NOTES

A NOTE ON SOME TRANSLATION-PARAMETER FAMILIES OF DENSITIES FOR WHICH THE MEDIAN IS AN M.L.E.

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SUMMARY. The following theorem is proved in the forthcoming book by Kagan, Linnik and Rao: If the median is an m.l.e. of θ for a sample of size $n = 4$ from a population with density $f(x-\theta)$ and $f(x)$ is lower semi-continuous at the origin then $f(x) = ke^{-\alpha|x|}$ $-\infty < x < \infty$, $\alpha > 0$. The object of this note is to show that the theorem is false if we require the median is an m.l.e. of θ for samples of size $n = 2$ and 3.

1. Introduction

Teicher (1961) has shown that under very mild conditions a translation-parameter family of densities must be normal if the m.l.e. is to be the sample mean. The following similar characterisation of the double exponential may be found in the forthcoming book by Kagan, Linnik and Rao: If the median is an m.l.e. of θ for a sample of size $n = 4$ from a population with density $f(x-\theta)$ and $f(x)$ is lower semi-continuous at the origin then $f(x) = ke^{-\alpha|x|}$ $-\infty < x < \infty$, $\alpha > 0$. The object of this note is to show that the theorem is false if we require the median is an m.l.e. of θ for samples of size $n = 2$ and 3.

2. A counter example

Let

$$\log f(y) = -\alpha_1 y + k_1 \qquad \text{if} \quad 0 \leqslant y \leqslant c_1$$

$$= -\alpha_2 y + k_2 \qquad \text{if} \quad c_1 \leqslant y < \infty$$

where $k_2 = (\alpha_2 - \alpha_1)c_1 + k_1$ for continuity at $y = c_1$.

Let

$$0 < \alpha_1 < \alpha_2 < 2\alpha_1.$$

$$\log f(-y) = \log f(y).$$

Then $-\log f(y)$ is a symmetric convex function.

Therefore

$$-\log f(-y-\theta) - \log f(y-\theta)$$

is a symmetric convex function of θ (for fixed y) and hence has a minimum at $\theta = 0$. This implies the median is an m.l.e. for a sample of size 2.

Consider now

$$\psi(\theta) = -\log f(-y_1-\theta) - \log f(-\theta) - \log f(y_2-\theta) \quad y_1 > 0, \;\; y_2 > 0$$

which is again a convex function and hence a local minimum is an absolute minimum. We show $\theta = 0$ is a local minimum.

Observe that for θ sufficiently small and $\geqslant 0$

$$\psi(\theta) = \alpha_i |-y_1-\theta| + \alpha_1 |\theta| + \alpha_j |y_2-\theta| - (k_i+k_1+k_j)$$

where

$$\alpha_i = \alpha_1 \quad \text{or} \quad \alpha_2$$

$$\alpha_j = \alpha_1 \quad \text{or} \quad \alpha_2$$

$$k_i = k_1 \quad \text{or} \quad k_2$$

$$k_j = k_1 \quad \text{or} \quad k_2$$

which is minimum at $\theta = 0$ since 0 is a median of the distribution

$$\begin{matrix} -y_1 & 0 & y_2 \\ \dfrac{\alpha_i}{\alpha_i+\alpha_1+\alpha_j} & \dfrac{\alpha_1}{\alpha_i+\alpha_1+\alpha_j} & \dfrac{\alpha_j}{\alpha_i+\alpha_1+\alpha_j}. \end{matrix}$$

Similarly $\psi(\theta)$ for θ sufficiently small and $\leqslant 0$ is minimum at $\theta = 0$. This shows $\psi(\theta)$ has a local minimum at $\theta = 0$. This implies for sample size $n = 3$, the median is an m.l.e.

Reference

Teicher, H. (1961): Maximum likelihood characterisation of distribution. *Ann. Math. Stat.*, 32, 1214-1222.

Paper received : September, 1970.