u) = r(u)\, e^{-\lambda u}$, λ being defined by $\sum_1^n \gamma_j e^{-B_j\lambda} = 1$ (so that $\lambda > 0$). r is continuous, and $r(u) = \sum_1^n p_j r(u+B_j)$, where $p_j = \gamma_j e^{-B_j\lambda}$, $\sum_1^n p_j = 1$, so that, by the intermediate value theorem, $r(u) = r[u+B(u)]$, where $B_* \leqslant B(u) \leqslant B^*$. Thus, a sequence $\{b_m\} \to \infty$ as $m \to \infty$, with $b_0 = 0$, exists such that $r(b_m) = r(0) = c_3$ (say), or $k(b_m) = c_3 e^{-\lambda b_m}$, where $B_* \leqslant b_{m+1} - b_m \leqslant B^*$ and $b_m \geqslant mB_*$ obviously, for all m. Hence, for $0 \leqslant x < \lambda$ and for all $m \geqslant 0$,

$$\begin{aligned}\int_{b_m}^{b_{m+1}} e^{xu} k(u)du &\leqslant e^{xb_{m+1}} k(b_m)\,(b_{m+1} - b_m) \\ &\leqslant c_3 B^* \exp[-\lambda b_m + x b_{m+1}] \\ &\leqslant c_3 B^* \exp[xB^* - (\lambda - x)b_m] \\ &\leqslant c_4(x).\exp[-(\lambda - x)mB_*]\end{aligned}$$

where $c_4(x)$ is a constant dependent on x but not on m. Hence

$$\int_0^\infty e^{xu} k(u)du = \sum_{m=0}^\infty \int_{b_m}^{b_{m+1}} < \infty \text{ if } x < \lambda, \text{ whence the lemma.}$$

Lemma 2 : *There exist entire functions $\sigma(z)$ and $K_g(z)$, given by the relations* (15) *and* (16) *below, such that*

$$X_g(z) = -K_g(z)/\sigma(z) \text{ for } Re\, z > -\lambda. \qquad \ldots \quad (12)$$

Remark: Our definitions and procedure in the proof below are the same as in Shimizu (1968)—except that the validity of (12) is claimed there only for $\operatorname{Re} z > 0$, while our stronger claim, made possible by Lemma 1, is precisely what makes our proof much simpler.

Proof: Taking the Laplace transforms over $[0, \infty)$ of both sides of each of the relations (4), we have, by Lemma 1, for $\operatorname{Re} z > -\lambda$,

$$\chi_g(z)\Big(1-\sum_1^p \gamma_j e^{B_j z}\Big) - \chi_h(z)\Big(\sum_{p+1}^n \gamma_j e^{B_j z}\Big) + E_g(z) = 0$$

$$\chi_g(z)\Big(-\sum_{p+1}^n \gamma_j e^{B_j z}\Big) + \chi_h(z)\Big(1-\sum_1^p \gamma_j e^{B_j z}\Big) + E_h(z) = 0 \qquad \ldots \quad (13)$$

where

$$E_g(z) = \sum_1^p \gamma_j e^{B_j z} \int_0^{B_j} e^{-zu} g(u)\,du + \sum_{p+1}^n \gamma_j e^{B_j z} \int_0^{B_j} e^{-zu} h(u)\,du,$$

$E_h(z) =$ the dual of the RHS above with g and h interchanged. ... (14)

If therefore

$$\left.\begin{aligned} \sigma_1(z) &= 1-\sum_1^n \gamma_j e^{B_j z} \\ \sigma_2(z) &= 1-\sum_1^p \gamma_j e^{B_j z} + \sum_{p+1}^n \gamma_j e^{B_j z} \text{ if } p < n \\ \text{and} \quad \sigma(z) &= \begin{cases} \sigma_1(z) & \text{if } p = n \\ \sigma_1(z)\sigma_2(z) & \text{if } p < n \end{cases} \end{aligned}\right\} \qquad \ldots \quad (15)$$

then, eliminating χ_h from the relations (13), we have for both the cases $p = n$ and $p < n$,

$$\sigma(z)\chi_g(z) + K_g(z) = 0 \text{ for } \operatorname{Re} z > -\lambda$$

where

$$K_g(z) = \begin{cases} E_g(z) & \text{if } p = n \\ \Big(1-\sum_1^p \gamma_j e^{B_j z}\Big)E_g(z) + \Big(\sum_{p+1}^n \gamma_j e^{B_j z}\Big)E_h(z) & \text{if } p < n. \end{cases} \qquad \ldots (16)$$

Since χ_g is analytic in $\operatorname{Re} z > -\lambda$, we obtain (12) at once from the above relation.

We now note several facts of importance:

(A) For $s = 1, 2$, $|\sigma_s(x+iy)| \geq 1-\sum_1^n \gamma_j e^{B_j x} > 0$ if $x < -\lambda$, y real, so that $\sigma_s(z) \neq 0$ and hence $\sigma(z) \neq 0$ if $\operatorname{Re} z < -\lambda$. Thus the only singularities of $K_g(z)/\sigma(z)$

are poles at zeros of $\sigma(z)$ lying on the line $\text{Re}\, z = -\lambda$, in view of the above fact and relation (12). Also,

$$|\sigma(z)| > c_5(\gamma) \text{ if } \text{Re}\, z \leqslant -\gamma < -\lambda. \quad \ldots \quad (17)$$

(B) For any fixed real $c, |K_g(z)| \leqslant c_6(c)$ for $\text{Re}\, z \leqslant c$. ... (18)

This follows at once from (14) and (16).

(C) $\sigma(z)$ has the following properties (noted by Linnik, 1953 and Shimizu, 1968) :

(i) the number of zeros of $\sigma(z)$ in any closed rectangle of the form $\{a \leqslant \text{Re}\, z \leqslant b,\ y \leqslant \text{Im}\, z \leqslant y+1\}$, is bounded by a number $N(a, b)$ which is *not* dependent on y.

(ii) for any given $\delta > 0$, if z_0 is *any* point whatever such that its distance from *every* zero of $\sigma(z)$ is $> \delta$, then $|\sigma(z_0)| > c_7(\delta)$, a positive constant independent of the particular point z_0 satisfying the above distance restriction.

The above two properties are consequences of the fact that $\sigma(z)$ is an entire almost periodic function : cf. Levin (1964), Chapter 6, Section 2, Lemmas 1 and 2. [$\sigma(z)$ also has the easily-verified property that all its zeros are located in some strip $-\lambda \leqslant \text{Re}\, z \leqslant \mu$($\lambda$ as defined above), but we shall not need this fact.] For a proof from first principles of these properties of $\sigma(z)$, one may refer to Linnik (1953, Sections 9 and 10).

C(i) implies the existence of some $\delta > 0$ independent of m and a sequence $\{T_m\} \to \infty$ as $m \to \infty$, with $m < T_m < m+1$, such that all the zeros of $\sigma(z)$ in $\{-\gamma \leqslant \text{Re}\, z \leqslant c,\ m \leqslant T_m < m+1\}$ lie at a (vertical) distance $> \delta$ from the line $\text{Im}\, z = T_m$, so that $|\sigma(z)| > c_8$ for all m if $\text{Im}\, z = T_m$, in view of C(ii) above and (17). Clearly, the same is true of $\text{Im}\, z = -T_m$ as well, for all m. Thus we finally have

$$|\sigma(z)| > c_8 \text{ for all } m \text{ if } |\text{Im}\, z| = T_m. \quad \ldots \quad (19)$$

We now invoke a simple lemma concerning the zeros of $\sigma(z)$ on $\text{Re}\, z = -\lambda$ (cf. Shimizu, 1968) :

Lemma 3 : *The zeros of $\sigma(z)$ on $\text{Re}\, z = -\lambda$ are all simple.*

Further,

(a) *if $B \epsilon A_n(0)$, then $-\lambda$ is the only such zero;*

(b) *if $B \epsilon B_n^p(\rho)$, $p \leqslant n$, then the set of such zeros of $\sigma_1(z)$ is $\{-\lambda+(2m\pi i/\rho),\ m$ integer$\}$, while $\sigma_2(z)$ has no such zero; and*

(c) *if $B \epsilon C_n^p(\rho)$, so that $p < n$, then the set of such zeros of $\sigma(z)$ is $\left\{-\lambda+\frac{2m\pi i}{\rho},\ m \text{ integer}\right\}$ and of $\sigma_2(z)$ is $\left\{-\lambda+\frac{(2m+1)\pi i}{\rho},\ m \text{ integer}\right\}$.*

Proof: Since $\sigma_1(-\lambda) = 0$, we have

$$\sigma_1(-\lambda+iy) = 0 \text{ iff } 1-\cos B_j y = 0 \text{ for } 1 \leqslant j \leqslant n \qquad \dots \quad (20)$$

$$\left.\begin{aligned} \sigma_2(-\lambda+iy) = 0 \text{ iff } 1-\cos B_j y = 0 \text{ for } 1 \leqslant j \leqslant p, \\ \text{and} \qquad 1+\cos B_j y = 0 \text{ for } p < j \leqslant n. \end{aligned}\right\} \qquad \dots \quad (21)$$

Hence σ_1 and σ_2 cannot vanish simultaneously at any point $-\lambda+iy$. Also, if, for $s = 1$ or 2, $\sigma_s(-\lambda+iy) = 0$, then, for that y,

$$\sigma_s'(-\lambda+iy) = -\Sigma\gamma_j B_j e^{-B_j\lambda} < 0,$$

in view of (20) and (21). Hence the assertion that the zeros of $\sigma(z)$ on $\text{Re}\, z = -\lambda$ are simple. Assertions (a)-(c) follow from (20) and (21).

Let now $G(z) = -K_g(z)/\sigma(z)$, so that G is analytic everywhere except possibly at the zeros of $\sigma(z)$ on $\text{Re}\, z = -\lambda$, and $G = \chi_g$ for $\text{Re}\, z > -\lambda$. By the complex inversion formula for the Laplace transform (Widder, 1946, p. 73), for $t > 0$,

$$\int_0^t g(u)du = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{e^{tz}\chi_g(z)}{z}\, dz \quad \text{for any } c > 0$$

$$= \lim_{m\to\infty} \frac{1}{2\pi i} \int_{c-iT_m}^{c+iT_m} H(z, t)dz \qquad \dots \quad (22)$$

where $H(z, t) = e^{tz}G(z)/z$, and $\{T_m\}$ is a sequence chosen to satisfy (19). If $S_m(t)$ denotes the sum of the residues of $-H(z, t)$ at the zeros of $\sigma(z)$ lying in the interval $\{\text{Re}\, z = -\lambda,\ |\text{Im}\, z| < T_m\}$, then, noting that the residue of $H(z, t)$ at the origin is $\chi_g(0)$, we have for any $R > \lambda$, by the residue theorem,

$$\frac{1}{2\pi i} \int_{c-iT_m}^{c+iT_m} H(z, t)dz = \frac{1}{2\pi i} \left[\int_{-R+iT_m}^{c+iT_m} - \int_{-R-iT_m}^{c-iT_m} + \int_{-R-iT_m}^{-R+iT_m} H(z, t)dz \right] - S_m(t) + \chi_g(0); \qquad \dots \quad (23)$$

it is easily checked that, as $R\to\infty$, for fixed m, $\int_{-R-iT_m}^{-R+iT_m} H(z, t)dz \to 0$ in view of (17) and (18). Hence the RHS above is

$$= s_m(t) - S_m(t) + \chi_g(0) \qquad \dots \quad (24)$$

where

$$s_m(t) = \frac{1}{2\pi i} \left[\int_{-\infty+iT_m}^{c+iT_m} - \int_{-\infty-iT_m}^{c-iT_m} H(z, t)dz \right].$$

It is again easy to check that, as $m\to\infty$, $s_m(t)\to 0$, in view of (18) and (19), so that we have from (22), (23) and (24), and noting that $\chi_g(0) = \int_0^\infty g(t)dt$, the fundamental relation

$$\int_t^\infty g(u)du = \lim_{m\to\infty} S_m(t). \qquad \dots \quad (25)$$

To prove the theorem, we need only compute the expression on the RHS of (25) in the different cases.

By 'relevant zero' below, we mean a zero of $\sigma(z)$ lying on $\text{Re } z = -\lambda$.

(a) Let $B\epsilon A_n(0)$. Then, $-\lambda$ is the only relevant zero, and, for $t > 0$, (25) gives:

$$\int_t^\infty g(u)du = \xi_1 e^{-\lambda t}, \quad \xi_1 \text{ a constant,}$$

or, differentiating with respect to t (g is continuous by assumption),

$$g(t) = \xi e^{-\lambda t}, \text{ similarly } h(t) = \eta e^{-\lambda t}. \qquad \dots \quad (26)$$

The validity of (26) for all t (and not merely for all $t > 0$) follows then from the fact that, to study g and h in $(-A, \infty)$, we need only consider the functions $g_A(t) = g(t-A)$ and $h_A(t) = h(t-A)$ in $(0, \infty)$, and since g_A and h_A satisfy (4), our above analysis applies to them.

Also, if $p < n$, substituting from (26) in (4), we see that $\xi = \eta$ in such a case.

(b) Let $B\epsilon B_n^*(\rho)$, $p \leqslant n$. The relevant zeros are

$$\alpha_k = -\lambda + (2k\pi i/\rho), \; k \text{ integer,}$$

and from (25), we have for $t > 0$,

$$\int_t^\infty g(u)du = \lim_{m\to\infty} \left\{ \sum_{|\text{Im}\,\alpha_k| < T_m} C_k e^{2k\pi t i/\rho} \right\} e^{-\lambda t} \qquad \dots \quad (27)$$

$$= \xi_0(t) e^{-\lambda t}$$

where ξ_0 is periodic with period ρ. The LHS above is differentiable, hence ξ_0' exists and we may differentiate both sides of (27) to obtain for $t > 0$

$$g(t) = \xi(t) e^{-\lambda t}, \text{ similarly } h(t) = \eta(t) e^{-\lambda t} \qquad \dots \quad (28)$$

where ξ and η are periodic with period ρ, and are also non-negative and right-continuous since g and h are so. The validity of (28) in intervals $(-m\rho, \infty)$, m positive integer, is argued out as in case (a), so that (28) holds for all t. Also, if $p < n$, then substituting from (28) in (4), we obtain $\xi(t) = \eta(t)$ in that case.

(c) If $B\epsilon C_n^*(\rho)$, so that $p < n$, the relevant zeros are

$$\alpha_k = -\lambda + (k\pi i/\rho), \; k \text{ integer,}$$

and from (25), we have for $t > 0$,

$$\int_t^\infty g(u)du = \lim_{m\to\infty} [\xi_m(t) + \eta_m(t)] e^{-\lambda t} \qquad \dots \quad (29)$$

where $\xi_m(t+\rho) = \xi_m(t)$ and $\eta_m(t+\rho) = -\eta_m(t)$. Writing $(t+\rho)$ in place of t in the above, we have therefore for $t > 0$

$$\int_{t+\rho}^\infty g(u)du = \lim_{m\to\infty} [\xi_m(t) - \eta_m(t)] e^{-\lambda(t+\rho)}. \qquad \dots \quad (30)$$

It follows that $\xi_m(t)$ and $\eta_m(t)$ converge separately as $m \to \infty$, so that

$$\int_t^{\infty} g(u)du = [\xi_0(t)+\eta_0(t)]e^{-\lambda t} \quad \ldots \quad (31)$$

where $\xi_0(t+\rho) = \xi_0(t)$ and $\eta_0(t+\rho) = -\eta_0(t)$, so that again

$$\int_{t+\rho}^{\infty} g(u)du = [\xi_0(t)-\eta_0(t)]e^{-\lambda(t+\rho)}. \quad \ldots \quad (32)$$

(31) and (32) imply that ξ_0 and η_0 are (individually) differentiable and we immediately obtain, for $t > 0$, the representations

$$g(t) = [\xi_1(t)+\eta_1(t)]e^{-\lambda t};\ h(t) = [\xi_2(t)+\eta_2(t)]e^{-\lambda t}. \quad \ldots \quad (33)$$

The validity of (33) for all t is argued out as before. Substituting from (33) in (4) and remembering that, in our present case, $m_1, \ldots, m_p$ are all even and $m_{p+1}, \ldots, m_n$ are all odd, we find that

$$\xi_1+\eta_1 = \xi_2-\eta_2 \text{ and } \xi_1-\eta_1 = \xi_2+\eta_2$$

whence $\xi_1 = \xi_2 = \xi(\text{say})$ and $\eta_1 = -\eta_2 = \eta(\text{say})$,

yielding the representation (10c).

In conclusion, we recall that the above argument has assumed the continuity of g and h, and that, as remarked immediately preceding Lemma 1, the general case follows from this. Thus Theorem 2 stands proved.

As an almost immediate consequence of Theorem 2 and of our earlier work, R-R (1968), we have :

Theorem 3 : *Let f be a non-vanishing and non-trivial c.f., satisfying for all real t the relation* (2). *Then (f is i.d. and), λ being the unique real number such that* $\sum_1^n \gamma_j\beta_j^{\lambda} = 1$,

(i) $0 < \lambda \leqslant 2$,

(ii) *f is a normal c.f. iff $\lambda = 2$,*

(iii) *if $0 < \lambda < 2$, then, in the Levy representation $L(\mu, \sigma, M, N)$ for $\log f$, we have $\sigma = 0$, and M and N have the following representations depending on the nature of the vector $B = (B_1, \ldots, B_n)$, where $B_j = -\log \beta_j$.*

(a) *If $B \epsilon A_n(0)$, then*

$$M(u) = \xi|u|^{-\lambda},\ N(u) = -\eta u^{-\lambda} \quad \ldots \quad (34)$$

where ξ and η are non-negative constants with $\xi+\eta > 0$. Further, $\xi = \eta$ if $p < n$, or if $\lambda = 1$ (whether $p < n$ or $p = n$), so that, for a suitable real c, $f(t)e^{ict}$ is the *c.f. of a symmetric stable law with exponent* λ; *if* $p = n$ *and* $\lambda \neq 1$, *then* f *is a (general) stable law with exponent* λ.

(b) *If* $B \epsilon B_n^p(k)$, $p \leqslant n$, *then*

$$M(u) = \frac{\xi(\log|u|)}{|u|^{\lambda}} \text{ and } N(u) = \frac{-\eta(\log u)}{u^{\lambda}} \qquad \ldots \quad (35)$$

where ξ *and* η *are non-negative, right-continuous functions on* R_1, *with period* ρ; *further, if* $p < n$, *then* $\xi = \eta$. *Of course, here and in* (c) *below,* ξ *and* η *must also be such that* M *and* N *are non-decreasing.*

(c) *If* $B \epsilon C_n^p(\rho)$, *so that* $p < n$, *then*

$$M(u) = \frac{\xi(\log|u|)+\eta(\log|u|)}{|u|^{\lambda}} \text{ and } N(u) = \frac{-\xi(\log u)+\eta(\log u)}{u^{\lambda}} \qquad \ldots \quad (36)$$

where ξ *and* η *are right-continuous functions on* R_1 *such that* $\xi(x+\rho) = \xi(x)$ *and* $\eta(x+\rho) = -\eta(x)$ *for all* x. ξ *is also non-negative.*

Proof: (i) and (ii) follow from R-R (1968). (iii) follows from Theorem 2 except for the assertion in (a) that $\lambda = 1$ yields a Cauchy law whether $p < n$ or $p = n$. If $p < n$, this is covered by Theorem 2. If $p = n$, then we use the explicit formula for $\log f$ as in the proof of Theorem 1 to conclude that f is the c.f. of a Cauchy distribution.

Remarks: (1) In R-R (1968), it has been proved further that if $0 < \lambda < 2$, then the corresponding d.f. has absolute moments of all orders $< \lambda$ but not of order λ. These assertions also follow from (34)-(36) above in view of the fact that the absolute moment of order δ exists iff

$$\int_{(-\infty,-1)} |u|^{\delta}\, dM(u) + \int_{(1,\infty)} u^{\delta} dN(u) < \infty$$

(see Ramachandran, 1969, Theorem 9). It was also established in R-R (1968) that the distributions in all these cases are absolutely continuous.

(2) Suppose the same c.f. f satisfies (2) for two sets of constants $(n_1, \beta_1, \gamma_1, p_1)$ and $(n_2, \beta_2, \gamma_2, p_2)$. Then, both must give rise to the same λ (Theorem 1 gives us such an example, where $\lambda = 1$). If one of the vectors β_1 and β_2 belong to $A_{n_1}(0)$ or to $A_{n_2}(0)$ respectively, or if $\beta_1 \epsilon A_{n_1}(\rho_1)$ and $\beta_2 \epsilon A_{n_2}(\rho_2)$ where ρ_1/ρ_2 is irrational then it follows that f must be a stable law (in the second case, we proceed as in the proof of Theorem 1). In special cases of such a situation, it may be possible to go further and state that f is a symmetric stable law except possibly for a location parameter (as in Theorem 1).

4. Functional equations involving infinite products

In this section, we consider solutions of equations of the form

$$f(t) = \prod_{j=1}^{\infty} [f(\pm\beta_j t)]^{\gamma_j} \qquad \ldots \quad (37)$$

where f is a non-vanishing, non-trivial c.f., $0 < \beta_j < 1$ and $\gamma_j > 0$ for all j. We may remark that if the γ_j are all positive integers, then the RHS above may be well-defined even if f vanishes; if in such a case, relation (37) holds, then we can *prove* that f is indeed non-vanishing; the proof is not quite trivial, however. We may first note (as in the proof of Theorem 4 below) that $\Sigma\gamma_j\beta_j^2 < \infty$, so that $\beta_j \to 0$ as $j \to \infty$ (since $\gamma_j \geqslant 1$ for all j), and then use the fact that the convergence of the sequence $\left\{\prod_1^n [f(\pm\beta_j t)]^{\gamma_j}\right\}$ of c.f.'s to the c.f. f is necessarily uniform on compact intervals to arrive at a contradiction to the assumption that f has zeros on the real line : cf. our proofs in Section 5.

We shall first establish (Theorem 4) a necessary and sufficient condition on the γ's and β's for f to be normal without imposing any further condition on the β's than those stated following relation (37). Then we prove (Theorem 5) the infinite divisibility of f if it satisfies (37) under the further restriction that $\beta_j \to 0$ as $j \to \infty$ (which is satisfied if the γ_j are bounded away from zero, and in particular if they are all positive integers). This restriction is satisfied in the regression problem of Section 4 to which we apply Theorems 4-6. Finally we obtain sufficient conditions (Theorem 6) on the β's and γ's under which statements analogous to Theorems 2 and 3 can be made.

Theorem 4 : *Let f be a non-trivial c.f. satisfying, in some interval around the origin where it does not vanish, the relation*

$$f(t) = \prod_{j=1}^{\infty} [f(\pm\beta_j t)]^{\gamma_j}$$

where $0 < \beta_j < 1$ and $\gamma_j > 0$ for all j. Then

(i) $\Sigma\gamma_j\beta_j^2 \leqslant 1$,

and (ii) *f is a normal c.f. iff $\Sigma\gamma_j\beta_j^2 = 1$.*

Note: In the interval concerned, $[f(\beta_j t)]^{\gamma_j}$ is of course defined as $\exp[\gamma_j \log f(\beta_j t)]$ where $\log f$ denotes that branch of the logarithm which is continuous there and vanishes at the origin.

Proof: The convergence of $\Sigma\,\gamma_j\beta_j^2$ is known in the case where the γ_j are all $= 1$. In certain presentations of that result, however, unnecessary restrictions such as the finiteness of the variance of the corresponding d.f. are made. In the interests of clarity therefore, we present below the (short) proof as needed in our case.

Let $g = |f|^2$ and G be the d.f. corresponding to g. Then (37) gives :

$$g(t) = \prod_{j=1}^{\infty} [g(\beta_j t)]^{\gamma_j}, \quad |t| < \delta. \qquad \ldots \quad (38)$$

Since F and G are non-degenerate, there exists an $A > 0$ such that $\int_{-A}^{A} x^2 dG(x) > 0$. Fixing such an A, setting $t_0 = \pi/A$ (A can be so chosen that t_0 lies in the interval concerned), noting that

$$g(t) = 1-2 \int \sin^2(tx/2)dG(x) \leqslant \exp[-2 \int \sin^2(tx/2)dG(x)]$$

and the well-known inequality : $(1 \geqslant) \sin\theta/\theta \geqslant 2/\pi$ if $0 \leqslant \theta \leqslant \pi/2$, we have from (38) for any positive integer n,

$$-\log g(t_0) \geqslant 2 \sum_1^n \gamma_j \int \sin^2(\beta_j t_0 x/2)dG(x)$$

$$\geqslant 2 \sum_1^n \gamma_j \int_{-A}^{A} \sin^2(\beta_j t_0 x/2)dG(x)$$

$$\geqslant (2t_0^2/\pi^2) \sum_1^n \gamma_j \beta_j^2 \int_{-A}^{A} x^2 dG(x)$$

whence the convergence of $\sum_1^\infty \gamma_j \beta_j^2$ follows.

The non-degeneracy of F and G then implies that the relation $\Sigma\gamma_j\beta_j^2 > 1$ cannot hold. For, suppose it does, and let n be chosen and fixed such that $\sum_1^n \gamma_j\beta_j^2 = 1+\delta, \delta > 0$. Then, letting $\psi(t) = -\log g(t)/t^2, t \neq 0$, we have for $t > 0$

$$\psi(t) \geqslant \sum_1^n \gamma_j\beta_j^2\psi(\beta_j t) = (1+\delta)\psi[t\beta(t)] \qquad \dots (39)$$

by the intermediate value theorem, where $\min_{1 \leqslant j \leqslant n} \beta_j \leqslant \beta(t) \leqslant \max_{1 \leqslant j \leqslant n} \beta_j$. Thus, there exists a sequence $\{b_m\} \to 0$, depending on the fixed n, such that

$$\psi(b_m) \leqslant (1+\delta)^{-m}\psi(1); \; m = 1, 2, \dots$$

so that $\psi(b_m) \to 0$ as $m \to \infty$. This implies that G and F are degenerate (R-R, 1968, Theorem 2.3d), contrary to assumption. Hence the assertion (i).

(ii) Let $\Sigma\gamma_j\beta_j^2 = 1$. We shall prove that f is a normal c.f. (The converse is obvious). This is proved for the case : $\gamma_j = 1$ for all j, by Laha and Lukacs (1965)—also see Lukacs (1968), pp.116-122, by first establishing the infinite divisibility of f and then examining its Levy-Khinchin representation. ψ being defined as above, we have for $t \neq 0$, $|t| < \delta$,

$$\psi(t) = \Sigma p_j\psi(\beta_j t) \text{ where } p_j = \Sigma\gamma_j\beta_j^2, \; \Sigma p_j = 1,$$

which we rewrite in the form

$$\Sigma p_j[\psi(t)-\psi(\beta_j t)] = 0, \quad 0 < |t| < \delta. \qquad \dots (40)$$

(40) implies (by contradiction) that, for any $(\delta >) t > 0$, there exists at least one β_j, $j = j(t)$, such that $\psi(t) \geqslant \psi(\beta_j t)$. Fix $t_0 > 0$, and let

$$S(t_0) = \{0 < t \leqslant t_0 | \psi(t) \leqslant \psi(t_0)\}.$$

$S(t_0)$ is non-empty; let $\tau = \inf S(t_0)$, so that $\tau \geqslant 0$. We claim that $\tau = 0$. For, suppose $\tau > 0$. Then the continuity of ψ implies that $\psi(t_0) \geqslant \psi(\tau)$. Also, there exists a

$k = k(\tau)$ such that $\psi(\tau) > \psi(\beta_k\tau)$, so that $\psi(t_0) > \psi(\beta_k\tau)$, i.e., $\beta_k\tau \epsilon S(t_0)$, contrary to the definition of τ. Hence $\tau = 0$ and therefore there exists a sequence $\{t_n\} \to 0$ such that $\psi(t_n) \leqslant \psi(t_0)$ for all n, so that, by R-R (1968), Theorem 2.3c, G has finite variance. This in turn implies that $\lim_{t \to 0} \psi(t)$ exists; denoting it by $\psi(0)$, we have $\psi(t_0) \geqslant \psi(0)$.

Again, (40) implies the existence of $l = l(t)$ for any $(\delta >) t > 0$ such that $\psi(t) \leqslant \psi(\beta_l t)$. This fact, on proceeding as above, yields the relation $\psi(t_0) \leqslant \psi(0)$ for any t_0 in $(0, \delta)$. Thus ψ is constant in $|t| < \delta$, whence it follows that g is normal; then, by the Levy-Cramer Theorem, f is also normal.

We pass to the examination of the solutions of (37) where $\Sigma\gamma_j\beta_j^2 < 1$. Our analysis here is not as complete as in the finite case, but we single out cases where an analysis similar to that in Section 2 can be made.

We begin with an analogue of Theorem 2, for which we rename the β's above as follows. Let $a_1, a_2, \ldots$ be the sequence of β's occurring in the above product with the positive sign (if any), and $b_1, b_2, \ldots,$ be the sequence of β's occurring there with the negative sign (if any). Let the exponents γ corresponding to a_j, b_j be renamed as δ_j, ϵ_j respectively. Let $A_j = -\log a_j$, $B_j = -\log b_j$. We consider the following classification of the infinite-vector-pairs (A, B) :

$\mathscr{A}(\rho)$: there exists a $\rho > 0$ such that $A_j = k_j/\rho$, $B_j = l_j/\rho$ where the k_j and l_j are all positive integers and further their greatest common factor $= 1$,

$\mathscr{A}(0)$: there exists no such ρ,

$\mathscr{B}(\rho)$: the subset of $\mathscr{A}(\rho)$ where at least one of the k_j is odd and/or at least one l_j is even,

$\mathscr{C}(\rho)$: the subset of $\mathscr{A}(\rho)$ where all the k_j are even, and all the l_j are odd.

Note that if the set of b's is empty, then $\mathscr{B}(\rho) = \mathscr{A}(\rho)$ and $\mathscr{C}(\rho)$ is empty, by the definition of ρ.

Theorem 5 : *Let g and h be non-negative, non-increasing right-continuous functions defined on R_1, with $g(+\infty) = h(+\infty) = 0$, and satisfying for all real u the relations :*

$$\left.\begin{aligned} g(u) &= \sum_1^\infty \delta_j g(u+A_j) + \sum_1^\infty \epsilon_j h(u+B_j) \\ h(u) &= \sum_1^\infty \delta_j h(u+A_j) + \sum_1^\infty \epsilon_j g(u+B_j) \end{aligned}\right\} \qquad \ldots \quad (41)$$

where the δ's and ϵ's are all positive, and the A's and B's are all positive and bounded away from zero as well. Then

(i) *$g \equiv h \equiv 0$ if $\Sigma\delta_j + \Sigma\epsilon_j \leqslant 1$ (this holds even if the A_j, B_j are merely positive and not necessarily bounded away from zero);*

(ii) *if $\Sigma\delta_j + \Sigma\epsilon_j$ converges to a sum > 1, let λ be the unique positive number such that $\Sigma\delta_j e^{-A_j\lambda} + \Sigma\epsilon_j e^{-B_j\lambda} = 1$. Then the assertions of Theorem 2 hold, with the phrase 'if $p < n$' being replaced by : 'if the set of B's is non-empty', and with $\mathscr{A}(0)$, $\mathscr{B}(\rho)$ and $\mathscr{C}(\rho)$ in place of $A_n(0)$, $B_n^p(\rho)$ and $C_n^p(\rho)$ respectively ;*

(iii) *if* $\Sigma\delta_j+\Sigma\epsilon_j$ *diverges, suppose however that for some* $\nu(>0)$, $\Sigma\delta_j e^{-A_j\nu}+\Sigma\epsilon_j e^{-B_j\nu}$ *converges to a sum* >1, *and let then* $\lambda(>\nu)$ *be the unique positive number such that* $\Sigma\delta_j e^{-A_j\lambda}+\Sigma\epsilon_j e^{-B_j\lambda}=1$. *Then again, the assertions of Theorem 2, with the modifications in* (ii) *above, hold.*

Note : As an example of case (ii) above, we may cite : $\delta_j=\epsilon_j=1$ for all j, $A_j=2jc$, $B_j=(2j+1)c$ for some $c>0$.

Proof : (i) Let $k=g+h$, so that

$$k(u)=\Sigma\delta_j k(u+A_j)+\Sigma\,\epsilon_j k(u+B_j). \qquad \ldots \quad (42)$$

If now $\Sigma\delta_j+\Sigma\epsilon_j\leqslant 1$, then $\Sigma\delta_j[k(u)-k(u+A_j)]+\Sigma\epsilon_j[k(u)-k(u+B_j)]\leqslant 0$ from the above, whereas, on the other hand, each term of the above sum is $\geqslant 0$ since k is non-increasing. Hence every term of the above sum must be zero whence we easily see that $k\equiv 0$ and consequently, so are g and h.

(ii) and (iii) : In both these cases, in (ii) necessarily and in (iii) by assumption, there exists a $\nu>0$ such that $\Sigma\gamma_j e^{-c_j\nu}$ converges to a sum >1 and $\lambda(>\nu)$ is the unique real number satisfying $\Sigma\gamma_j e^{-c_j\lambda}=1$. Choose and fix an N such that $\sum_1^N\gamma_j=1+\delta$, $\delta>0$. If $c_N^*=\max\{c_j:1\leqslant j\leqslant N\}$, then, as in Section 2, $k(0)\geqslant(1+\delta)^n k(nC_N^*)$ for all positive integers n, whence

$$k(u)\leqslant D_1\exp(-D_2u) \quad \text{for all } n\geqslant 0$$

where $D_1>0$ and $D_2=[\log(1+\delta)]/C_N^*>0$. Hence $\int_u^\infty k(t)dt$ exists for all real u, and the same is true obviously then of g or h in place of k. As in Section 2, one consequence of this fact is that it suffices to prove our theorem in the case where g and h are both continuous, in addition to their other properties assumed in the statement of the theorem; we shall henceforth assume that they are so, consequently k is also continuous.

Lemma 4 : $\int_0^\infty e^{xu}k(u)du<\infty$ *for* $x<\lambda$ *so that* $\chi_g(z)=\int_0^\infty e^{-zu}\;g(u)du$ *and* $\chi_h(z)=\int_0^\infty e^{-zu}h(u)du$ *are defined and analytic in* $Re\,z>-\lambda$.

Proof : Setting $k(u)=r(u)e^{-\lambda u}$, so that r is also continuous, we have from (42) that

$$r(u)=\sum_1^\infty\gamma_j e^{-C_j\lambda}r(u+C_j)=\sum_1^\infty p_j\,r(u+C_j), \quad \text{say,}$$

where $\Sigma p_j=1$. For any fixed N,

$$r(u)\geqslant\sum_1^N p_j\,r(u+C_j)=\Big(\sum_1^N p_j\Big)r[u+C(u)]$$

where $C \leqslant C(u) \leqslant C_N^* = \max(C_1, \ldots, C_N)$, by the intermediate value theorem, so that, for a sequence $\{b_m\}$ depending on N, we have $r(b_m) \leqslant r(0) \cdot q_N^{-m}$, where $q_N = \sum_1^N p_j$, whence, as in Section 2, $\int_0^\infty e^{xt} k(t)dt < \infty$ for x such that $C(\lambda - x) + \log q_N > 0$. Since N is arbitrary, $q_N \to 1$ as $N \to \infty$, and C is independent of N, our assertion follows.

Lemma 5: *The series* $\sum \gamma_j e^{C_j x} \int_0^{C_j} e^{-xu} k(u)du$ *converges for all real* $x < -\nu$, *so that*

$$\sum \delta_j e^{A_j x} \int_0^{A_j} e^{-zu} g(u)du$$

and

$$\sum \epsilon_j e^{B_j x} \int_0^{B_j} e^{-zu} g(u)du,$$

and the corresponding formal sums with h in place of g, are all defined and analytic in $Re\, z < -\nu$. *Further all these functions are bounded in any half-plane* $x = Re\, z \leqslant -\gamma$ *where* $\gamma > \nu$.

Proof: Let $x < -\nu$ be fixed; choose and fix α with $0 \leqslant \alpha < \lambda$ and $\theta > 0$ such that $-x \neq \dfrac{\alpha}{1+\theta} > \nu$, this being possible since $\lambda > \nu$. Then, if

$$\int_0^\infty e^{\alpha u} k(u)du = D_3(\alpha) < \infty \quad \text{(Lemma 4)},$$

we have

$$\int_u^{u(1+\theta)} e^{\alpha v} k(v)dv \leqslant D_3(\alpha)$$

whence, since k is non-increasing, we obtain

$$k(u) \leqslant D_4(\alpha, \theta) u^{-1} \exp[-\alpha u/(1+\theta)] \quad \text{for all } u > 0.$$

We shall use the above estimate for $u > 1$; for $0 \leqslant u \leqslant 1$, since k is bounded there, we have

$$k(u) \leqslant D_5(\alpha, \theta) \exp[-\alpha u/(1+\theta)], \quad 0 \leqslant u \leqslant 1$$

so that we finally have

$$k(u) \leqslant D_6(\alpha, \theta) \exp[-\alpha u/(1+\theta)] \text{ for all } u \geqslant 0.$$

Remembering that $x + \alpha/(1+\theta) \neq 0$, we have

$$\sum_1^\infty \gamma_j e^{C_j x} \int_0^{C_j} e^{-xu} k(u)du \leqslant D_6(\alpha, \theta) \sum_1^\infty \gamma_j e^{C_j x} \int_0^{C_j} \exp\left[-\left(x + \frac{\alpha}{1+\theta}\right)u\right] du$$

$$= D_6(\alpha, \theta) \sum_1^\infty \gamma_j \left[e^{C_j x} - e^{-C_j \alpha/(1+\theta)}\right] \Big/ \left(x + \frac{\alpha}{1+\theta}\right)$$

$$< \infty \text{ since } x < -\nu \text{ and } \frac{\alpha}{1+\theta} > \nu.$$

Hence the main statement of the lemma. The other statements of the lemma are direct consequences thereof or of the above estimate.

Since (i) χ_g and χ_h are analytic in $\mathrm{Re}\, z > -\lambda$, (ii) $1-\Sigma\delta_j e^{A_j z} \pm \Sigma\epsilon_j e^{B_j z}$ are analytic in $\mathrm{Re}\, z < -\nu$ and (iii) in view of Lemma 5, we may proceed as in the proof of Lemma 2 in Section 2, for z such that $-\lambda < \mathrm{Re}\, z < -\nu$, defining in particular, for $\mathrm{Re}\, z < -\nu$,

$$\sigma_1(z) = 1 - \sum_1^\infty \delta_j e^{A_j z} - \sum_1^\infty \epsilon_j e^{B_j z}$$

$$\sigma_2(z) = 1 - \sum_1^\infty \delta_j e^{A_j z} + \sum_1^\infty \epsilon_j e^{B_j z} \quad \text{if the set of } B\text{'s is non-empty,}$$

and

$$\sigma(z) = \begin{cases} \sigma_1(z) & \text{if the set of } B\text{'s is empty} \\ \sigma_1(z)\sigma_2(z) & \text{if the set of } B\text{'s is non-empty,} \end{cases}$$

$$E_g(z) = \sum_1^\infty \delta_j e^{A_j z} \int_0^{A_j} e^{-zu} g(u)du + \sum_1^\infty \epsilon_j e^{B_j z} \int_0^{B_j} e^{-zu} h(u)du$$

$E_h(z)$ = the dual of the RHS above with g and h interchanged

$$K_g(z) = \begin{cases} E_g(z) & \text{if the set of } B\text{'s is empty} \\ \left(1-\sum_1^\infty \delta_j e^{A_j z}\right) E_g(z) + \left(\sum_1^\infty \epsilon_j e^{B_j z}\right) E_h(z) & \text{if the set of } B\text{'s is non-empty} \end{cases}$$

to obtain finally the fundamental relation (valid whether the set of B's is empty or not)

$$\chi_g(z) = -K_g(z)/\sigma(z) \quad \text{for } -\lambda < \mathrm{Re}\, z < -\nu.$$

We recall that χ_g is analytic in $\mathrm{Re}\, z > -\lambda$ while K_g and σ are both analytic in $\mathrm{Re}\, z < -\nu$. Applying the complex inversion formula, we have for $t > 0$

$$\int_0^t g(u)du = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{e^{tz}\chi_g(z)}{z} dz \text{ for any fixed } c > 0$$

$$= \lim_{T\to\infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{e^{tz}\chi_g(z)}{z} dz + \text{the residue at the origin of the integrand (this residue being obviously } = \chi_g(0)),$$

where $-\lambda < a < 0$, noting that $\chi_g(z)$ is bounded in $\mathrm{Re}\, z \geqslant a$, so that

$$\int_t^\infty g(u)du = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{e^{tz}\chi_g(z)}{z} dz \ (-\lambda < a < 0).$$

We may then choose and fix a such that $-\lambda < a < -\nu$, and proceed as before, noting that $\sigma(z)$ being an analytic almost periodic function in $\mathrm{Re}\, z < -\nu$, the properties of $\sigma(z)$ quoted and used in Section 2 apply to our present $\sigma(z)$ also — in particular, there exists a sequence $\{T_m\} \to \infty$ as $m \to \infty$ with the property (19). We have then an obvious analogue of Lemma 3 for the present case, and the rest of the discussion proceeds easily along the lines of Section 2. Hence Theorem 5 stands proved.

We proceed to establish the infinite divisibility of c.f.'s f satisfying (37) in the case where $\beta_j \to 0$ as $j \to \infty$. In view of Theorem 4, this condition is satisfied if (f is non-trivial and) the γ_j are all positive integers; in which case, as we have already noted, the non-vanishing nature of f *follows* from (37), and does not have to be postulated as an assumption.

Theorem 6 : *Let f be a non-vanishing, non-trivial c.f. satisfying for all real t the relation*

$$f(t) = \prod_{j=1}^{\infty} [f(\pm\beta_j t)]^{\gamma_j} \qquad \dots \quad (43)$$

(for some fixed sequence of positive and negative signs), where $0 < \beta_j < 1$, and $\gamma_j > 0$ for all j, and $\beta_j \to 0$ as $j \to \infty$. Then f is infinitely divisible. Further, assertions similar to those of Theorem 3 can be made regarding the Levy functions M and N in the Levy representation for $\log f$, if the β's and γ's satisfy suitable conditions (namely one of those stated in Theorem 5).

Proof : We shall prove our assertion regarding infinite divisibility for the case where the positive sign occurs throughout; the same argument goes through with obvious necessary modifications in the general case. We write

$$\phi_n(t) = \prod_{n+1}^{\infty} [f(\beta_j t)]^{\gamma_j} \qquad \dots \quad (44)$$

so that

$$f(t) = \prod_{1}^{n} [f(\beta_j t)]^{\gamma_j} \phi_n(t). \qquad \dots \quad (45)$$

Iterating the above relation n times, we have

$$f(t) = \xi_n(t) . \psi_n(t), \text{ say} \qquad \dots \quad (46)$$

where

$$\xi_n(t) = \prod_{j_1+\dots+j_n=n} [f(\beta_1^{j_1} \dots \beta_n^{j_n} t)]^{(n; j_1, \dots, j_n)\gamma_1^{j_1} \dots \gamma_n^{j_n}} \qquad \dots \quad (46a)$$

$(n; j_1, \dots, j_n)$ denoting the multinomial coefficient

$$\frac{n!}{j_1!, \dots, j_n!}$$

and

$$\psi_n(t) = \phi_n(t) \times \{[\phi_n(\beta_1 t)]^{\gamma_1} \dots [\phi_n(\beta_n t)]^{\gamma_n}\} \times \dots \times \{[\phi_n(\beta_1^{n-1} t)]^{\gamma_1^{n-1}} \dots [\phi_n(\beta_n^{n-1} t)]^{\gamma_n^{n-1}}\} \qquad \dots \quad (46b)$$

so that

$$\log \psi_n(t) = \lim_{M\to\infty} \sum_{j=n+1}^{M} \gamma_j \left\{ \log f(\beta_j t) + \sum_{r=1}^{n-1} [\Sigma_r^*(r; k_1, \ldots, k_n) \gamma_1^{k_1} \ldots \gamma_n^{k_n} \log f(\beta_j \beta_1^{k_1} \ldots \beta_n^{k_n} t)] \right\}$$

where Σ_1^* runs over all non-negative integer n-tuples $(k_1, \ldots, k_n)$ such that $\Sigma k_i = r$. We write

$$\log \psi_n(t) = \lim_{M\to\infty} \sum_{j=n+1}^{M} \{\Sigma_j \gamma_{j:k_1,\ldots,k_n} \log f(\beta_{j:k_1,\ldots,k_n} t)\} \qquad \ldots \quad (47)$$

where

$$\beta_{j:k_1,\ldots,k_n} = \beta_j \beta_1^{k_1} \ldots \beta_n^{k_n}$$

$$\gamma_{j:k_1,\ldots,k_n} = \frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!} \gamma_j \gamma_1^{k_1} \ldots \gamma_n^{k_n}$$

and, for fixed j, Σ_j runs through all non-negative-integer n-tuples $(k_1, \ldots, k_n)$ with $0 \leqslant \Sigma k_i \leqslant n-1$. We drop for convenience the individual suffixes and rewrite (47) simply as

$$\log \psi_n(t) = \lim_{M\to\infty} \sum_{j=n+1}^{M} [\Sigma_j \gamma_* \log f_*(t)] \qquad \ldots \quad (48)$$

where $f_*(t) = f(\beta_* t)$. In view of the facts that $\beta_j \to 0$ as $j \to \infty$ and (consequently) $0 < \max_j \beta_j < 1$, we see that $\beta_* \to 0$ uniformly in $*$ as $n \to \infty$, so that the c.f.'s f_* satisfy the "uniform asymptotic negligibility" condition : for any fixed $T > 0$,

$$\lim_{n\to\infty} [\max_{|t|\leqslant T, *} |f_*(t)-1|] = 0. \qquad \ldots \quad (49)$$

Fix a $\tau > 0$; let F_* denote the d.f. corresponding to f_*; define

$$a_* = \int_{|x|<\tau} x dF_*(x), \ \tilde{F}_*(\cdot) = F_*(\cdot + a_*), \ \tilde{f}_* = \text{c.f. of } \tilde{F}_*. \qquad \ldots \quad (50)$$

We claim that η_n given by

$$\log \eta_n(t) = \sum_{n+1}^{\infty} [\Sigma_j \gamma_* \{ia_* t + \int (e^{itu}-1) d\tilde{F}_*(u)\}] \qquad \ldots \quad (51)$$

is well-defined and is an i.d.c.f. which provides an "accompanying i.d. law" for ψ_n; in fact we prove below that $\log \psi_n(t) - \log \eta_n(t) \to 0$ as $n \to \infty$ uniformly in every interval $|t| \leqslant T$.

Since $b_* \to 0$ as $n \to \infty$, uniformly in $*$, we have the usual central limit theorem estimates (Loeve, 1963, "central inequalities", p. 304) : given any $\varepsilon > 0$, there exists a positive constant $c(T, \tau)$ such that for all $|t| \leqslant T$ and $M > n \geqslant N = N(T, \tau)$,

$$\max_{M, *, |t|\leqslant T} |\tilde{f}_*(t) - 1| < \varepsilon$$

and

$$\sum_{j=n+1}^{M} \Sigma_j \gamma_s |\tilde{f}_s(t)-1|^2 \leqslant c(T,\tau)\int_0^T \left\{ \sum_{j=n+1}^{M} \Sigma_j \gamma_s |\log |f_s(t)||\right\} dt$$

$$\leqslant c(T,\tau)\int_0^T |\log|\psi_n(t)||\,dt$$

so that

$$\sum_{j=n+1}^{M} \Sigma_j \gamma_s |\log \tilde{f}_s(t)+1-\tilde{f}_s(t)| \leqslant \sum_{j=n+1}^{M} \Sigma_j \gamma_s |\tilde{f}_s(t)-1|^2$$

$$\leqslant \varepsilon\, c(T,\tau)\int_0^T |\log|\psi_n(t)||\,dt.$$

Now, fix $T \geqslant 1$ and $n \geqslant N(T,\tau)$. Then

$$\sum_{j=n+1}^{\infty} \Sigma_j \gamma_s \{\log \tilde{f}_s(t)+1-\tilde{f}_s(t)\}$$

boundedly converges for all t with $|t| \leqslant T$, so that, so does the series

$$\sum_{j=n+1}^{\infty} \Sigma_j \gamma_s \ \{\log f_s\ (t)-i\alpha_s t+1-\tilde{f}_s(t)\}.$$

Hence $\sum_{j=n+1}^{\infty} \Sigma_j \gamma_s\{-i\alpha_s t+1-\tilde{f}_s(t)\}$ and consequently

$$\sum_{j=n+1}^{\infty} \Sigma_j \gamma_s \int (1-\cos tx)\, d\tilde{F}_s(x) \text{ converge boundedly.}$$

Let

$$\tilde{G}_s(x) = \int_{(-\infty,x]} \frac{u^2}{1+u^2}\, d\tilde{F}_s(u),$$

so that $\sum_{j=n+1}^{\infty} \Sigma_j \gamma_s \int (1-\cos tx)\, \frac{1+x^2}{x^2}\, d\tilde{G}_s(x)$ converges for all t. Integrating termwise between 0 and 1, we have

$$\sum_{j=n+1}^{\infty} \Sigma_j \gamma_s \int \left(1-\frac{\sin x}{x}\right) \frac{1+x^2}{x^2}\, d\tilde{G}_s(x) < \infty$$

so that, $\left(1-\frac{\sin x}{x}\right) \frac{1+x^2}{x^2}$ being bounded away from zero for all x, we have

$$\sum_{j=n+1}^{\infty} \Sigma_j \gamma_s \tilde{G}_s(+\infty) < \infty.$$

Defining $H_n(x) = \sum_{j=n+1}^{\infty} \Sigma_j\gamma_s\widetilde{G}_s(x)$, we have

$$\sum_{j=n+1}^{\infty} \Sigma_j\gamma_s\{\log \bar{f}_s(t)+1-\bar{f}_s(t)\}$$

$$= \sum_{j=n+1}^{\infty} \Sigma_j\gamma_s\{\log f_s(t)-i\alpha_s t+\int(1-e^{itx})d\widetilde{F}_s(x)\}$$

$$= \sum_{j=n+1}^{\infty} \Sigma_j\gamma_s\{[\log f_s(t)-i(\alpha_s+\theta_s)t]-\int\left(e^{itx}-1-\frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}\,d\widetilde{G}_s(x)\}$$

where $\theta_s = \int \frac{x}{1+x^2}\,d\widetilde{F}_s(x)$, so that, from the convergence of the LHS, it follows that $\sum_{j=n+1}^{\infty} \Sigma_j\gamma_s(\alpha_s+\theta_s)$ converges; call the sum of the series C_n, so that for all $n > N(T,\tau)$ and $|t| \leqslant T$,

$$|\log \psi_n(t)-iC_nt-\int\left(e^{itx}-1-\frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}\,dH_n(x)| < \varepsilon.C(T,\tau)\,.\int_0^T |\log|\psi_n(t)||\,dt. \quad \ldots \quad (52)$$

Rewriting (46a) as $\log \xi_n(t) = \Sigma\,\gamma_s \log f_s(t)$, a finite sum depending on n, and adopting definitions similar to (50), we obtain, for $|t| \leqslant T$, $n > N' = N'(T,\tau)$,

$$|\log \xi_n(t)-it(\Sigma\,\gamma_s\alpha_s)-\Sigma\int(e^{itu}-1)d\widetilde{F}_s(u)| < \varepsilon.\ C(T,\tau)\int_0^T |\log|\xi_n(t)||\,dt \quad \ldots \quad (53)$$

so that, for $n > \max(N, N')$ and $|t| \leqslant T$, we have the sum of the RHS's of (52) and (53) is $< \varepsilon\,.\,c(T,\tau)$. $\int_0^T |\log|f(t)||\,dt$. This shows that there is a suitable 'accompanying i.d. law' for the product $\xi_n\,.\,\psi_n$, so that f is i.d. Hence the theorem.

If then $L(\mu,\sigma,M,N)$ be the Levy representation for $\log f$, then $\{a_n\}$ being the sequence of positive β's (if any) and $\{b_n\}$ being the sequence of negative β's (if any), and $\{\delta_n\}$ and $\{\varepsilon_n\}$ being the corresponding subsequences of the γ's, we have the relations:

$$\sigma^2(1-\Sigma\gamma_j\beta_j^2) = 0$$

$$M(u) = \Sigma\,\delta_j M\left(\frac{u}{a_j}\right)-\Sigma\varepsilon_j N\left(-\frac{u}{b_j}\right) \text{ for } u < 0,$$

$$N(u) = \Sigma\,\delta_j N\left(\frac{u}{a_j}\right)-\Sigma\,\varepsilon_j M\left(-\frac{u}{b_j}\right) \text{ for } u > 0,$$

M and N being assumed (without loss of generality) to be respectively left-and right-continuous on $(-\infty, 0)$ and $(0,\infty)$. Setting $g(u) = -N(e^u)$ and $h(u) = M(-e^u)$, $A_j = -\log a_j$, $B_j = -\log b_j$, we have the relations (41); since $\beta_j \to 0$, max $|\beta_j|$ exists and is < 1, so that the condition that the A_j and B_j be bounded away from zero, in addition to being positive, is satisfied. Hence Theorem 5 can be applied to our situation to yield further information (analogous to Theorem 3) regarding the Levy functions M and N. We omit the formal statement which bears the same relationship to Theorem 3 as Theorem 5 does to Theorem 2.

5. Application to regression problems

In this section, we point out cases where our discussion in the preceding sections can be applied to obtain solutions of the regression equation

$$E\left(\sum_1^\infty a_jX_j \,\Big|\, \sum_1^\infty b_jX_j\right) = 0 \quad \text{a.s.,} \qquad \dots \quad (54)$$

where the X_j are i.i.d.r.v.'s with $EX_1 = 0$.

Theorem 7 : *Let $X_1, X_2, \dots$ be an infinite sequence of non-degenerate i.i.d.r.v.'s with $EX_1 = 0$. Suppose they satisfy (54) where $\{a_j\}$ and $\{b_j\}$ are sequences of real constants such that $\Sigma|a_j| < \infty$ and Σb_jX_j converges almost surely to a r.v., and suppose further that*

(i) $a_1 \neq 0$, $|b_j| < |b_1|$ *for all* $j < 1$,

and (ii) $a_jb_1/a_1b_j < 0$ *for all* $j > 1$ *for which* $a_jb_j \neq 0$.

Then X_1 follows an infinitely divisible law. More precise statements about the Levy functions in the Levy representation of the log c.f. of that 'generalized stable law' can be had if the a_j, b_j satisfy the conditions postulated in any one of the Theorems 4 to 6. In particular, X_1 is normally distributed under the above conditions iff $\sum_1^\infty a_jb_j = 0$.

Proof: Since $\Sigma|a_j| < \infty$ and $E|X_1| < \infty$ by assumption, $\Sigma E|a_jX_j| < \infty$ and hence Σa_jX_j converges a.s. (see, for instance, Rao, 1965, p. 91 or Lukacs, 1968, p. 72). Also, for the same reasons,

$$\begin{aligned} 0 &= E(\Sigma a_jX_je^{it\Sigma b_jX_j}) \\ &= \Sigma E(a_jX_je^{it\Sigma b_jX_j}) \\ &= \Sigma a_j[f'(b_jt).\prod_{k\neq j} f(b_kt)] \qquad \dots \quad (55) \end{aligned}$$

where f is the c.f. of X_1.

Since Σb_jX_j converges a.s. to a r.v., $\Pi f(b_jt)$ represents a c.f., so that $\prod_{n+1}^\infty f(b_kt) \to 1$ as $n \to \infty$, in fact uniformly in every compact t-interval, so that $\prod_1^\infty f(b_kt) = 0$ is possible only if $f(b_kt) = 0$ for some k. Now, if $I : |t| < \delta$ be the largest interval around the origin in which f does not vanish, we may then divide through by $\prod_1^\infty f(b_kt)$ in equation (55) to obtain

$$\sum_1^\infty a_j\frac{f'(b_jt)}{f(b_jt)} = 0, \quad t \in I. \qquad \dots \quad (56)$$

In any compact sub-interval of I, $f(b_jt)$ is bounded away from zero uniformly for all j, and $|f'(b_jt)| \leqslant E|X_1|$ while $\Sigma|a_j| < \infty$ by assumption, so that we may integrate the LHS of (56) term by term over any interval $[0, t]$, $|t| < \delta$, to get

$$\Sigma \frac{a_j}{b_j} \log f(b_jt) = \text{constant} = 0, \; t\epsilon I,$$

the summation running over all j such that $a_jb_j \neq 0$. We rewrite the above as

$$f(t) = \prod_1^{\infty} [f(\beta_jt)]^{\gamma_j}, \; t\epsilon I \qquad \ldots \quad (57)$$

where $|\beta_j| < 1$ and $\gamma_j > 0$ for all j, $(\beta_j = b_j/b_1)$.

We now claim that f is non-vanishing for all t and that (57) holds for all t. Suppose in the above that $\delta < \infty$, so that $f(\pm\delta) = 0$. Now, by assumption, f is non-trivial and Σb_jX_j converges a.s., and these imply, by Theorem 4, that $\Sigma b_j^2 < \infty$, i.e., $\Sigma\beta_j^2 < \infty$, so that, in particular, $\beta_j \to 0$ as $j \to \infty$. We may therefore speak of $\max_j |\beta_j|$ and assume without loss of generality that $|\beta_1| = \max_j |\beta_j|$ and let $\delta_1 = \delta/|\beta_1|$, so that

$$f(\beta_jt) \neq 0 \text{ for any } j \text{ provided } |t| < \delta_1. \qquad \ldots \quad (58)$$

Let now $g = |f|^2$ so that $g(t) = \prod_1^{\infty} [g(\beta_jt)]^{\gamma_j}$ for $|t| < \delta$. We note that g is real-valued, with $g(-t) = g(t)$, and hence, in particular, satisfies the elementary inequality valid for such c.f.'s

$$1-g(2t) \leqslant 4[1-g(t)]. \qquad \ldots \quad (59)$$

Lemma 6 : *If* $\prod_1^{\infty} (g(\beta_jt))^{\gamma_j}$ *converges uniformly for* $|t| \leqslant \epsilon$ *and* $g(\beta_jt) \neq 0$ *for* $|t| \leqslant 2\epsilon$ *for all* j, *then* $\prod_1^{\infty} [g(\beta_jt)]^{\gamma_j}$ *converges uniformly for* $|t| \leqslant 2\epsilon$.

Proof : Since $0 \leqslant -\log(1-\theta)$ for $0 \leqslant \theta < 1$, and relation (59) holds, the uniform convergence of $\Sigma\gamma_j \log g(\beta_jt)$ for $|t| \leqslant \epsilon$ implies that of $\Sigma\gamma_j[1-g(2\beta_jt)]$ for $|t| \leqslant \epsilon$. Also since $g(2\beta_jt) \neq 0$ for $|t| \leqslant \epsilon$, $\log g(2\beta_jt)$ is defined for such t (with the usual choice of the logarithm) and since $\beta_j \to 0$ as $j \to \infty$, $g(2\beta_jt) \geqslant 1/2$ for all sufficiently large j uniformly for all such t, i.e., there exists $J(\epsilon)$ such that this relation is true for $|t| \leqslant \epsilon$ if $j \geqslant J(\epsilon)$. For such j and t, we have

$$-\log g(2\beta_jt) \leqslant \frac{1-g(2\beta_jt)}{g(2\beta_jt)} \leqslant 2[1-g(2\beta_jt)]$$

so that $\Sigma\gamma_j \log g(2\beta_jt)$ converges uniformly for $|t| \leqslant \epsilon$, so that the sum-function is continuous there. Thus $\Pi[g(\beta_jt)]^{\gamma_j}$ converges uniformly to a continuous function in $|t| \leqslant 2\epsilon$.

Now the fact that $\prod_1^\infty [g(\beta_j t)]^{\gamma_j}$ converges to the continuous function $g(t)$ in $|t| < \delta$ implies that the convergence is uniform in any compact subinterval $|t| \leqslant \epsilon$ thereof; this is a consequence of the fact that for a series of (real and) *non-negative* functions of a real variable defined on a compact interval, uniform convergence is equivalent to the continuity of the sum-function (see, for instance, Titchmarsh, 1939, p. 13). This fact together with relation (58) and Lemma 6 implies that

$$0 = g(\delta) = \lim_{t \uparrow \delta} g(t) = \lim_{t \uparrow \delta} \Pi [g(\beta_j t)]^{\gamma_j}$$

$$= \Pi [g(\beta_j \delta)]^{\gamma_j} \neq 0.$$

This contradiction shows that g, and hence f, cannot vanish at any point of the real line. Then, as we have already proved, relation (57) holds at all points and our theorem stands proved. The particular assertion about conditions for the normality of f follows from Theorem 4.

A well-known theorem of Marcinkiewicz (see for instance Lukacs, 1968, pp. 112-116) connects the identity of distribution of two linear forms in a sequence of i.i.d.r.v.'s with the normality of those r.v.'s. An extension thereof enables us to state sufficient conditions under which relation (54) would imply the normality of the X_j.

Theorem 8 : *Let $X_1, X_2, \ldots$ be a sequence of non-degenerate i.i.d.r.v.'s having moments of all orders, with $EX_1 = 0$. Suppose they satisfy* (54) *where the $\{a_j\}$ and $\{b_j\}$ are sequences of real constants satisfying the following conditions* :

(i) $\Sigma |a_j| < \infty$;

(ii) *$\Sigma b_j X_j$ converges almost surely to a r.v.,*

and (iii) *if $\{\beta_j\}$, $\{\beta_j^*\}$ be the subsequences of $\{b_j\}$ for which respectively $a_j b_j > 0$ and $a_j b_j < 0$ then $\{|\beta_j|\}$ and $\{|\beta_j^*|\}$ are not permutations of each other.*

Then the X_j are normal.

Remark : The X_j need not be normal if the moments of all orders do not exist, even if all the other conditions above are satisfied, as pointed out by Linnik (1953) in respect of Marcinkiewicz's theorem.

Proof : Proceeding as in the proof of Theorem 7, we arrive at the relation

$$\Sigma(a_j/b_j) \log f(b_j t) = 0, \ t \epsilon I$$

I being any interval around the origin where the c.f. $\Pi f(b_j t)$ does not vanish, the summation running over all j such that $a_j b_j \neq 0$. We thus have

$$\Sigma \gamma_j \log f(\beta_j t) = \Sigma \gamma_j^* \log f(\beta_j^* t), \ t \epsilon I \qquad \ldots \quad (60)$$

where the γ's are all positive.

We note that (since the convergence a.s. of $\Sigma b_j X_j$ implies that $\Sigma b_j^2 < \infty$) β_j and $\beta_j^* \to 0$ as $j \to \infty$, so that we may speak of max $|\beta_j|$ and max $|\beta_j^*|$. As in the proof of Theorem 4, it follows from the convergence of the two members of relation (60) that $\Sigma \gamma_j \beta_j^2 < \infty$ and $\Sigma \gamma_j^* (\beta_j^*)^2 < \infty$; since β_j and $\beta_j^* \to 0$, the above implies the convergence of $\Sigma \gamma_j \beta_j^{2s}$ and $\Sigma \gamma_j^* (\beta_j^*)^{2s}$ for all positive integers s; it further follows then that

$$\lim_{s \to \infty} (\Sigma \gamma_j \beta_j^{2s})^{1/2s} = \max_j |\beta_j|.$$

Proceeding then along the lines of the proof of Marcinkiewicz's theorem, we obtain our assertion.

6. Characterizations of the Wiener process with linear mean value function defined on a compact real interval

We assume familiarity with all the concepts involved in the phrases : 'a continuous (in-probability) homogeneous stochastic process with independent increments defined on a compact interval $[A, B]$' and 'a Wiener process with linear mean value function (m.v.f.) defined on $[A, B]$'. We also assume familiarity with the basic concepts associated with second order random processes. For all these we may refer the reader, for instance, to Lukacs (1968, Sections 5.1 and 5.2, pp. 100-109).

We state below four results characterizing the Wiener process with linear m.v.f., all of them being straightforward extensions of known results, these extensions being in keeping with the spirit of the earlier sections of this paper. We omit the proofs of three of them, since they may be obtained by proceeding along the lines of Lukacs (1968); our proof of the remaining proposition (Theorem 11) follows along those of our proof of Theorem 4 (ii) of the present paper.

Let $\alpha(\cdot)$ be a non-constant, non-decreasing right-continuous function defined on a compact real interval $[a, b]$, with $\alpha(a) = A$, $\alpha(b) = B$, and let $X(t)$ be a continuous homogeneous process with independent increments defined on $[A, B]$. Let then

$$f(u, \tau) = E \exp\{iu[X(t+\tau) - X(t)]\}, \ A \leqslant t < t+\tau \leqslant B$$

and

$$f(u) = [f(u, \tau)]^{1/\tau}, \quad \phi(u) = \log f(u).$$

Let g and h below denote functions continuous on $[a, b]$. The random variable

$$Y_g = \int_a^b g(t) dX[\alpha(t)]$$

is well-defined in the sense of

$$\operatorname*{plim}_{||\Delta|| \to 0} \sum_{r=0}^{n-1} g(\tau_r)\{X[\alpha(t_{r+1})] - X[\alpha(t_r)]\}$$

where

$$\Delta : \{t_0 = a < t_1 < \dots < t_n = b\}$$

is an arbitrary sub-division of $[a, b]$, $\|\Delta\| = \max(t_{r+1}-t_r)$, and $\tau_r \epsilon [t_r, t_{r+1}]$. The c.f. ψ_g of Y_g is given by

$$\log \psi_g(u) = \int_a^b \phi[ug(t)]d\alpha(t)$$

and the c.f. $\psi_{g,h}$ of Y_g and Y_h is given by

$$\log \psi_{g,h}(u, v) = \int_a^b \phi[ug(t)+vh(t)]d\alpha(t).$$

For the proofs of the above statements, which run along lines similar to those for the case $\alpha(t) = t$, we may refer to Lukacs (1968), Section 5.2. Also given there are sufficient conditions under which

$$Y_g = \int_a^b g(t)dX[\alpha(t)]$$

is well-defined in the sense of a limit in quadratic mean when $X(t)$ is a second-order random process, and under which the above expressions for the c.f.'s ψ_g and $\psi_{g,h}$ are valid : these are proved there for the case $\alpha(t) = t$ and are easily carried over to the case of general α.

We shall refer to the finite measure induced by the point-function α on the Borel subsets of $[a, b]$ as the *α-measure* thereon.

Theorem 9 (Characterization by independence of two stochastic integrals) : *Let $X(t)$ be a continuous homogeneous process with independent increments defined on $[A, B]$ and g and h be continuous functions on $[a, b]$ such that each of them is non-vanishing on a set of positive α-measure there (not necessarily the same for both), and at least one of them vanishes nowhere on $[a, b]$. Y_g and Y_h are independent iff*

(i) *$X(t)$ is a Wiener process with linear m.v.f.*

and (ii) $\int_a^b gh d\alpha = 0$ *in case $X(t)$ is not degenerate.*

Theorem 10 (Identical distribution of two stochastic integrals) : *Let $X(t)$ be a continuous homogeneous process with independent increments on $[A, B]$. Suppose g and h are continuous functions on $[a, b]$ with $\max|g(t)| \neq \max|h(t)|$ there. Y_g and Y_h are identically distributed then iff*

(ii) *$X(t)$ is a Wiener process with linear m.v.f.,*

(ii) $\int_a^b g d\alpha = \int_a^b h d\alpha$ *if the m.v.f.* $\neq 0$,

and (iii) $\int_a^b g^2 d\alpha = \int_a^b h^2 d\alpha$.

For the proof, we note that the identical distribution of Y_g and Y_h is equivalent to the relation

$$\int_a^b \phi[ug(t)]d\alpha(t) = \int_a^b \phi[uh(t)]d\alpha(t) \text{ for all real } u,$$

and that, k running through positive integral values,

$$\lim_{k\to\infty} \left\{ \int_a^b [g(t)]^{2k} d\alpha(t) \right\}^{1/2k} \text{ exists and } = \max |g(t)| \text{ on } [a, b],$$

referring the reader for the rest of the argument to Lukacs (1968).

Theorem 11 (Identical distribution of a stochastic integral and a r.v.): *Let $X(t)$ be a continuous homogeneous process with independent increments on $[A, B]$, and g a continuous function on $[a, b]$ satisfying one or other of the following conditions :*

(i) $|g(t)| < 1$ *for all t in $[a, b]$ and g has at most a finite number of zeros there;*

or (ii) $|g(t)| > 1$ *for all t in $[a, b]$: this implies that g is of constant sign.*

Suppose for some (and hence for every) positive integer $n \geqslant 1/(B-A)$ the distribution of Y_g is the same as the n-th convolution of the d.f. of $X\left(t+\frac{1}{n}\right)-X(t)$, $A \leqslant t < t+\frac{1}{n} \leqslant B$. (If $B-A \geqslant 1$, this condition simplifies into : Y_g and $X(t+1)-X(t)$, $A \leqslant t < t+1 \leqslant B$, have the same distribution.) Then $X(t)$ is a Wiener process with linear m.v.f. iff

$$\int_a^b [g(t)]^2 d\alpha(t) = 1. \qquad \dots \quad (61)$$

Further, in that case, $$\int_a^b g(t)d\alpha(t) = 0 \text{ or the m.v.f. is } \equiv 0. \qquad \dots \quad (62)$$

Proof : The 'only if' part and (62) follow from the fact that the identical distribution of the above r.v.'s is equivalent to the relation

$$\int_a^b \phi[ug(t)]d\alpha(t) = \phi(u) \text{ for all real } u. \qquad \dots \quad (63)$$

As for the 'if' part, suppose now that (63) and (61) hold. If $\theta(u) = \phi(u)+\phi(-u)$, we obtain from (63) the relation

$$\int_a^b \theta[ug(t)]d\alpha(t) = \theta(u) \text{ for all real } u. \qquad \dots \quad (64)$$

(61) then implies, with $c(t) = |g(t)|$, $F(u, t) = \theta[uc(t)]-[c(t)]^2\theta(u)$,

$$\int_a^b F(u, t)d\alpha(t) = 0. \qquad \dots \quad (65)$$

We now consider g satisfying condition (i). Let $u \neq 0$ be fixed. If $(a = a_0 \leqslant)$ $a_1 < a_2 < \ldots < a_n (\leqslant a_{n+1} = b)$ be the points where g vanishes, then $F(u, a_r) = 0$ for $1 \leqslant r \leqslant n$, so that (65) may be rewritten as

$$\sum_{r=0}^{n} \int_{[a_r, a_{r+1}]} F(u, t) d\alpha(t) = 0. \qquad \ldots \quad (66)$$

In the above sum, we can ignore those r for which $\int_{[a_r, a_{r+1}]} d\alpha(t) = 0$, so that (66) implies the existence of some r, r depending on u, $0 \leqslant r \leqslant n$, such that

$$\int_{[a_r, a_{r+1}]} d\alpha(t) > 0, \quad \int_{[a_r, a_{r+1}]} F(u, t) d\alpha(t) \geqslant 0.$$

F being continuous in t, the above implies the impossibility of $F(u, t)$ being strictly negative on $[a_r, a_{r+1}]$, so that there exists some t_u there, with $c(t_u) > 0$, such that $F(u, t_u) \geqslant 0$. Similarly there exists a t_u^* in $[a, b]$ such that $c(t_u^*) > 0$ and $F(u, t_u^*) \leqslant 0$. Thus, for every $u \neq 0$, there correspond $\xi_u = uc(t_u)$ and $\xi_u^* = uc(t_u^*)$—so that $0 < |\xi_u| < |u|$, $0 < |\xi_u^*| < |u|$—such that, if $\eta(u) = \theta(u)/u^2$ for $u \neq 0$, then

$$\eta(u) \geqslant \eta(\xi_u) \text{ and } \eta(u) \leqslant \eta(\xi_u^*).$$

It then follows as in the proof of Theorem 4(ii) that η is constant so that $\theta(u) \equiv -cu^2$ and, by the Levy-Cramer theorem, f is a normal c.f. (63) then yields (62).

Considering now g satisfying (ii), g is of constant sign and $|g|$ is bounded below by a constant $1+\partial > 1$ as well as bounded. We then obtain straightaway from (65) in this case the existence, for any fixed $u \neq 0$, of t_u and t_u^* in $[a, b]$ such that $F(u, t_u) \geqslant 0$, $F(u, t_u^*) \leqslant 0$, where of course $c(t_u) \geqslant 1+\partial$, $c(t_u^*) \geqslant 1+\partial$. Thus, for every fixed $u > 0$, there exists a sequence $u = u_0 < u_1 < u_2 < \ldots \to \infty$ such that $\eta(u_{k+1}) \geqslant \eta(u_k)$ for all k, and a sequence $u = u_0^* < u_1^* < u_2^* < \ldots \to \infty$ such that $\eta(u_{k+1}^*) \leqslant \eta(u_k^*)$. But f being an i.d.c.f., if γ be the Gaussian constant in the Levy representation for $\phi = \log f$, then $\lim_{u \to \infty} \eta(u)$ exists and $= -2\gamma$. Thus we have from the above that $\eta(u) \leqslant -2\gamma \leqslant \eta(u)$ for all $u > 0$, so that $\theta(u) \equiv -2\gamma u^2$. It follows from the Levy-Cramer theorem then that f is a normal c.f. and then, as before, (63) yields (62).

We proceed to our final result.

Theorem 12 (Linearity and homoscedasticity of the regression of one stochastic integral on another) : *Let $X(t)$ be a continuous homogeneous process with independent increments defined on $[A, B]$. Assume further that it is a second-order process with its m.v.f. and covariance function both of bounded variation on $[A, B]$. Let g and h be two continuous functions on $[a, b]$, and let there be a compact sub-interval of $[a, b]$ in which* (i) $gh \neq 0$ *and* (ii) *g is not proportional to h. Let $Y_g = \int_a^b g(t) dX[\alpha(t)]$ and $Y_h = \int_a^b h(t) dX[\alpha(t)]$, taken to be in the sense of limits in quadratic mean. Then the process $X(t)$ is a Wiener process with linear m.v.f. iff Y_g has linear regression and constant scatter on Y_h (the regression of Y_g on Y_h is linear and homoscedastic).*

References

Eaton, M. L. (1966): Characterization of distributions by the identical distribution of linear forms. *J. Appl. Prob.*, **3**, 481-494.

Levin, B. Ja. (1964): *Distribution of Zeros of Entire Functions*. Translated from the Russian original by the Amer. Math. Soc., Providence.

Linnik, Yu. V. (1953): Linear forms and statistical criteria, I and II (in Russian), *Ukrain. Mat. Zhurnal*, **5**, 207-243 and 247-290 (available in '*Selected Translations in Math. Stat. Prob.*', Vol. 3, Amer. Math. Soc., 1962).

Lukacs, E. (1960): *Characteristic Functions*, Griffin.

——— (1968): *Stochastic Convergence*, Heath.

Lukacs, E. and Laha, R. G. (1965): On a linear form whose distribution is identical with that of a monomial. *Pacific J. Math.*, **15**, 207-214.

——— (1968): On a property of the Wiener process. *Ann. Inst. Statist. Math.*, **20**, 383-389.

Loeve, M. (1963): *Probability Theory*, Van Nostrand (III edition).

Rao, C. Radhakrishna (1965): *Linear Statistical Inference and its Applications*, John Wiley.

Ramachandran, B. and Rao, C. Radhakrishna (1968): Some results on characteristic functions and characterizations of the normal and generalized stable laws. *Sankhyā*, Series A, **30**, 125-140.

Ramachandran, B. (1969): On characteristic functions and moments. *Sankhyā*, Series A, **31**, 1-12.

Shimizu, R. (1968): Characteristic functions satisfying a functional equation (I). *Ann. Inst. Statist. Math.*, **20**, 187-209.

Titchmarsh, E. C. (1939): *The Theory of Functions*, Oxford (II edition).

Widder, D. V. (1946): *The Laplace Transform*, Princeton Univ. Press.

Paper received : August, 1969.