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BOUNDS ON MOMENTS OF CERTAIN RANDOM VARIABLES

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1. Summary. Let $\{X_n, n \geqq 1\}$ be a sequence of random variables and let $S_n = \sum_{i=1}^n X_i$. Under the condition that $\{S_n\}$ forms a martingale sequence, it was shown in [2] that, for $\nu \geqq 2$,

$$E(|S_n)^\nu) \leqq C_\nu n^{(\nu/2)-1} \sum_{i=1}^n E|X_i|^\nu,$$

where

$$C_\nu = [8(\nu - 1) \max (1, 2^{\nu-3})]^\nu. \tag{1}$$

The purpose of this paper is to show that the constant C_ν can be replaced by a much smaller constant in the following two cases: (i) ν is an even integer and the martingale dependence condition is replaced by one which is more explicit in terms of moments (Theorem 1); (ii) the X_n's are independent with zero means (Theorem 2). For case (i) we give for $E(|S_n|^\nu)$ a bound which is a polynomial in n. This last bound does not appear to be too exhorbitant because, as shown by an example, it is not valid for all martingales $\{S_n\}$.

2. The results. We first prove the following

THEOREM 1. *Suppose that for every integer $p \geqq 1$ and for every choice of positive integers $i_1, \cdots, i_p$, $k_1, \cdots, k_p$, the condition* min $(k_1, \cdots, k_p) = 1 \Rightarrow$ *$E(X_{i_1}^{k_1} \cdots X_{i_p}^{k_p})$, if it exists, equals zero. Then, for $m = 1, 2, \cdots$,*

$$E(|S_n|^{2m}) \leqq D_{2m} n^{m-1} \sum_{i=1}^n E|X_i|^{2m},$$

where

$$D_{2m} = \sum_{p=1}^m p^{2m-1}/(p - 1)!. \tag{2}$$

PROOF. To make the writing simpler we write $\gamma_{\nu,n} = E|X_n|^\nu$ and $\beta_{\nu,n} = \sum_{i=1}^n \gamma_{\nu,i}/n$. Keep n and m fixed. The result holds if $\beta_{2m,n} = \infty$.

Suppose therefore that $\beta_{2m,n} < \infty$. For $1 \leqq p \leqq 2m$, let A_p denote the set of all p-tuples $\mathbf{k} = (k_1, \cdots, k_p)$ such that the k's are positive integers satisfying $(k_1 + \cdots + k_p) = 2m$. Let

$$T(i_1, \cdots, i_p) = \sum (2m)!/(k_1! \cdots k_p!)E(X_{i_1}^{k_1} \cdots X_{i_p}^{k_p})$$

where the summation is over $\mathbf{k} \, \varepsilon \, A_p$. Then

$$E(|S_n|^{2m}) = E(S_n^{\ 2m}) = \sum_{p=1}^{2m} \sum{}' T(i_1, \cdots, i_p), \tag{3}$$

where $\sum'$ denotes summation over the region $1 \leqq i_1 < \cdots < i_p \leqq n$. If $p > m$ and $\mathbf{k} \, \varepsilon \, A_p$ then min $(k_1, \cdots, k_p) = 1$. Thus $p > m \Rightarrow > T(i_1, \cdots, i_p) = 0$.

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Moreover by Hölder's inequality

$$|E(X_{i_1}^{k_1} \cdots X_{i_p}^{k_p})| \leqq \gamma_{2m,i_1}^{k_1/2m} \cdots \gamma_{2m,i_p}^{k_p/2m} .$$

Therefore

$$\begin{aligned} |T(i_1, \cdots, i_p)| &\leqq (\gamma_{2m,i_1}^{1/2m} + \cdots + \gamma_{2m,i_p}^{1/2m})^{2m} \\ &\leqq p^{2m-1}(\gamma_{2m,i_1} + \cdots + \gamma_{2m,i_p}). \end{aligned}$$

Thus, from (3),

$$\begin{aligned} E(|S_n|^{2m}) &\leqq \sum_{p=1}^{m} p^{2m-1} \sum{}' (\gamma_{2m,i_1} + \cdots + \gamma_{2m,i_p}) \\ &= \sum_{p=1}^{m} p^{2m-1} \binom{n-1}{p-1} \sum_{j=1}^{n} \gamma_{2m,j} \\ &\leqq \sum_{p=1}^{m} p^{2m-1} n^{p-1}/(p-1)! \cdot n\beta_{2m,n} \\ &\leqq n^m \beta_{2m,n} D_{2m}. \end{aligned}$$

This completes the proof of the theorem.

REMARK 1. Berman [1] calls a sequence $\{X_n\}$ of random variables *sign-invariant* if, for every n and for every choice of $\epsilon_1, \cdots, \epsilon_n$ each equal to $+1$ or -1, the joint distribution of $\epsilon_1 X_1, \cdots, \epsilon_r X_n$ is the same as that of $X_1, \cdots, X_n$. It is easy to see that the moment condition of the above theorem is satisfied if $\{X_n\}$ is sign-invariant. It also holds if the X_n's are independent with zero means.

THEOREM 2. *Suppose that the X_n's are independent random variables with zero means. Then, for $\nu \geqq 2$,*

$$E(|S_n|^\nu) \leqq F_\nu n^{\nu/2-1} \sum_{i=1}^{n} E|X_i|^\nu,$$

where

$$F_\nu = \tfrac{1}{2}\nu(\nu-1) \max(1, 2^{\nu-3})[1 + 2\nu^{-1} D_{2m}^{(\nu-2)/2m}],$$

where the integer m satisfies $2m \leqq \nu < 2m + 2$ and the constant D_{2m} is given by (2).

PROOF. We will use the notation introduced at the beginning of the proof of Theorem 1. Keep ν and n fixed. Again the result holds if $\beta_{\nu,n} = \infty$. Suppose therefore that $\beta_{\nu,n} < \infty$.

Let $\Delta_n(\nu) = E(|S_n|^\nu - |S_{n-1}|^\nu)$. Then from the relation (3.5) of [2], we get

$$\Delta_n(\nu) \leqq \tfrac{1}{2}\nu\delta_\nu[\gamma_{2,n} E(|S_{n-1}|^{\nu-2}) + \gamma_{\nu,n}], \tag{4}$$

where $\delta_\nu = (\nu - 1)\max(1, 2^{\nu-3})$. Now $\nu - 2 < 2m$. Therefore, using Theorem 1, we get

$$\begin{aligned} E(|S_{n-1}|^{\nu-2}) &\leqq [E(|S_{n-1}|^{2m})]^{(\nu-2)/2m} \\ &\leqq D_{2m}^{(\nu-2)/2m}(n-1)^{(\nu-2)/2} \beta_{2m,n-1}^{(\nu-2)/2m}. \end{aligned} \tag{5}$$

Using the inequality $\beta_{2m,n-1} \leqq \beta_{\nu,n-1}^{2m/\nu}$ in (5) and the inequality $\gamma_{2,n} \leqq \gamma_{\nu,n}^{2/\nu}$ in (4), we obtain

$$\Delta_n(\nu) = \tfrac{1}{2}\nu\delta_\nu[D_{2m}^{(\nu-2)/2m}(n-1)^{(\nu-2)/2} \beta_{\nu,n-1}^{(\nu-2)/\nu} \gamma_{\nu,n}^{2/\nu} + \gamma_{\nu,n}]. \tag{6}$$

From the corollary to Lemma 2 of [2], we have

$$\sum_{j=1}^{n} (j-1)^{(\nu-2)/2} \beta_{\nu,j-1}^{(\nu-2)/\nu} \gamma_{\nu,j}^{2/\nu} \leqq 2\nu^{-1} n^{\nu/2} \beta_{\nu,n} . \tag{7}$$

Finally, inequalities (6) and (7) yield

$$\begin{aligned} E(|S_n|^\nu) &= \sum_{j=1}^{n} \Delta_j(\nu) \\ &\leqq \tfrac{1}{2}\nu\delta_\nu[D_{2m}^{(\nu-2)/2m} 2\nu^{-1} n^{\nu/2} \beta_{\nu,n} + n\beta_{\nu,n}] \\ &\leqq n^{\nu/2}\beta_{\nu,n} F_\nu . \end{aligned}$$

This completes the proof of the theorem.

REMARK 2. Suppose that the X_n's are independent with zero means. Then $\{S_n\}$ is a martingale and the Theorem of [2] is applicable. However, Theorem 2 above also applies and gives a better bound. If, moreover, ν is an even integer, then Theorem 1 gives a still better bound.

REMARK 3. Let $\{X_n, n \geqq 1\}$ be an exchangeable process with $E(X_1X_2) = 0$. Then the proof of Theorem 2 breaks down. However, the conclusion is valid because of the de Finetti theorem (see Section 4 of [2]).

REMARK 4. Let $\beta'_{\nu,n} = \max\{E|X_i|^\nu, 1 \leqq i \leqq n\}$. If the moment condition of Theorem 1 holds then minor modifications of the proof of that theorem show that

$$E(|S_n|^{2m}) \leqq \beta'_{2m,n} \sum_{p=1}^{n} \binom{n}{p} \sum{}'' (2m)!/(k_1! \cdots k_p!), \tag{8}$$

where $\sum''$ denotes summation over the region $k_i \geqq 2$ and $k_1 + \cdots + k_p = 2m$. Thus

$$\begin{aligned} E(|S_n|^{2m}) \leqq \beta'_{2m,n}[&(2m)!/(2^m \cdot m!)n^m \\ &+ (2m)!(m-5)/(9\cdot 2^m \cdot (m-2)!)n^{m-1} + o(n^{m-1})] \end{aligned}$$

Note that the leading term has coefficient $(2m)!/(2^m \cdot m!)$, which is natural in view of the central limit theorem.

REMARK 5. Suppose the X_n's satisfy the moment condition of Theorem 1 and in addition are identically distributed. Then the bound (8) is better than that given by Theorem 1.

REMARK 6. The bound (8) does not appear to be exorbitant in that it is not valid for all martingales, as seen from the following example. Let the basic probability space be $\{1, \cdots, 6\}$ with the points 1, 2, 5 and 6 getting the mass $\frac{1}{8}$ each and the points 3 and 4 getting the mass $\frac{1}{4}$ each We define three random variables X_1, X_2 and X_3 on the space with values given in the following table.

Random Variable	Point 1	2	3	4	5	6
X_1	1	1	1	-1	-1	-1
X_2	1	1	-1	1	-1	-1
X_3	$2^{\frac{1}{2}}$	$-2^{\frac{1}{2}}$	0	0	$2^{\frac{1}{2}}$	$-2^{\frac{1}{2}}$

The sequence S_1, S_2, S_3 of partial sums forms a martingale. Further $ES_3^4 = 9 + 12 \cdot 2^{1/2}$ which exceeds the bound 21, given by (8).

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