# SIMULTANEOUS REDUCTION OF SEVERAL HERMITIAN FORMS

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*SUMMARY.* In this paper the author obtains in several cases necessary and sufficient conditions for simultaneous reduction of several hermitian forms to diagonal forms by a single non-singular linear transformation. A characterisation of a semi-simple matrix with real eigen values is obtained. It is also shown that several semisimple matrices commute pairwise if and ony if they can be expressed as polynomials in a common semi-simple matrix.

## 1. Introduction

Simultaneous reduction of two hermitian forms one of which is positive definite to diagonal forms by means of a nonsingular linear transformation is well-known (See Rao, 1965). In an interesting paper Mitra and Rao (1968) considered the problem of simultanous reduction of a pair of hermitian forms. They obtained in several classified cases neat necessary and sufficient conditions for simultaneous reduction of a pair of hermitian forms to diagonal forms by means of cogredient and contragredient transformations. They also gave a necessary and sufficient condition for several hermitian forms to be reduced simultaneously to diagonal forms by a single unitary transformation.

In this paper we obtain in several cases necessary and sufficient conditions for simultaneous reduction of several hermitian forms to diagonal forms by a single non-singular linear transformation. These are obtained in Sections 3 and 4. In Section 2, we give a characterisation of semisimple matrices with real eigen values and show that several semisimple matrices commute pairwise if and only if they can be expressed as polynomials in a common semisimple matrix.

We follow the same notations as in Rao and Mitra (1971).

## 2. Some preliminary theorems

In this section we prove two theorems in linear algebra which are also of independent interest.

Theorem 1 : *Let $A_1, A_2, \ldots, A_k$ be semisimple matrices of the same order. Then $A_1, A_2, \ldots, A_k$ commute pairwise if and only if they can be expressed as polynomials in a common semisimple matix.*

*Proof :* Proof of 'if' part is trivial. To prove the 'only if' part we proceed follows.

Since $A_i$, $i = 1, 2, \ldots, k$ are semisimple matrices it follows that (See Theorem 9.5 of Perlis, 1952)

$$A_i = a_{i1} E_{i1} + \ldots + a_{it_i} E_{it_i}, \quad i = 1, 2, \ldots, k$$

where

$$\left.\begin{array}{ll} \text{(i)} & E_{ij} \text{ is idempotent } \forall i, j \\ \text{(ii)} & E_{ij}E_{ij'} = 0 \quad \text{if } j \neq j' \forall i \end{array}\right\} \qquad \ldots \quad (2.1)$$

and (iii) $\sum_{j=1}^{t_i} E_{ij} = I \, \forall i$

Further, since $A_1, \ldots, A_k$ commute pairwise it follows that (again see Theorem 9.5 of Perlis, 1952)

$$E_{ij}E_{lm} = E_{lm}E_{ij} \vee i, j, l, m. \qquad \ldots \quad (2.2)$$

Define
$$B = \sum_{i_1, \ldots, i_k} c_{i_1, i_2, \ldots, i_k} E_{1i_1} E_{2i_2} \ldots E_{ki_k}$$

where the $l_1 \ldots l_k$ complex numbers $c_{i_1, i_2, \ldots, i_k}$ are all distinct.

First observe that

$$(E_{1i_1} E_{2i_2} \ldots E_{ki_k})(E_{1j_1} \ldots E_{kj_k}) = 0 \text{ if } i_l \neq j_l \text{ for some } l.$$

This clearly follows from (2.1) and (2.2). Again from (2.1) and (2.2) if follows that

$$(E_{1i_1} \ldots E_{ki_k}).(E_{1i_1} \ldots E_{ki_k}) = E_{1i_1} \ldots E_{ki_k}$$

and
$$\sum_{i_1, \ldots, i_k} E_{1i_1} E_{2i_2} \ldots E_{ki_k} = I.$$

Now, an appeal to Theorem 9.5 of Perlis (1952) yields that $B$ is semisimple.

Further observe that $\sum_{i_2, \ldots, i_k} E_{2i_2} \ldots E_{ki_k} = I$ and also sum of all such similar products by omitting for example the principal idempotent of $A_l$ is the identity matrix.

Let $p(\cdot)$ be a polynomial function with complex coefficients. From (2.1) and (2.2) it follows that

$$p(B) = \sum_{i_1, \ldots, i_k} p(c_{i_1, i_2, \ldots, i_k}) E_{1i_1} E_{2i_2} \ldots E_{ki_k}$$

Now there exist polynomials $p_m, m = 1, 2, \ldots, k$ such that

$$p_m(c_{i_1, i_2, \ldots, i_k}) = a_{mi_m} \text{ for all } m, i_1, i_2, \ldots, i_k.$$

[Lagrange's method can be used for constructing such polynomials.]

Hence

$$p_m(B) = \sum_{i_1, \ldots, i_k} p_m(c_{i_1, \ldots, i_k}) E_{1i_1} \ldots E_{ki_k}$$

$$= \Big(\sum_{i_j \neq i_m} E_{1i_1} \ldots E_{ki_k}\Big)\Big(\sum_{i_m = 1} a_{mi_m} E_{mi_m}\Big)$$

$$= A_m \quad \text{for} \quad m = 1, 2, \ldots, k.$$

This completes the proof of Theorem 1.

Corollary : *If $A_1, A_2, \ldots, A_k$ are semisimple matrices which commute pairwise then there exists a nonsingular matrix $T$ such that $TA_iT^{-1}$ is diagonal for $i = 1, 2, \ldots, k$.*

We now obtain a characterisation of semisimple matrices with real eigen values. In later sections we come across conditions involving semisimple matrices with real eigen values and thus this theorem may not be quite out of place here. We prove

Theorem 2 : *$A$ is semisimple with real eigen values if and only if there exists a positive definite matrix $M$ such that $MAM^{-1} = A^*$.*

*Proof :*

*Proof of 'if' part :* Let $M$ be a positive definite matrix such that $MAM^{-1} = A^*$. Since $M$ is positive definite $M = BB^*$ where $B$ is a nonsingular matrix. Thus

$$BB^*AB^{*-1}B^{-1} = A^* \implies B^*AB^{*-1} = B^{-1}A^*B$$

$\implies B^*AB^{*-1}$ is hermitian $\implies B^*AB^{*-1}$ is semisimple with real eigen values.

This completes the proof of 'if' part.

*Proof of 'only if' part :* If $A$ is semisimple with real eigen values then there exists a nonsingular matrix $B$ such that $BAB^{-1} = D$ where $D$ is a real diagonal matrix. Hence $BAB^{-1} = D = D^* = B^{-1*}A^*B^*$. This in turn implies that $B^*BAB^{-1}B^{*-1} = A^*$. Clearly $B^*B$ is positive definite. This completes the proof of 'only if' part.

The following interesting result of Mitra (1968) follows immediately as an idempotent matrix is indeed a semisimple matrix with real eigen values.

Corollary (Mitra, 1968) : *If $A$ is an idempoent matrix, then there exists a positive definite matrix $M$ such that $MAM^{-1} = A^*$.*

We state below a theorem given in Mitra and Rao (1968), for completeness.

Theorem 3 : *Let $A_1, A_2, \ldots, A_k$ be hermitian matrices of the same order. Then there exists a unitary matrix $T$ such that $TA_iT^*$ is diagonal for $i = 1, 2, \ldots, k$ if and only if $A_1, \ldots, A_k$ commute pairwise.*

## 3. Simultaneous reduction when one of the matrices is nonsingular

We prove

Theorem 4 : *Let $A_1, A_2, \ldots, A_k$ be hermitian matrices of the same order and let $A_1$ be nonsingular. Then there exists a nonsingular matrix $T$ such that $TA_i\,T^*$ is is diagonal, $i = 1, 2, \ldots, k$ if and only if*

(a) *$A_iA_1^{-1}$ is semisimple with real eigen values for $i = 1, \ldots, k$*

*and* (b) *$A_iA_1^{-1}A_j = A_jA_1^{-1}A_i$ for all $i$ and $j$.*

*Proof :* Proof of 'only if' part is trivial. To prove the 'if' part we proceed as follows.

$$\forall\, i, j,\ A_iA_1^{-1}A_j = A_jA_1^{-1}A_i \implies A_1A_1^{-1}, A_2A_1^{-1}, \ldots, A_kA_1^{-1} \text{ commute pairwise.}$$

Hence (a) and (b), in view of Theorem 1 and the corollary after Theorem 1, imply that there exists a nonsingular matrix $M$ such that for each $i$, $MA_iA_1^{-1}M^{-1} = D_i$ where $D_i$ is a diagonal matrix with real elements. Now, $\forall i$, $MA_iA_1^{-1}M^{-1} = D_i \implies \forall\, i$, $MA_iM^* = D_iMA_1M^* \implies D_i$ and $MA_1M^*$ commute for all $i$.

Also observe that $D_1, \ldots, D_k$, being diagonal matrices, commute pairwise.

Hence by Theorem 3, there exists a unitary matrix $L$ such that $LMA_1M^*L^*$ and $L\,D_iL^*$ are diagonal for all $i$.

Thus,

$$LMA_iM^*L^* = LD_iL^*LMA_1M^*L^* \text{ is diagonal for } i = 1, 2, \ldots, k.$$

Put $T = LM$ and observe that $T$ is nonsingular and $TA_iT^*$ is diagonal for all $i$.

This completes the proof of Theorem 4.

Corollary : *Let $A_1, A_2, \ldots, A_k$ be hermitian matrices of the same order and let $A_1$ be positive definite. Then there exists a nonsingular matrix $T$ such that $TA_iT^*$ is diagonal for all $i$ if and only if $A_iA_1^{-1}A_j = A_jA_1^{-1}A_i$ for all $i, j$*

*Proof :* Corollary follows trivially from Theorem 4 once it is observed that $A_iA_1^{-1}$ is semisimple with real eigen values (being similar to $A_i$) if $A_i$ is hermitian and $A_1$ is positive definite.

We now state a theorem analogous to the theorem corresponding to simultaneous reduction of a pair of hermitian matrices by contragredient transformations (See Mitra and Rao, 1968). The proof of this theorem follows on similar lines to that of Theorem 4 and is omitted.

Theorem 5 : *Let $A_1, A_2, \ldots, A_k$ be hermitian matrices of the same order and let $A_1$ be nonsingular. Then there exists a nonsingular matrix $T$ such that $TA_1T^*$ and $(T^*)^{-1}A_iT^{-1}$ are diagonal for each $i \geqslant 2$, if and only if*

(a) $A_iA_1$ is semisimple with real eigen values for $i = 2, \ldots, k$; and

(b) $A_iA_1A_j = A_jA_1A_i$ for $i, j, = 2, \ldots, k$.

## 4. Simultaneous reduction of several arbitrary hermitian forms

We prove

Theorem 6 : *Let $A_1, A_2, \ldots, A_k$ be hermitian matrices of order $n \times n$ such that $\mathcal{M}(A_i) \subset \mathcal{M}(A_1)$ for $i = 2, \ldots, k$. Then there exists a nonsingular matrix $T$ such that $TA_iT^*$ is diagonal for all $i$ if and only if*

(a) *$A_iA_1^-$ is semisimple with real eigen values for all $i$,*

*and* (b) *$A_iA_1^-A_j = A_jA_1^-A_i$ for all $i, j$*

*where $A_1^-$ is some g-inverse of $A_1$.*

*Proof :*

*Proof of 'if' part :* (a) and (b), in view of Theorem 1 and the corollary after Theorem 1, imply that there exists a nonsingular matrix $M$ such that $MA_iA_1^-M^{-1} = D_i$ where $D_i$ is a real diagonal matrix.

$\mathcal{M}(A_i) \subset \mathcal{M}(A_1)$ and $MA_iA_1^-M^{-1} = D_i \Longrightarrow MA_iM^* = D_iMA_1M^*$
$\Longrightarrow D_1, \ldots, D_k$ and $MA_1M^*$ commute pairwise (since $D_1, \ldots, D_k$ are diagonal they commute pairwise).

The rest of the proof of 'if' part follows on the same lines as that of Theorem 4.

*Proof of 'only if' part :* Let $R(A_1) = r$ and without loss of generality let $TA_1T^* = \begin{pmatrix} \Delta_1 & 0 \\ 0 & 0 \end{pmatrix}$ where $\Delta_1$ is a nonsingular diagonal matrix of order $r \times r$.

Partition $T^* = (T_1^* : T_2^*)$ where $T_1^*$ is of order $n \times r$. Observe that $A_1 T_2^* = 0$ which in turn implies that $A_i T_2^* = 0$ for $i = 2, \ldots, k$. (This is so because $\mathcal{M}(A_i) \subset \mathcal{M}(A_1)$.)

Hence $TA_iT^* = \begin{pmatrix} \Delta_i & 0 \\ 0 & 0 \end{pmatrix}$ where $\Delta_i$ is a diagonal matrix of order $r \times r$.

Clearly, $A_1^- = T^* \begin{pmatrix} \Delta_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} T$ is a g-inverse of $A_1$.

Now,

$$A_i A_1^- = T^{-1} \begin{pmatrix} \Delta_i & 0 \\ 0 & 0 \end{pmatrix} T^{*-1} T^* \begin{pmatrix} \Delta_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} T$$
$$= T^{-1} \begin{pmatrix} \Delta_i \Delta_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} T$$

which is clearly semisimple with real eigen values.

It is easy to check that $A_i A_1^- A_j = A_j A_1^- A_i$ for all $i, j$.

*Note* : Theorem 4 can be deduced from Theorem 6.

We state below a few interesting corollaries which are easy to prove.

Corollary 1 : *Let $A_1, A_2, \ldots, A_k$ be hermitian matrices such that $A = \sum_{i=1}^{k} A_i$ is nonnegative definite and $\mathcal{M}(A_i) \subset \mathcal{M}(A)$ for $i = 1, 2, \ldots, k$. Then there exists a nonsingular matrix $T$ such that $TA_iT^*$ is diagonal for $i = 1, 2, \ldots, k$ if and only if $A_i A^- A_j = A_j A^- A_i$ for all $i, j$ where $A^-$ is any g-inverse of $A$.*

Corollary 2 : *Let $A_1, A_2, \ldots, A_k$ be n.n.d. (nonnegative definite) matrices of the same order. Then there exists a nonsingular matix $T$ such that $TA_iT^*$ is diagonal for each $i$ if and only if $A_i A^- A_j = A_j A^- A_i$ for all $i, j$ where $A = \sum_{i=1}^{k} A_i$ and $A^-$ is any g-inverse of $A$.*

We now give a sufficient condition for hermitian matrices $A_1, A_2, \ldots, A_k$ to be reduced simultaneously diagonal forms by a nonsingular linear transformation. We prove

Theorem 7 : *Let $A_1, A_2, \ldots, A_k$ be hermitian matrices of the same order. Then there exists a nonsingular matrix $T$ such that $TA_iT^*$ is diagonal if*

(a) $R(NA) = R(NAN^*) = R(\sum_{i=1}^{k} NA_i)$

*where* $A = \sum_{i=1}^{k} A_i$ *and* $N^* = A_1^-$,

(b) $\Gamma_i A_1^-$ *is semisimple with real eigen values*
*where* $\Gamma_i = A_i - AN^*(NAN^*)^- NA_i$ *and* $A_1^-$ *is some g-inverse of* $A_1$.

(c) $\Gamma_i A_1^- \Gamma_j = \Gamma_j A_1^- \Gamma_i$ *for* $i, j = 1, \ldots, k$,

(d) $NA_iN^*(NAN^*)^-$ *is semisimple with real eigen values*

*and* (e) $NA_iN^*(NAN^*)^- NA_jN^* = NA_jN^*(NAN^*)^- NA_iN^*$ *for* $i, j = 2, \ldots, k$.

*Proof:* First observe $\Gamma_i N^* = 0$ for $i = 1, 2, \ldots, k$. Hence $\Gamma_i = J_i A_1$ and $\Gamma_i A_1^- = \Gamma_i A_{1r}^-$ where $J_i$ is some matrix and $A_{1r}^-$ is a reflexive hermitian g-inverse of $A_1$, for $i = 1, 2, \ldots, k$. Let $A_1 = CDC^*$ where $C^* C = I_r$ and $D$ is a nonsingular diagonal matrix with real diagonal elements. Let $A_{1r}^- = Y\Lambda Y^*$ where $Y^* Y = I_r$ and $\Lambda$ is a nonsingular diagonal matrix. Then $A_{1r}^- = ZD^{-1}Z^*$ where $Z = Y(C^*Y)^{-1}$. Clearly $C^*Z = I_r$. Now consider

$$S = \begin{pmatrix} Z^* - GN \\ N \end{pmatrix}$$

where
$$G = Z^*AN^*(NAN^*)^-.$$

Observe that $(Z^* - GN)A_i = Z^*\Gamma_i, \quad i = 1, 2, \ldots, k$.

Thus
$$SA_iS^* = \begin{pmatrix} E_i & 0 \\ 0 & NA_iN^* \end{pmatrix}$$

where $E_i = Z^* \Gamma_i Z$ for $i = 2, \ldots, k$

and
$$SA_1S^* = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.$$

Further for each $i$, $E_iD^{-1}$ is semisimple with real eigen values as $\Gamma_i A_1^-$ is semisimple with real eigen values. Also $\Gamma_i A_1^- \Gamma_j = \Gamma_j A_1^- \Gamma_i$ for all $i, j \Longrightarrow E_iD^{-1}E_j = E_jD^{-1}E_i$ for all $i, j$. Hence by Theorem 4 it follows that there exists a nonsingular matrix $L$ such that $LEL^*$ is diagonal for $i = 2, \ldots, k$ and $LDL^*$ is diagonal. Further (a), (d) and (e) together with Theorem 6 imply that there exists a nonsingular matrix $M$ such that $MNA_iN^*M^*$ is diagonal for $i = 2, \ldots, k$.

Let $T = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix} S$. Then clearly $TA_iT^*$ is diagonal for $i = 1, 2, \ldots, k$.

Further $T$ is nonsingular as $L, M$ and $S$ are nonsingular.

This completes the proof of Theorem 7.

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