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THE PASSAGE FROM RANDOM WALK TO DIFFUSION IN QUANTUM PROBABILITY II

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SUMMARY. In the framework of the tensor product of an initial Hilbert space with the boson Fock space over L_2 (R_+) a quantum random walk is constructed and its convergence to a quantum diffusion limit is exhibited

1. Introduction

An approach to the passage from random walk to a diffusion limit in the Schrodinger picture of quantum probability was outlined in Parthasarathy (1987). Such an approach leads to a limit theorem for unitary operator valued adapted processes in a sequence of varying Hilbert spaces (Parthasarathy, 1987 ; Accardi and Bach). To be more in tune with the classical approach of formulating limit theorems in a single sample space of right continuous paths with left limits we now reformulate the notion of a quantum random walk int he algebraic language of Accardi, Frigerio and Lewis (1982) by making the elementary observation that a classical Markov chain can always be constructed by first choosing a sequence of independent random maps on the state space into itself and making successive compositions. Such a reformulation enables us to examine the passage to quantum diffusion limits in the Heisenberg picture in a single Hilbert space at least in one interesting example.

2. A quantum random walk

We begin with the description of a classical Markov chain in algebraic terms and then formulate the notion of a quantum random walk by analogy. To this end consider a Markov chain with a finite or countable state space $S = \{1, 2, \ldots\}$ and stationary transition probability matrix $P = ((p_{ij}))$.

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Denote by $M(S)$ the set of all maps from S into itself and by $\mathcal{F}$ the smallest σ-algebra generated by all subsets of $M(S)$ of the form $\{f : f(i) = j\}$, $i, j \in S$. Under the composition operation o the set $M(S)$ becomes a measurable semi-group with identity and acting on S.

Proposition 2.1 : *There exists a probability measure μ on $(M(S), \mathcal{F})$ such that*

$$\mu\{f : f(i) = j\} = p_{ij} \quad \textit{for all } i, j \in S. \qquad \ldots \quad (2.1)$$

Proof : Let μ_i denote the probability distribution $p_{i1}, p_{i2}, \ldots$ on the set S for each $i \in S$. Define $\mu = \mu_1 \times \mu_2 \times \ldots$, the cartesian product of μ_1, $\mu_2, \ldots$. If we look at μ as a measure on $(M(S), \mathcal{F})$ it is clear from definitions that μ satisfies (2.1). □

Remark 1 : Suppose we choose $x_0 \in S$ according to a distribution λ, independent and identically distributed random elements $g_1, g_2, \ldots$ from $M(S)$ with distribution μ as in Proposition 2.1 and put $x_n = g_n \circ g_{n-1} \circ \ldots \circ g_1 (x_0)$ for $n = 1, 2, \ldots$ then $\{x_n\}_{n \geq 0}$ is a Markov chain with initial distribution λ and stationary transition probability matrix P.

Remark 2 : If P is a doubly stochastic matrix in the sense that $\sum_i p_{ij} = 1$ for each j and $\sum_j p_{ij} = 1$ for each i in S, and if S is finite, then the semigroup $M(S)$ in Proposition 2.1 can be replaced by the group $G(S)$ of all bijective maps of S onto itself. This follows from Birkhoff's theorem that every doubly stochastic matrix is a convex combination of permutation matrices (Berman and Plemmons, 1979, page 50.)

Remark 3 : If the state space S is the real line or, more generally, an uncountable Polish space and the transition probability P is described by $P(x, E)$ which is measurable in x for fixed E and a probability measure on the Borel σ-algebra of S for fixed x then Proposition 2.1 holds with $M(S)$ being the semigroup of all Borel measurable maps of S into itself, $\mathcal{F}$ the smallest σ-algebra generated by all subsets of the form $\{f : f(x) \in E\}$, $x \in S$, E a Borel subset of S and (2.1) being replaced by the relation

$$\mu\{f : f(x) \in E\} = P(x, E).$$

Indeed, assume without loss of generality that S is the real line and denote by $G(x, t)$, $x \in \mathbf{R}$, $t \in [0, 1]$ the left continuous nondecreasing inverse of the distribution function of the measure $P(x, .)$. Then G is measurable in (x, t). Choose i.i.d. random variables $\xi_1, \xi_2, \ldots$ with uniform distribution in [0, 1] and put $G_j(x) = G(x, \xi_j)$. Then $x, G_1(x), G_2 \circ G_1(x), \ldots, G_n \circ G_{n-1} \circ \ldots \circ G_1(x), \ldots$ is a

Markov sequence with initial state x and transition probability $P(\cdot, \cdot)$. This shows once again that Markov chains with an arbitrary initial distribution λ on S and transition probability $P(\cdot, \cdot)$ can be generated according to the recipe in Remark 1.

We shall now impart a more algebraic character to the Markov chain described in Remark 1. For any Borel space Ω let $B(\Omega)$ denote the algebra of all complex valued bounded Borel functions on Ω. Let

$$\Omega = S \times M(S) \times M(S) \ldots$$

$$\rho = \lambda \times \mu \times \mu \times \ldots$$

where λ, μ are as in Remark 1. Let $B_{n]} = B(S \times M(S) \times M(S) \times \ldots \times M(S))$ where $M(S)$ appears n-fold in the cartesian product. Then $B_{n]}$ is an increasing sequence of subalgebras of $B(\Omega)$. Define the sequence $\{J_n\}$ of homomorphisms from $B(S)$ into $B(\Omega)$ by putting

$$J_0\phi = \phi, \ (J_n\phi)(x, f_1, f_2, \ldots) = \phi(f_n \circ f_{n-1} \circ \ldots \circ f_1(x)) \quad \ldots \quad (2.2)$$

for $n = 1, 2, \ldots$. The sequence $\{J_n\}$ is adapted to $\{B_{n]}\}$ in the sense that $J_n \phi \in B_{n]} \ \forall \ n$ and $\phi \in B(S)$. Furthermore

$$\rho(J_n \phi) = \int (J_n \phi) \, d\rho = \int (P^n\phi)(x) \lambda(dx)$$

where

$$(P \phi)(x) = \sum_{j \in S} p_{xj} \, \phi(j)$$

is the transition operator of the Markov chain. Thus the quadruple $(B(\Omega), \{B_{n]}\}, \{J_n\}, \rho)$ describes the Markov chain in the algebraic sense of Accardi, Frigerio and Lewis (1982). In view of (2.2) the dynamics of the chain is determined by the measure ρ and the one step homomorphism J_1.

We are now ready to introduce a special class of quantum random walks in analogy with the description of classical Markov chains given above. Let h_0, $\mathcal{H}$ be complex separable Hilbert spaces and let $\mathcal{S}$, $\mathcal{N}$ be respectively W^* algebras of operators acting on h_0, $\mathcal{H}$. We assume that both $\mathcal{S}$, $\mathcal{N}$ include the identity operator. Furthermore we assume that $\mathcal{N}$ is a finite dimensional vector space of dimension d. Let $J : \mathcal{S} \to \mathcal{S} \otimes \mathcal{N}$ be a W^* homomorphism preserving identity. If $\mathcal{S}$ and $\mathcal{N}$ are called the *system* and *noise* algebras then J may be interpreted as a one step random walk and compared with the homomorphism J_1 in (2.2).

Choose a basis $N_1, N_2, \ldots, N_d$ for $\mathcal{N}$. Then there exist (structure) constants b_j, c^k_{ij}, e_{ij} such that

$$1 = \sum_j b_j N_j, \quad \ldots \quad (2.3)$$

$$N_i N_j = \sum_k c^k_{ij} N_k, \quad \ldots \quad (2.4)$$

$$N_i^\dagger = \sum_j e_{ij} N_j \quad \ldots \quad (2.5)$$

where † denotes adjoint and 1 denotes the identity operator. Define the maps $\alpha_j : \mathcal{S} \to \mathcal{S}$, $1 \leqslant j \leqslant d$ by the relations

$$J(x) = \sum_{j=1}^{d} \alpha_j(x) \otimes N_j \quad \dots \quad (2.6)$$

The fact that J is a W^* homomorphism implies that each α_j is a linear map and the following identities hold :

$$\alpha_k(xy) = \sum_{i,j} c_{ij}^k \alpha_i(x)\, \alpha_j(y), \quad \dots \quad (2.7)$$

$$\alpha_k(x^\dagger) = \sum_i e_{ik}\, \alpha_i(x)^\dagger, \quad \dots \quad (2.8)$$

$$\alpha_k(1) = b_k, \quad \dots \quad (2.9)$$

where b_k stands for b_k times the identity. The identities (2.7)—(2.9) may be compared with the cohomological identities derived by Hudson (1986). Consider the increasing sequence $\{\mathcal{B}_{n]}\}$, $n \geqslant 0$ of W^* algebras defined by

$$\mathcal{B}_{0]} = \mathcal{S},\ \mathcal{B}_{n]} = \mathcal{S} \otimes \underbrace{\mathcal{N} \otimes \dots \otimes \mathcal{N}}_{n\text{-fold}},\ n = 1, 2, \dots \quad \dots \quad (2.10)$$

where $\mathcal{B}_{n]}$ is looked upon as a subalgebra of $\mathcal{B}_{n+1]}$ by identifying $\xi \in \mathcal{B}_n$ with $\xi \otimes 1$ in $\mathcal{B}_{n+1]}$. For any $x \in \mathcal{S}$ let

$$J_0(x) = x,\ J_1(x) = J(x),$$

$$J_n(x) = \sum_{j=1}^{d} J_{n-1}(\alpha_j(x)) \otimes N_j,\ n = 2, 3, \dots \quad \dots \quad (2.11)$$

Proposition 2.2 : *J_n is an identity preserving W^* homomorphism from $\mathcal{S}$ into $\mathcal{B}_{n]}$ for each $n \geqslant 0$.*

Proof : The proposition holds for $n = 0, 1$ by definition. We proceed by induction starting with the assumption that the proposition holds for all J_j, $j \leqslant n-1$. It is clear from the linearity of the maps α_j that J_n is linear. From (2.4) and (2.7) we have

$$J_n(x)J_n(y) = \sum_{i,j} J_{n-1}(\alpha_i(x)\alpha_j(y)) \otimes N_i N_j$$

$$= \sum_k J_{n-1}\Big(\sum_{ij} c_{ij}^k\, \alpha_i(x)\alpha_j(y)\Big) \otimes N_k$$

$$= \sum_k J_{n-1}(\alpha_k(xy)) \otimes N_k$$

$$= J_n(xy).$$

By a similar argument using (2.5) and (2.8) we obtain

$$\begin{aligned} J_n(x)^\dagger &= \Sigma\, J_{n-1}\,(\alpha_k(x))^\dagger \otimes N_k^\dagger \\ &= \sum_j J_{n-1}\left(\sum_k e_{kj}\alpha_k(x)^\dagger\right) \otimes N_j \\ &= \sum_j\; J_{n-1}\,(\alpha_j(x^\dagger)) \otimes N_j \\ &= J_n(x^\dagger). \end{aligned}$$

From (2.3) and (2.9) we have

$$\begin{aligned} J_n(1) &= \sum_k\; J_{n-1}\,(\alpha_k(1)) \otimes N_k \\ &= \Sigma\; b_k \otimes N_k \\ &= 1 \otimes \Sigma\; b_k N_k = 1. \;\; \square \end{aligned}$$

Remark 1 : Let λ be a state in $\mathcal{S}$ and let $\{\mu_j\}$, $j = 1, 2, \ldots$ be a sequence of states in $\mathcal{N}$. Suppose there exists a W^* algebra $\mathcal{B}$ with a state ρ on it such that $\bigcup_{n \geqslant 0} \mathcal{B}_{n]}$ is dense in $\mathcal{B}$ and for any $x \in \mathcal{S}$, $y_j \in \mathcal{N}_j$, $j = 1, 2, \ldots$

$$\rho(x \otimes y_1 \otimes \ldots \otimes y_k) = \lambda(x) \prod_{j=1}^{k} \mu_j(y_j) \text{ for every } k. \quad \ldots \quad (2.12)$$

Then the quadruple $(\mathcal{B}, \{\mathcal{B}_{n]}\}, \{J_n\}, \rho)$ is a discrete time quantum stochastic process in the sense of Accardi, Frigerio and Lewis (1982). Define the (transition) operators $T_{n-1,n}$ on $\mathcal{S}$ by

$$T_{n-1,n}x = \sum_{j=1}^{d} \mu_n(N_j)\, \alpha_j(x),\; n = 1, 2, \ldots \quad \ldots \quad (2.13)$$

Then

$$\rho(J_n\, x) = \lambda(T_{0,1}\, T_{1,2} \ldots T_{n-1,n}x). \quad \ldots \quad (2.14)$$

We call the process $(\mathcal{B}, \{\mathcal{B}_{n]}\}\; \{J_n\}, \rho)$ a (inhomogeneous) *quantum random walk* with $T_{n-1,n}$ as the (one step) transition operator from time $n-1$ to n. Each transition operator $T_{n-1,n}$ is a completely positive map on $\mathcal{S}$ preserving identity, in the sense that for any state λ on $\mathcal{S}$ and finite sequences $\{x_r\}$, $\{y_r\}$, $r = 1, 2, \ldots, k$ in $\mathcal{S}$ the following inequality holds.

$$\lambda\left(\sum_{r,s} x_r^\dagger T_{n-1,n}(y_r^\dagger y_s)\, x_s\right) \geqslant 0$$

Indeed, this follows from Proposition 2.2, (2.13) and (2.14). If $\mu_j = \mu$ for all j then one obtains a homogeneous quantum random walk with transition operator T given by

$$Tx = \sum_{j=1}^{d} \mu(N_j)\, \alpha_j(x).$$

Remark 2 : Replacing W^* algebra by C^* algebra one obtains a parallel notion of a quantum random walk.

Proposition 2.3 : *Let* J_n, ρ *be as in* (2.11) *and* (2.12) *respectively and let*

$$E(n, x, \rho) = \rho(J_n\, x). \quad \ldots \quad (2.15)$$

Then the following difference equation holds

$$E(n, x, \rho)-E(n-1, x, \rho) = \sum_{k=1}^{d} \mu_n(N_k)E(n-1, \alpha_k(x)-b_k x, \rho) \quad \ldots \quad (2.16)$$

where the constants b_k *are as in* (2.3).

Proof : By definitions

$$E(n, x, \rho) = \sum_{k=1}^{d} E(n-1, \alpha_k(x), \rho)\mu_n(N_k)$$

and

$$\Sigma\, b_k\mu_n(N_k) = \mu_n(\Sigma\, b_k N_k) = 1.$$

Now (2.16) follows immediately from these two equations. □

Example 1 : Let $\boldsymbol{h}_0$, $\mathcal{H}$ be complex separable Hilbert spaces, dim $\mathcal{H} < \infty$ and let $\mathcal{S} = \mathcal{B}(\boldsymbol{h}_0)$, $\mathcal{N} = \mathcal{B}(\mathcal{H})$ where for any Hilbert space $\boldsymbol{k}$, $\mathcal{B}(\boldsymbol{k})$ denotes the algebra of all bounded operators on $\boldsymbol{k}$. Let λ be a density matrix in $\boldsymbol{h}_0$ and let $\{\phi_j\}$, $j = 1, 2, \ldots$, be a sequence of unit vectors in $\mathcal{H}$. For any unitary or antiunitary operator U on $\boldsymbol{h}_0 \otimes \mathcal{H}$ define the homomorphism J from $\mathcal{S}$ into $\mathcal{S} \otimes \mathcal{N}$ by

$$J(x) = U^{\dagger} x \otimes 1\; U.$$

Let $\mathcal{H}^{\infty} = \mathcal{H} \otimes \mathcal{H} \otimes \ldots$ be the countable tensor product defined with respect to the stabilising sequence $\{\phi_j\}$ of unit vectors. Put $\mathcal{B} = \mathcal{B}(\boldsymbol{h}_0 \otimes \mathcal{H}^{\infty})$, $\mathcal{B}_{n]} = \mathcal{B}(\boldsymbol{h}_0 \otimes \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H})$ where $\mathcal{H}$ appears n-fold in the tensor product. There exists a state ρ on $\mathcal{B}$ such that

$$\rho(x \otimes y_1 \otimes \ldots \otimes y_k) = (tr\; \lambda x) \prod_{j=1}^{k} < \phi_j, y_j\phi_j >$$

for all $x \in \mathcal{B}(\boldsymbol{h}_0)$, $y_j \in \mathcal{B}(\mathcal{H})$. The sequence $\{J_n\}$ is defined by (2.11). Thus one obtains a quantum random walk $(\mathcal{B}, \{\mathcal{B}_{n]}\}\, \{J_n\}, \rho)$.

Write

$$\boldsymbol{h}_0 \otimes \mathcal{H}^{\infty} = \boldsymbol{h}_0 \otimes \mathcal{H}_1 \otimes, \ldots, \otimes \mathcal{H}_j \otimes \ldots\ldots$$

where $\mathcal{H}_j$ denotes the j-th copy of $\mathcal{H}$. Consider the unitary isomorphism

$$\sigma_j : \boldsymbol{h}_0 \otimes \mathcal{H}_1 \otimes \ldots \mathcal{H}_j \otimes \ldots \rightarrow \boldsymbol{h}_0 \otimes \mathcal{H}_j \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_{j-1} \otimes \mathcal{H}_{j+1} \ldots$$

induced by the appropriate permutation of indices and denote

$$U_j = \sigma_j^{-1} U \otimes 1 \sigma_j,$$

where on the right hand side U operates in $h_0 \otimes \mathcal{H}_j$ and 1 is the identity operator in $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_{j-1} \otimes \mathcal{H}_{j+1} \otimes \dots$. Then U_j is called the j-th *ampliation* of U to the Hilbert space $h_0 \otimes \mathcal{H}^\infty$. Then

$$J_n(x) = W_n^\dagger x \otimes 1 \; W_n$$

where $W_n = U_n \, U_{n-1} \dots\dots U_1$. If we use this itself as the definition of J_n then the assumption that dim $\mathcal{H} < \infty$ can be dropped.

Example 2 : Let $\mathcal{S}$ be the algebra of all bounded complex valued measurable functions on $\boldsymbol{R}$ and let $\mathcal{H} = \mathcal{B}(C^2)$ the algebra of all 2×2 complex matrices. Let α, β, θ be real valued measurable functions on $\boldsymbol{R}$. For any $\phi \in \mathcal{S}$ let

$$J(\phi)(x) = \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{pmatrix} \begin{pmatrix} \phi(x+\alpha(x)) & 0 \\ 0 & \phi(x+\beta(x)) \end{pmatrix} \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix} \quad \dots \quad (2.17)$$

If λ is a probability measure on $\boldsymbol{R}$ and Φ is the unit vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $C^2 = \mathcal{H}$ we define $\mathcal{H}^\infty$ with respect to the constant sequence $\{\Phi\}$ of unit vectors. Viewing $\mathcal{S}$ as $L^\infty(\lambda)$ by going to λ-equivalence classes we see that there exists a state ρ on $\mathcal{B} = L^\infty(\lambda) \otimes \mathcal{B}(\mathcal{H}^\infty)$ satisfying

$$\rho(\phi \otimes y_1 \otimes \dots \otimes y_k) = (\int \phi d\lambda) \prod_{j=1}^{k} < \Phi, y_j \Phi >.$$

Putting $\mathcal{B}_{n]} = L^\infty(\lambda) \otimes \mathcal{B}(\mathcal{H}^{\otimes n})$ and defining $\{J_n\}$ by (2.11) one obtains a homogeneous quantum random walk $(\mathcal{B}, \mathcal{B}_{n]}, \{J_n\}, \rho)$.

If we write

$$N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, N_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, N_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \dots \quad (2.18)$$

then from (2.17) it follows that

$$\left.\begin{aligned} \alpha_1(\phi) &= \phi(x+\alpha(x)) \cos^2\theta(x) + \phi(x+\beta(x)) \sin^2\theta(x) \\ \alpha_2(\phi) &= [\phi(x+\alpha(x)) - \phi(x+\beta(x))] \sin\theta(x) \cos\theta(x) \\ \alpha_3(\phi) &= \alpha_2(\phi) \\ \alpha_4(\phi) &= \phi(x+\alpha(x)) \sin^2\theta(x) + \phi(x+\beta(x)) \cos^2\theta(x) \end{aligned}\right\} \quad \dots \quad (2.19)$$

where α_j, $j = 1, 2, 3, 4$ are the maps on $\mathcal{S}$ determined by (2.6) when applied to the homomorphism (2.17) from $\mathcal{S}$ into $\mathcal{S} \otimes \mathcal{B}(C^2)$.

3. Passage to diffusion limits

For any complex Hilbert space k let $\Gamma(k)$ denote the boson Fock space over k (Hudson and Parthasarathy, 1984 ; Meyer, 1986). defined by

$$\Gamma(k) = C \oplus k \oplus \dots\dots \oplus (\otimes^{(n)}_{Sym.} k) \oplus \dots\dots$$

where $\otimes^{(n)}_{Sym.}$ denotes n-fold symmetric tensor product. Inner products $< .,. >$ are to be understood as conjugate linear in the first and linear in the second variable. For any $u \in \boldsymbol{k}$ we denote by $\psi(u)$ the coherent vector defined by

$$\psi(u) = 1 \oplus u \oplus (2\,!)^{-1/2}\, u \otimes u \oplus \ldots\ldots \oplus (n\,!)^{-1/2} \otimes^{(n)} u \oplus \ldots\ldots$$

Let $\boldsymbol{h}$ be a fixed complex separable Hilbert space and let

$$\mathcal{H} = \Gamma(L_2(\mathcal{R}_+) \otimes \boldsymbol{h}),$$

$$\mathcal{H}[a, b] = \Gamma(L_2[a, b] \otimes \boldsymbol{h}),$$

$$\mathcal{B}[a, b] = \mathcal{B}(\mathcal{H}[a, b])$$

where $\mathcal{B}(\boldsymbol{k})$ denotes the algebra of all bounded operators on any Hilbert space $\boldsymbol{k}$ and $\boldsymbol{R}_+ = [0, \infty)$. Denote by $\theta(a, b)$ the unitary isomorphism from $L_2[0, 1] \otimes \boldsymbol{h} \to L_2[a, b] \otimes \boldsymbol{h}$ defined by

$$[\theta(a, b) f]\,(t) = (b-a)^{-1/2} f((b-a)^{-1}\,(t-a))$$

where f is looked upon as an $\boldsymbol{h}$-valued square integrable function on $[0, 1]$. Let $\Theta(a, b)$ be the second quantization of $\theta(a, b)$ defined by

$$\Theta(a, b)\,\psi(f) = \psi(\theta(a, b) f)$$

on coherent vectors and extended uniquely as a unitary isomorphism from $\mathcal{H}[0, 1]$ onto $\mathcal{H}[a, b]$. Suppose that $\mathcal{N} \subset \mathcal{B}[0, 1]$ is a finite dimensional W^* algebra of dimension d containing the identity operator. For any $N \in \mathcal{N}$ let

$$N(a, b) = \Theta(a, b)\, N\, \Theta\,(a, b)^{-1}, \quad \ldots \quad (3.1)$$

$$\mathcal{N}[a, b] = \{N(a, b),\ N \in \mathcal{N}\}. \quad \ldots \quad (3.2)$$

Choose and fix a basis $N_1, N_2, \ldots, N_d$ in the vector space $\mathcal{N}$ and fix the structure constants b_j, c^k_{ij}, e_{ij}, $1 \leqslant i, j, k \leqslant d$ for $\mathcal{N}$ according to this basis as determined by (2.3)—(2.5). For any $h > 0$, $n = 1, 2, \ldots$ let

$$\mathcal{B}^h_{n h]} = \mathcal{S} \otimes \mathcal{N}\,(0, h) \otimes \mathcal{N}\,(h, 2h) \otimes \ldots \otimes \mathcal{N}((n-1)h, nh)$$

$$\subset \mathcal{S} \otimes \mathcal{B}\,[0, nh] \subset \mathcal{S} \otimes \mathcal{B}(\mathcal{H}) \quad \ldots \quad (3.3)$$

where $\mathcal{S}$ is a fixed W^* algebra of operators on some initial Hilbert space $\boldsymbol{h}_0$. In identifying the various algebras $\mathcal{B}[a, b]$ as subalgebras of $\mathcal{B}(\mathcal{H})$ we have used the basic properties of boson Fock spaces as mentioned in Hudson and Parthasarathy, (1984) and Meyer, (1986).

Suppose that for every $h > 0$ there is given a W^* homomorphism $J^{(h)}$: $\mathcal{S} \to \mathcal{S} \otimes \mathcal{N}$ preserving identity. Let

$$J^{(h)}\,(x) = \sum_{k=1}^{d} \alpha_k(h, x) \otimes N_k \quad \ldots \quad (3.4)$$

Define homorphisms $J^{(h)}_{nh} : \mathcal{S} \to \mathcal{B}^{h}_{nh]}$, $n = 0, 1, 2, \ldots$ inductively by putting

$$J^{(h)}_{0}(x) = x,\ J^{(h)}_{h}(x) = \sum_{k=1}^{d} \alpha_k(h, x) \otimes N_k\ (0, h), \qquad \ldots \quad (3.5)$$

$$J^{(h)}_{nh}(x) = \sum_{k=1}^{d} J^{(h)}_{\overline{n-1}h}(\alpha_k(h, x)) \otimes N_k\,(\overline{n-1}h, nh). \qquad \ldots \quad (3.6)$$

It follows from the arguments in the proof of Proposition 2.2 that $J^{(h)}_{nh}$ is, indeed, an identity preserving W^* homomorphism.

For any $f \in L_2\ (\boldsymbol{R}_+) \otimes \boldsymbol{h}$ considered as an $\boldsymbol{h}$-valued square integrable function in $\boldsymbol{R}_+$ and $0 \leqslant a < b < \infty$ let $f_{[a,b]}$ denote the restriction of f to the interval $[a, b]$. Let ρ_f denote the pure state determined by the unit vector $(\exp - 1/2\|f\|^2)\psi(f)$ in $\mathcal{H}$. Let $\rho_f(a, b)$ denote the pure state determined by $(\exp - 1/2\ \|f_{[a,b]}\|^2)\ \psi(f_{[a,b]})$ in $\mathcal{H}[a, b]$. Then for any $h > 0$ we have

$$\rho_f = \rho_f(0, h) \otimes \rho_f(h, 2h) \otimes \ldots \otimes \rho_f(n-1h, nh) \otimes \ldots \qquad \ldots \quad (3.7)$$

For any initial state λ on $\mathcal{S}$ we obtain a quantum random walk in terms of the quadruple $(\mathcal{S} \otimes \mathcal{B}(\mathcal{H}), \{\mathcal{S} \otimes \mathcal{B}[0, nh]\}, J^{(h)}_{nh}, \lambda \otimes \rho_f)$ for every $h > 0$. Thus one obtains a family of random walks indexed by a small parameter h in a single algebra, namely $\mathcal{S} \otimes \mathcal{B}(\mathcal{H})$ and therefore it is natural to ask the following question : as $h \to 0$, $n \to \infty$ and $nh \to t$ do the homomorphisms $J^{(h)}_{nh}$ converge to limiting homomorphisms J_t yielding thereby a continuous time quantum stochastic process in the sense of Accardi, Frigerio and Lewis ? If, in addition, such a limiting process has the property that $\{J_t(x)\}$ obeys a quantum stochastic differential equation for every x in the sense of Hudson and Parthasarathy (1984) then one may ask whether the limit is a diffusion in the sense of Hudson (1986).

For any $f, g \in L_2(\boldsymbol{R}_+) \otimes \boldsymbol{h}$, $h > 0$, $n = 0, 1, 2, \ldots$, $x \in \mathcal{S}$ define the operators $T^{(h)}\ (nh, x, f, g)$ on the initial Hilbert space $\boldsymbol{h}_0$ by

$$\begin{aligned} &< u, T^{(h)}\,(nh, x, f, g)v > \\ &= < u \otimes \psi(f), J^{(h)}_{nh}(x)\, v \otimes \psi(g) >,\ u, v \in \boldsymbol{h}_0 \end{aligned} \qquad \ldots \quad (3.8)$$

where $J^{(h)}_{nh}\ (x)$ is determined by (3.5), (3.6).

Proposition 3.1 : *The operators* $\{T^{(h)}\ (nh, x, f, g)\}$ *obey the difference equations*

$$\begin{aligned} &T^{(h)}\,(nh, x, f, g) - T^{(h)}\,(\overline{n-1}h, x, f, g) \\ &= \sum_{k=1}^{d} \{\exp - < f_{[\overline{n-1}h, nh]}, g_{[\overline{n-1}h, nh]} > \} \\ &< \psi(f_{[\overline{n-1}h, nh]}), N_k(\overline{n-1}h, nh)\psi(g_{[\overline{n-1}h, nh]}) > \\ &\times T^{(h)}(\overline{n-1}h, \alpha_k(h, x) - b_k x, f, g). \end{aligned} \qquad \ldots \quad (3.9)$$

Proof: Let $f_{[a}$ denote the restriction of f to $[a, \infty)$. From (3.8) and (3.6) and the basic properties of coherent vectors in a boson Fock space we have

$$< u, T^{(h)} (nh, x, f, g) v >$$
$$= \sum_{k=1}^{d} < u \otimes \psi(f_{[0,\overline{n-1}h]}), J^{(h)}_{\overline{n-1}h} (\alpha_k(h, x))v \otimes \psi(g_{[0,\overline{n-1}h]}) >$$
$$\times < \psi(f_{[\overline{n-1}h,nh]}), N_k(\overline{n-1}h, nh)\psi(g_{[\overline{n-1}h,nh]})> < \psi(f_{[nh}), \psi(g_{[nh})>$$

which is equivalent to

$$T^{(h)} (nh, x, f, g) = \sum_k \{\exp - < f_{[\overline{n-1}h,nh]}, g_{[\overline{n-1}h,nh]} >\}$$
$$< \psi(f_{[\overline{n-1}h,nh]}), N_k(\overline{n-1}h, nh)\psi(g_{[\overline{n-1}h,nh]}) >$$
$$\times T^{(h)}(\overline{n-1}h, \alpha_k(h, x), f, g). \qquad \ldots (3.10)$$

By definition $T^{(h)} (nh, x, f, g)$ is linear in x. Using the relation $\sum_k b_k N_k(\overline{n-1}h, nh) = 1$ in $\mathcal{B}[\overline{n-1}h, nh]$ we obtain

$$T^{(h)}(\overline{n-1}h, x, f, g)$$
$$= \{\exp - < f_{[\overline{n-1}h,nh]}, g_{[\overline{n-1}h,nh]} >\}$$
$$< \psi(f_{[\overline{n-1}h,nh]}), \sum_k b_k N_k(\overline{n-1}h, nh)\psi(g_{[\overline{n-1}h,nh]})> T^{(h)}(\overline{n-1}h, x, f, g)$$
$$= \sum_k \{\exp - < f_{[\overline{n-1}h,nh]}, g_{[\overline{n-1}h,nh]} >\}$$
$$\times < \psi(f_{[\overline{n-1}h,nh]}), N_k(\overline{n-1}h, nh)\psi(g_{[\overline{n-1}h,nh]})> T^{(h)}(\overline{n-1}h, b_k x, f, g). \qquad \ldots (3.11)$$

Subtracting (3.11) from (3.10) we obtain (3.9). □

Remark 1 : Suppose that there exist constants $0 \leqslant \epsilon_k \leqslant 1$, $k = 1, 2, \ldots, d$ such that the limits

(1) $$\beta_k(x) = \lim_{h \to 0} h^{-\epsilon_k}(\alpha_k(h, x) - b_k x) \qquad \ldots (3.12)$$

(2) $$p_k(t, f, g) = \lim_{\substack{h \to 0 \\ nh \to t}} h^{-1+\epsilon_k} < \psi(f_{[\overline{n-1}h,nh]}), N_k(\overline{n-1}h, nh)\psi(g_{[\overline{n-1}h,nh]})> \qquad \ldots (3.13)$$

exist for every $t > 0$. Then it is significant that the difference equations (3.9) assume the form of differential equations in the limit :

$$\frac{d}{dt} T(t, x, f, g) = \sum_{k=1}^{d} p_k(t, f, g)\, T(t, \beta_k(x), f, g). \qquad \ldots (3.14)$$

For each fixed h, the maps $\{\alpha_k(h, .)\}$ obey the identities (2.7)—(2.9). These may assume a limiting form for the maps $\{\beta_k\}$. We shall see in the next section that these are the cocycle identities of Hudson (1986) at least in one important example.

The next two rather standard and elementary propositions are useful in the study of convergence of quantum random walks to diffusions. We include their proofs more for the sake of completeness than for their newness.

Proposition 3.2 : *Let X be a complex Banach space and let $\mathcal{B}(X)$ be the Banach space of all bounded operators on X. Suppose that $\beta_j \in \mathcal{B}(X)$, $1 \leqslant j \leqslant k$ and f_j, $1 \leqslant j \leqslant k$ are complex valued locally essentially bounded functions on $\boldsymbol{R}_+$. Then the ordinary differential equation*

$$\frac{dT}{dt} = T \sum_{j=1}^{k} f(t)\beta_j, \quad (T(t) \in \mathcal{B}(X),\ t \geqslant 0) \qquad \ldots \quad (3.15)$$

with initial condition $T(0) = T_0$ has a unique solution in $\mathcal{B}(X)$.

Proof : This is proved by the routine Picard's iteration procedure after defining the $\mathcal{B}(X)$-valued continuous maps $T_n(\cdot)$ through the equations

$$T_0(t) = T_0 \text{ for all } t \geqslant 0,$$

$$T_n(t) = T_0 + \int_0^t T_{n-1}(s) \sum_j f_j(s)\beta_j(s)ds,\ n = 1, 2, \ldots$$

and using the inequalities

$$\|T_n(t) - T_{n-1}(t)\| \leqslant C_a \int_0^t \|T_{n-1}(s) - T_{n-2}(s)\| ds \text{ for all } 0 \leqslant t \leqslant a$$

where

$$C_a = \left(\max_j \|\beta_j\|\right) \operatorname*{ess.\,sup}_{0 \leqslant s \leqslant a} \sum_{j=1}^{k} |f_j(s)|$$

for each fixed $a > 0$. □

Proposition 3.3 : *Let X, $\mathcal{B}(X)$ be as in Proposition 3.2 Suppose β_j, $\beta_j^{(h)} \in \mathcal{B}(X)$, $1 \leqslant j \leqslant k$, $h > 0$ are such that*

$$\lim_{h \to 0} \|\beta_j^{(h)} - \beta_j\| = 0 \textit{ for each } j.$$

Let f_j, $1 \leqslant j \leqslant k$ be continuous functions on $\boldsymbol{R}_+$ and let $f_j^{(h)}$, $1 \leqslant j \leqslant k$, $h > 0$ be locally bounded measurable functions on $\boldsymbol{R}_+$ such that

$$\lim_{h \to 0} \sup_{0 \leqslant t \leqslant a} |f_j^{(h)}(t) - f_j(t)| = 0 \text{ for every } a > 0.$$

Define operators $T^{(h)}(t)$ on X by

$$T^{(h)}(0) = T_0,$$

$$T^{(h)}(rh) = T^{(h)}(\overline{r-1}h)\,\{1 + h\sum_j f_j^{(h)}(\overline{r-1}h)\beta_j^{(h)}\},\ r = 1, 2 \ldots,$$

$$T^{(h)}(t) = T_0 \text{ if } t = 0,$$

$$= T^{(h)}(rh) \text{ if } (\overline{r-1})\,h < t \leqslant rh.$$

Let $T(t)$ be the solution of the initial value problem (3.15). Then

$$\lim_{h \to 0} \sup_{0 \leqslant t \leqslant a} \|T^{(h)}(t) - T(t)\| = 0 \text{ for all } a > 0. \qquad \ldots \quad (3.16)$$

Proof: Denote by ν_h the discrete σ-finite measure with support $\{0, h, 2h, \ldots\}$ and mass h at each point of this set.

Then $T^{(h)}(t)$ obeys the integral equation

$$T^{(h)}(t) = T_0 + \int_{[0, t)} T^{(h)}(s) \left(\sum_j f_j^{(h)}(s) \beta_j \right) \nu_h(ds), \; t \geqslant 0$$

where $[0, t)$ is the half open interval containing 0 but not t and $[0, 0)$ is the empty set. Then

$$\begin{aligned} &T^{(h)}(t) - T(t) \\ &= \int_{[0, t)} \{T^{(h)}(s) - T(s)\} \left\{ \sum_j f_j^{(h)}(s) \beta_j^{(h)} \right\} \nu_h(ds) \\ &+ \int_{[0, t)} T(s) \left\{ \sum_j f_j^{(h)}(s) \beta_j^{(h)} - \sum_j f_j(s) \beta_j \right\} \nu_h(ds) \\ &+ \int_{[0, t)} T(s) \left\{ \sum_j f_j(s) \beta_j \right\} (\nu_h(ds) - ds). \qquad \ldots \quad (3.17) \end{aligned}$$

Let

$$\varepsilon(a, h) = \sup_{0 \leqslant t \leqslant a} \|S(t)\|,$$

where $S(t)$ denotes the sum of the last two terms on the right hand side of (3.17). Put

$$w(t, h) = \|T^{(h)}(t) - T(t)\|$$

$$C(a, h) = \sup_{0 \leqslant s \leqslant a} \left\| \sum_j f_j^{(h)}(s) \beta_j^{(h)} \right\| :$$

Then (3.17) implies that for $0 \leqslant t \leqslant a$

$$w(t, h) \leqslant C(a, h) \int_{[0, t)} w(s, h) \nu_h(ds) + \varepsilon(a, h) \qquad \ldots \quad (3.18)$$

We now claim that

$$w(rh, h) \leqslant [1 + hC(a, h)]^{r-1} \varepsilon(a, h), \; r = 1, 2, \ldots \qquad \ldots \quad (3.19)$$

Indeed, since $w(0, h) = 0$, (3.18) implies (3.19) for $r = 1$. Suppose (3.19) holds for $r \leqslant n$. Then (3.18) implies

$$\begin{aligned} w(\overline{n+1}h, h) &\leqslant \left\{ h\, C(a, h) \sum_{r=1}^{n} [1 + h\, C(a, h)]^{r-1} + 1 \right\} \varepsilon(a, h) \\ &= [1 + h\, C(a, h)]^n \varepsilon(a, h). \end{aligned}$$

This proves the claim (3.19). As $h \to 0$, $C(a, h)$ remains bounded and $\epsilon(a, h) \to 0$. Hence (3.16) follows from (3.18) and (3.19). □

We now go back to the homomorphisms $J_{nh}^{(h)}$ defined by (3.5), (3.6) and put

$$J_0^{(h)}(x) = x$$

$$J_t^{(h)}(x) = J_{nh}^{(h)}(x),\ x \in \mathcal{S},\ \overline{n-1}h \leqslant t \leqslant nh,$$

$$\beta_j^{(h)}(x) = h^{-\epsilon_j}\{\alpha_j(h, x) - b_j x\}, \qquad \ldots \quad (3.20)$$

$$p_j^{(h)}(nh, f, g) = h^{-1+\epsilon_j} < \psi(f_{[\overline{n-1}h,\, nh]}),\ N_j(\overline{n-1}h, nh)\psi(g_{[\overline{n-1}h,\, nh]}) >$$
$$\times \{\exp - < f_{[\overline{n-1}h,\, nh]},\ g_{[\overline{n-1}h,\, nh]} > \},\ n = 1, 2, \ldots \qquad \ldots \quad (3.21)$$

$$p_j^{(h)}(t, f, g) = p_j^{(h)}(h, f, g),\ 0 \leqslant t \leqslant h$$
$$= p_j^{(h)}(nh, f, g),\ \overline{n-1}h < t \leqslant nh,\ n = 2, 3, \ldots \qquad \ldots \quad (3.22)$$

for $f, g \in L_2(\mathbf{R}_+) \otimes \boldsymbol{h}$.

Proposition 3.4 : *Let $f,\ g\ \in\ L_2(\mathbf{R}_+) \otimes \boldsymbol{h}$ be fixed. Suppose that there exist $\beta_j \in \mathcal{B}(\mathcal{S})$: $1 \leqslant j \leqslant k$ such that*

$$\lim_{h \to 0} \|\beta_j^{(h)} - \beta_j\| = 0 \text{ for each } j.$$

Let there exist continuous functions $p_j(t, f, g)$, $t \geqslant 0$ such that

$$\lim_{h \to 0} \sup_{0 \leqslant t \leqslant a} |p_j^{(h)}(t, f, g) - p_j(t, f, g)| = 0 \text{ for each } j,\ a > 0.$$

Define the operators $T^{(h)}(t, f, g)$ on $\mathcal{S}$ by the identity

$$< u, [T^{(h)}(t, f, g)x]v > = < u \otimes \psi(f),\ J_t^{(h)}(x)\, v \otimes \psi(g) >$$

for all $x \in \mathcal{S}$, $u, v \in \boldsymbol{h}_0$. Then there exist operators $T(t, f, g) \in \mathcal{B}(\mathcal{S})$ satisfying the following conditions :

(i) $\lim\limits_{h \to 0} \sup\limits_{0 \leqslant t \leqslant a} \|T^{(h)}(t, f, g) - T(t, f, g)\| = 0$ for each $a > 0$.

(ii) $T(0, f, g) = \exp < f, g >$, $\dfrac{dT}{dt} = \sum\limits_{j=1}^{k} p_j(t, f, g) T\beta_j$.

Proof : This is an immediate consequence of Propositions 3.1—3.3.

Corollary : *Suppose there exist W^* homomorphisms $J_t : \mathcal{S} \to \mathcal{S} \otimes \mathcal{B}_{t]}$ preserving identity and a dense domain $D \subset L_2(\mathbf{R}_+) \otimes \boldsymbol{h}$ such that the operators $T(t, f, g)$ on $\mathcal{S}$ defined by Proposition 3.4 satisfy the identity*

$$< u, [T(t, f, g)x]v > = < u \otimes \psi(f),\ J_t(x) v \otimes \psi(g) >$$

for all $x \in \mathcal{S}$, $u, v \in \boldsymbol{h}_0$, $f,\ g \in D$.

Then

$$J_t(x) = st. \lim_{h \to 0} J_t^{(h)}(x) \text{ for each } x \in \mathcal{S}, t \geq 0.$$

where st. lim denotes strong limit.

Proof: Our argument is inspired by that of Evans (1987). First of all let $x \in \mathcal{S}$ be a unitary operator on $\boldsymbol{h}$. Then $J_t^{(h)}(x)$ and $J_t(x)$ are unitary. Since the set $\{u \otimes \psi(f), u \in \boldsymbol{h}_0, f \in D\}$ is total in $\boldsymbol{h}_0 \otimes \mathcal{H}$ and by Proposition 3.4.

$$\lim_{h \to 0} < u \otimes \psi(f), J_t^{(h)}(x)\, v \otimes \psi(g) >$$

$$= < u \otimes \psi(f), J_t(x) v \otimes \psi(g) >$$

for all $u, v \in \boldsymbol{h}_0$, $f, g \in D$ it follows that $J_t^{(h)}(x)$ converges weakly to $J_t(x)$ and hence strongly as $h \to 0$. If $x \in \mathcal{S}$ is arbitrary we can write $x = \frac{1}{2}(x+x^\dagger) - \frac{1}{2}(x-x^\dagger)$ and then express any selfadjoint element $y \in \mathcal{S}$ as

$$y = \frac{1}{2}\|y\|(\exp i \cos^{-1}\|y\|^{-1} y + \exp -i \cos^{-1}\|y\|^{-1} y).$$

In other words any $x \in \mathcal{S}$ is a linear combination of four unitary elements in $\mathcal{S}$. This proves the corollary. □

4. Convergence of a spin random walk

Until now our discussions have been somewhat general in character. We shall now construct a spin random walk when the algebra $\mathcal{A}$ is isomorphic to the algebra of 2×2 complex matrices and all the conditions of Proposition 3.4 and its corollary are fulfilled. To this end we put

$$\mathcal{S} = \mathcal{B}(\boldsymbol{h}_0), \mathcal{H} = \Gamma(L_2(\boldsymbol{R}_+)),$$

$$a = \int_0^1 \Gamma(R_s) dA(s) \qquad \ldots \quad (4.1)$$

where R_s is the reflection operator in $L_2(\boldsymbol{R}_+)$ defined by

$$(R_s f)(t) = \begin{cases} -f(t) \text{ if } t \leq s, \\ f(t) \text{ if } t > s, \end{cases}$$

$\Gamma(R_s)$ is the unitary operator in $\mathcal{H}$ obtained by second quantizing R_s and the right hand side of (4.1) is the quantum stochastic integral of $\Gamma(R_s)$ with respect to the annihilation process A in the sense of Hudson and Parthasarathy (1984). By the methods of Fock space stochastic calculus in

Hudson and Parthasarathy (1984, 1986) we know that a extends to an element of $\mathcal{B}[0, 1]$ satisfying the canonical anticommutation relations

$$a^2 = a^{\dagger 2} = 0, \; aa^\dagger + a^\dagger a = 1.$$

The four elements

$$N_1 = aa^\dagger, N_2 = a, N_3 = a^\dagger, N_4 = a^\dagger a \qquad \ldots \quad (4.2)$$

constitute a vector space basis for a 4 dimensional algebra $\mathcal{N} \subset \mathcal{B}[0, 1]$ which is isomorphic to the algebra of 2×2 complex matrices through the correspondence

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to N_1, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to N_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to N_3, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to N_4 \ldots \quad (4.3)$$

extended linearly. Putting

$$\varepsilon_1 = 1, \varepsilon_2 = \frac{1}{2}, \varepsilon_3 = \frac{1}{2}, \varepsilon_4 = 0, \qquad \ldots \quad (4.4)$$

routine computations based on the methods of Fock space stochastic calculus show that for $f, g \in L_2(\boldsymbol{R}_+)$ the quantities $p_j^{(h)}(nh, f, g), j = 1, 2, 3, 4$ defined by (3.21) are now given by

$$p_2^{(h)}(nh, f, g) = h^{-1} \int_{\overline{n-1}h}^{nh} g(s) \left\{ \exp - \int_{\overline{n-1}h}^{s} + \int_{s}^{nh} \bar{f} g(\tau) d\tau \right\} ds, \quad \ldots \quad (4.5)$$

$$p_3^{(h)}(nh, f, g) = \overline{p_2^{(h)}(nh, g, f)} \qquad \ldots \quad (4.6)$$

$$p_4^{(h)}(nh, f, g) = h^{-2} \int_{\overline{n-1}h}^{nh} \int_{\overline{n-1}h}^{nh} \bar{f}(s) g(s') \exp < f_{[\overline{n-1}h, nh]}, R_{|s'-s|} g_{[\overline{n-1}h, nh]} > ds ds' \qquad \ldots \quad (4.7)$$

$$p_1^{(h)}(nh, f, g) = h p_4^{(h)}(nh, f, g) + \exp < f_{[\overline{n-1}h, nh]}, g_{[\overline{n-1}h, nh]} > \qquad \ldots \quad (4.8)$$

where in the right hand side of the third equation $f_{[a, b]}$ denotes also the function $\chi_{[a, b]} f$ in $L_2(\boldsymbol{R})$, χ denoting indicator.

Proposition 4.1 : *Let $f, g \in L_2(\boldsymbol{R}_+)$ be continuous and let*

$$p_1(t, f, g) = 1, \; p_2(t, f, g) = g(t)$$

$$p_3(t, f, g) = \bar{f}(t), \; p_4(t, f, g) = \bar{f}(t) g(t).$$

Let $p_j^{(h)}(t, f, g)$ be defined by (4.5)–(4.8) *and* (3.22). *Then*

$$\lim_{h \to 0} \sup_{0 \leq t \leq b} |p_j^{(h)}(t, f, g) - p_j(t, f, g)| = 0$$

for each $j = 1, 2, 3, 4$.

Proof : Immediate from definitions. □

Let V be a unitary operator in $\boldsymbol{h}_0 \otimes \boldsymbol{C}^2$. Define the homomorphism $J_V : \mathcal{B}(\boldsymbol{h}_0) \to \mathcal{B}(\boldsymbol{h}_0 \otimes \boldsymbol{C}^2)$ by putting

$$J_V(x) = V^\dagger x \otimes 1 \; V. \qquad \ldots \quad (4.9)$$

Since $\boldsymbol{h}_0 \otimes \boldsymbol{C}^2 = \boldsymbol{h}_0 \oplus \boldsymbol{h}_0$ we can express V as a matrix $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ with elements from $\mathcal{B}(\boldsymbol{h}_0)$ and rewrite (4.9) as

$$J_V(x) = \begin{pmatrix} P^\dagger & R^\dagger \\ Q^\dagger & S^\dagger \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} \alpha_1(x) & \alpha_2(x) \\ \alpha_3(x) & \alpha_4(x) \end{pmatrix} \quad \ldots \quad (4.10)$$

where

$$\left.\begin{aligned} \alpha_1(x) &= P^\dagger x P + R^\dagger x R, \quad \alpha_2(x) = P^\dagger x Q + R^\dagger x S \\ \alpha_3(x) &= Q^\dagger x P + S^\dagger x R, \quad \alpha_4(x) = Q^\dagger x Q + S^\dagger x S \end{aligned}\right\} \quad \ldots \quad (4.11)$$

Now let $V = V_h$ depend on a small parameter $h > 0$ and be defined by

$$V_h = \begin{pmatrix} P_h & Q_h \\ R_h & S_h \end{pmatrix} = \begin{pmatrix} (1 - hL^\dagger L)^{\frac{1}{2}} e^{-ihH} & -h^{\frac{1}{2}} L^\dagger Z e^{-ihH} \\ h^{\frac{1}{2}} L e^{-ihH} & (1 - hLL^\dagger)^{\frac{1}{2}} Z e^{-ihH} \end{pmatrix} \quad \ldots \quad (4.12)$$

where $L, Z, H \in \mathcal{B}(\boldsymbol{h}_0)$, Z is unitary and H is selfadjoint. For all sufficiently small $h > 0$, V_h is a unitary operator in $\boldsymbol{h}_0 \otimes \boldsymbol{C}^2$. We now write $\alpha_j(h, x)$ for $\alpha_j(x)$ in (4.11) when $P, Q, R, S,$ have the suffix h. Using (4.2) define the W^* homorphisms $J^{(h)} : \mathcal{B}(\boldsymbol{h}_0) \to \mathcal{B}(\boldsymbol{h}_0) \otimes \mathcal{N}$ by

$$J^{(h)}(x) = \sum_{j=1}^{4} \alpha_j(h, x) \otimes N_j. \qquad \ldots \quad (4.13)$$

These homomorphisms initiate a quantum random walk according to the scheme described in (3.4)—(3.6) where the successive steps of the random walk are taken at times $h, 2h, 3h, \ldots$ We call it a *spin random walk* induced by the parameters H, L, Z and h.

Proposition 4.2 : *Let* $\{\varepsilon_j\}, \{\alpha_j(h, \ldots)\}$ *be as in* (4.4) *and* (4.12). *Define the maps* $\beta_j^{(h)}$ *on* $\mathcal{B}(\boldsymbol{h}_0)$ *by* (3.20) *where*

$$b_1 = b_4 = 1, \; b_2 = b_3 = 0.$$

Then

$$\lim_{h \to 0} \| \beta_j^{(h)}(x) - \beta_j(x) \| = 0 \textit{ for all } x \in \mathcal{B}(\boldsymbol{h}_0)$$

where

$$\begin{aligned} \beta_1(x) &= i[H, x] - \frac{1}{2}(L^\dagger L x + x L^\dagger L - 2L^\dagger x L), \\ \beta_2(x) &= [L^\dagger, x] Z, \; \beta_3(x) = -Z^\dagger [L, x], \qquad \ldots \quad (4.14) \\ \beta_4(x) &= Z^\dagger x Z - x. \end{aligned}$$

Proof: This is immediate from the definitions in (4.11)—(4.13) and straightforward computations. □

Remark: The fact that (4.10) defines an identity preserving W^* homomorphism implies that the limit maps β_j defined in Proposition 4.2 satisfy the following identities :

$$\begin{aligned} \beta_1(xy) &= \beta_1(x)y+x\beta_1(y)+\beta_2(x)\beta_3(y) \\ \beta_2(xy) &= x\beta_2(y)+\beta_2(x)y+\beta_2(x)\beta_4(y) \\ \beta_3(x^\dagger) &= \beta_2(x)^\dagger \\ (x+\beta_4(x))(y+\beta_4(y)) &= xy+\beta_4(xy). \end{aligned} \quad \ldots \quad (4.15)$$

These are special cases of cocycle identities that appear in the cohomological approach to quantum diffusions outlined by Hudson (1986). In this context there arises the following question. Suppose $\mathcal{S}$ is a W^* algebra with identity and $\{\beta_j, j = 1, 2, 3, 4\}$ is a quadruple of operators on $\mathcal{S}$ satisfying the identities (4.15). Does there exist a family of W^* homomorphisms $J^{(h)} : \mathcal{S} \to \mathcal{S} \otimes \mathcal{B}(C^2)$ such that

$$J^{(h)}(x) = \begin{pmatrix} \alpha_1(h, x) & \alpha_2(h, x) \\ \alpha_3(h, x) & \alpha_4(h, x) \end{pmatrix},$$

$$\beta_1(x) = \lim_{h \to 0} h^{-1}(\alpha_1(h, x) - x),$$

$$\beta_j(x) = \lim_{h \to 0} h^{-1/2}\, \alpha\, (h, x),\ j = 2, 3,$$

$$\beta_4(x) = x + \lim_{h \to 0}\ \alpha_4(h, x)$$

for all $x \in \mathcal{S}$?

Proposition 4.3 : *Let L, Z, $H \in \mathcal{B}(\boldsymbol{h}_0)$ where Z is unitary and H is self-adjoint and let β_j, $j = 1, 2, 3, 4$ be operators on $\mathcal{B}(\boldsymbol{h}_0)$ defined by* (4.14). *Then there exist W^* homomorphisms $J_t : \mathcal{B}(h_0) \to \mathcal{B}(h_0) \otimes \mathcal{B}(\Gamma(L_2(\boldsymbol{R}_+)))$ such that the family of operators $\{T(t, f, g),\ t \geq 0,\ f,\ g \in L_2\ (\boldsymbol{R}_+)\}$ on $\mathcal{B}(\boldsymbol{h}_0)$ defined by*

$$< u \otimes \psi(f),\ J_t(x)v \otimes \psi(g) > \, = \, < u,\ [T(t, f, g)x]v >$$

for all $x \in \mathcal{B}(\boldsymbol{h}_0)$, u, $v \in \boldsymbol{h}_0$ obeys the ordinary differential equations

$$\frac{dT}{dt} = T(\beta_1 + g\beta_2 + \bar{f}\beta_3 + \bar{f}g\beta_4),\ T(0, f, g) = exp < f, g >$$

whenever f, g are locally bounded.

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Proof : This is essentially a restatement of the results in Section 8 of Hudson and Parthasarathy (1984). Indeed, consider the quantum stochastic differential equation

$$dU = \{LdA^{\dagger} + (Z-1)d\Lambda - L^{\dagger}Z\, dA + (iH - \frac{1}{2} L^{\dagger}L)dt\}U,$$

with initial condition $U(0) = 1$. This has a unique unitary solution. Putting $J_t(x) = U_t^{\dagger} x \otimes 1\, U_t$ we obtain the required result.

Remark : The family of operators $\{J_t(x)\}$ obeys the quantum diffusion equations :

$$d\, J_t(x) = J_t(\beta_3(x))dA^{\dagger} + J_t(\beta_4(x))d\Lambda + J_t(\beta_2(x))dA + J_t(\beta_1(x))dt.$$

Theorem 4.4 : *Let $\boldsymbol{h}_0$ be a complex separable Hilbert space and let $\mathcal{H} = \Gamma(L_2(\boldsymbol{R}_+))$ be the boson Fock space over $L_2(\boldsymbol{R}_+)$. Suppose $L, H, Z \in \mathcal{B}(\boldsymbol{h}_0)$ are such that H is selfadjoint and Z is unitary. Let $J_t : \mathcal{B}(\boldsymbol{h}_0) \to \mathcal{B}(\boldsymbol{h}_0 \otimes \mathcal{H})$ be homomorphisms defined by $J_t(x) = U_t^{\dagger} x \otimes 1\, U_t$ where $\{U_t\}\ t \geqslant 0$ is the unique unitary solution of the quantum stochastic differential equation*

$$dU = \{LdA^{\dagger} + (Z-1)d\Lambda - L^{\dagger}ZdA + (iH - \frac{1}{2} L^{\dagger}L)dt\}U$$

with the initial condition $U(0) = 1$. Suppose $\{J_{nh}^{(h)}, n = 0, 1, 2, \ldots\}$ is the spin random walk initiated by the homomorphism

$$J^{(h)}(x) = \sum_{j=1}^{4} \alpha_j(h, x) \otimes N_j,\ x \in \mathcal{B}(\boldsymbol{h}_0)$$

where $N_j, j = 1, 2, 3, 4$ are defined by (4.1) *and* (4.2) *and*

$$\begin{pmatrix} \alpha_1(h, x) & \alpha_2(h, x) \\ \alpha_3(h, x) & \alpha_4(h, x) \end{pmatrix} = \begin{pmatrix} P_h^{\dagger} & R_h^{\dagger} \\ Q_h^{\dagger} & S_h^{\dagger} \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} P_h & Q_h \\ R_h & S_h \end{pmatrix}$$

P_h, Q_h, R_h, S_h being determined by (4.12). *Define $J_0^{(h)} =$ identity, $J_t^{(h)}(x) = J_{nh}^{(h)}(x)$ if $\overline{n-1}h < t \leqslant nh$, $n = 1, 2, \ldots$. Then*

$$st. \lim_{h \to 0} J_t^{(h)}(x) = J_t(x) \text{ for each } x \in \mathcal{B}(\boldsymbol{h}_0),\ t \geqslant 0,$$

where st. lim denotes strong limit in $\mathcal{B}(\boldsymbol{h}_0)$.

Proof : This is immediate from Corollary to Proposition 3.4 and Proposition 4.1—4.3. □

We conclude with some remarks on the spin random walk when the algebra $\mathcal{B}(\boldsymbol{h}_0)$ in Theorem 4.4 is replaced by the algebra $\mathcal{S} = C_b^{\infty}(\boldsymbol{R}_+)$ of all bounded C^∞ functions on $\boldsymbol{R}_+$. It is to be noted that $\mathcal{S}$ is neither a C^* nor a W^* algebra. To initiate the random walk we follow the equations (2.17)—(2.19) in Example 2 of Section 2. Let a, b be real elements of $\mathcal{S}$ such that $ba^{-1} \in \mathcal{S}$. Define $\theta(h, x)$ by

$$\cos^2\theta(h, x) = \tfrac{1}{2}(1+h^{\frac{1}{2}}\, b(x)\, a(x)^{-1})$$

for all sufficiently small $h > 0$ and the maps $\alpha_j(h,.)$ on $\mathcal{S}$ by

$$\alpha_1(h, \phi)(x) = \phi(x+h^{\frac{1}{2}}a(x)) \cos^2\theta(h, x)+\phi(x-h^{\frac{1}{2}}\, a(x)) \sin^2\theta(h, x)$$

$$\alpha_2(h, \phi)(x) = [\phi(x+h^{\frac{1}{2}}a(x))-\phi(x-h^{\frac{1}{2}}\, a(x))] \sin\theta(h, x) \cos\theta(h, x)$$

$$\alpha_3(h, \phi) = \alpha_2(h, \phi)$$

$$\alpha_4(h, \phi)(x) = \phi(x+h^{\frac{1}{2}}\, a(x)) \sin^2\theta(h, x)+\phi(x-h^{\frac{1}{2}}\, a(x)) \cos^2\theta(h, x)$$

Then the map $J^{(h)} : \mathcal{S} \to \mathcal{S} \otimes \mathcal{n}$ defined by

$$J^{(h)}(\phi) = \sum_{j=1}^{4} \alpha_j(h, \phi) \otimes N_j$$

is a $*$ homomorphism from $\mathcal{S}$ into $\mathcal{S} \otimes \mathcal{B}[0, 1]$. We may identify $\mathcal{S}$ with the algebra of multiplication operators in $\boldsymbol{h}_0 = L_2(\boldsymbol{R})$ (by bounded C^∞ functions). The homomorphism $J^{(h)}$ initiates a spin random walk in $\mathcal{S} \otimes \mathcal{B}(\Gamma_2(\boldsymbol{R}_+))$. We have the following relations

$$\beta_1(\phi) = \lim_{h \to 0} h^{-1}[\alpha_1(h, \phi)-\phi] = \frac{1}{2}\, a^2\phi'' + b\,\phi',$$

$$\beta_2(\phi) = \beta_3(\phi) = \lim_{h \to 0} h^{-\frac{1}{2}}\, \alpha_2(h, \phi) = a\phi',$$

$$\beta_4(\phi) = \lim_{h \to 0} \alpha_4(h, \phi) = 0.$$

A heuristic argument shows that as $h \to 0$ the spin random walk described above converges to a quantum diffusion (in the sense of Hudson) $\{J_t\}$ described by the stochastic differential equations

$$d\, J_t(\phi) = J_t(\beta_1(\phi))\, dt + J_t(\beta_2(\phi))\, d(A(t)+A^\dagger(t)),$$

A being the annihilation process. Since $A+A^\dagger$ is the classical Brownian motion process we are tempted to conclude that the spin random walk described by $J^{(h)}$ converges to a classical diffusion with generator $\frac{1}{2}\, a^2 \frac{d^2}{dx^2} + b\, \frac{d}{dx}$. This remark arose from conversations with K. B. Sinha.

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References

Accardi, L., Frigerio, A., Lewis, J. T. (1982): Quantum stochastic processes. *Publ. R.I.M.S. Kyoto Univ.*, **18**, 97-133.

Accardi, L., Bach, A.: Private communication.

Berman, A., Plemmons, R. J. (1979): *Nonnegative Matrices in the Mathematical Sciences*, New York: Academic Press.

Evans, M. D. (1987): Existence of quantum diffusions. Nottingham *preprint*.

Hudson, R. L. (1986): Quantum diffusions and cohomology of algebras, University of Nottingham *preprint*,.

Hudson, R. L., Parthasarathy, K. R. (1984): Quantum Itô's formula and stochastic evolutions. *Commun. Math. Phys.*, **93**, 301-323.

——— (1986): Unification of fermion and boson stochastic calculus. *Cummun. Math. Phys.*, **104**, 457-470.

Meyer, P. A. (1986): Eléments de probabilités quantiques. in Séminaire de probabilités XX 1984/85, 186-312, *Lecture Notes in Mathematics* **1204**, Berlin, Heidelberg, New York.

Parthasarathy, K. R. (1987): The passage from random walk to diffusion in quantum probability. University of Rome II *preprint* (To appear in the Celeboration Volume on Applied Probability, Ed. J. Gani)

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