A CHARACTERISATION OF MOORE-PENROSE INVERSE AND RELATED RESULTS

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SUMMARY. In this paper we prove the following results.

1(a) $G = A^+$ if $G = A_r^-$ and $[GG^{\#}]^r = [(A^{\#}A)^r]^-$ for some positive integer $r > 2$.

(b) $G = A^+$ if $G = A_r^-$ and $G^{\#}(GG^{\#})^r = [(A^{\#}A)^r A^{\#}]^-$ for some positive integer r.

2. $G = A_l^-$ and $G = A_m^-$ if and only if $G = A^-$ and $GG^{\#} = (A^{\#}A)_l^-$.

3. $G = A^+$ if and only if $G = A^-$ and $GG^{\#} = (A^{\#}A)^+$.

4. If $GG^{\#} = (A^{\#}A)_m^-$ and $G^{\#}(GG^{\#})^r = [(A^{\#}A)^r A^{\#}]^-$ then $G = A_l^-$ and $G = A_m^-$.

5. Let A be a matrix such that $R(A) = R(A^2)$. Let Jordan form of A be $A = L\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}L^{-1}$ where C is nonsingular. Then G is a g-inverse of A with power property if and only if $G = L\begin{pmatrix} C^{-1} & J \\ F & FCJ \end{pmatrix}L^{-1}$ where J and F are arbitrary subject to the condition $JF = 0$. [$A^{\#}$ is defined as follows: $(Ax, y)_m = (x, A^{\#}y)_n$ for all $x \in E^n$ and $y \in E^m$ where $(., .)_r$ is a valid inner product in E^r].

1. Notations

We use the following notations. The vector space of n-tuples over the field of complex numbers is denoted by $\mathcal{E}^n$. Matrices are denoted by bold face capital letters such as A, B, C, G, H etc. Throughout the paper we consider matrices over the field of complex numbers. 0 denotes a null matrix. A^*, $\mathcal{M}(A)$ and $R(A)$ denote respectively the complex conjugate transpose, column space and rank of a matrix A. If A is square $|A|$ denotes the determinant of A. Let A be a matrix of order $m \times n$. $A^{\#}$ denotes the adjoint of A. The adjoint matrix is defined by the condition $(Ax, y)_m = (x, A^{\#} y)_n$ for all $x \in \mathcal{E}^n$ and $y \in \mathcal{E}^m$ where $(\; , \;)_m$ and $(\; , \;)_n$ are valid inner products in $\mathcal{E}^m$ and $\mathcal{E}^n$ respectively. If $(x, y)_m = y^* M x$ and $(x, y)_n = y^* N x$, then $A^{\#} = N^{-1}A^*M$.

Various g-inverses considered in this paper are described in the following table :

	symbol	conditions
g-inverse	A^-	$AGA = A$
Reflexive g-inverse	A_r^-	$AGA = A, \; GAG = G$
Least square g-inverse	A_l^-	$AGA = A, \; (AG)^{\#} = AG$
minimum norm g-inverse	A_m^-	$AGA = A, \; (GA)^{\#} = GA$
Minimum norm least squares g-inverse (Moore-Penrose inverse)	A^+	$AGA = A, \; GAG = G,$ $(AG)^{\#} = AG, \; (GA)^{\#} = GA$

A matrix which is both a minimum norm g-inverse and a least squares g-inverse of A is denoted by A^-_{lm}. We sometimes use the notation $A^-_{l(M)}$, $A^-_{m(N)}$ and A^+_{MN} to indicate specifically the various norms that are involved.

2. A characterisation of Moore Penrose inverse

We prove

Theorem 2.1 : *$G = A^+$ iff either one of the following equivalent conditions hold*

(i) $G = A^-_r$ *and* $(GG^*)^\nu = [(A^*A)^\nu]^-$ *for some positive integer* $\nu \geqslant 2$.

(ii) $G = A^-_r$ *and* $G^*(GG^*)^\nu = [(A^*A)^\nu A^*]^-$ *for some positive integer* $\nu \geqslant 1$.

Proof : The "only if" part is trivial and indeed too modest. For the 'if' part consider a general singular value decomposition of A (see Rao and Mitra, 1970)

$$A = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^* \qquad \ldots \quad (2.1)$$

where U and V are matrices of order $m \times m$ and $n \times n$ respectively such that $U^*MU = I$ and $V^*N^{-1}V = I$ and D is a diagonal matrix of order $r \times r$ with diagonal elements as the positive square roots of the nonnull eigen values of A^*A where $r = R(A)$.

Now,

$$G = A^-_r \iff G = N^{-1}V \begin{pmatrix} D^{-1} & J \\ F & FDG \end{pmatrix} U^*M$$

$$\iff G^* = U \begin{pmatrix} D^{-1} & F^* \\ J^* & J^*DF^* \end{pmatrix} V^* \qquad \ldots \quad (2.2)$$

Define

$$X_\nu = D^{-1/2+\nu} \; Y_\nu D^{-\nu+3/2} \qquad \ldots \quad (2.3)$$

where $Y_\nu = JJ^*$ if ν is odd, and $= F^*F$ is ν is even.

Observe that

$$\begin{aligned}(GG^*)^\nu &= [(A^*A)^\nu]^- \\ &\implies D^{-2\nu+3/2}\,\{(I+X_{2\nu-1}) \ldots (I+X_1)-I\}D^{-3/2} = 0 \\ &\implies (I+X_{2\nu-1}) \ldots (I+X_1) = I \\ &\implies \prod_{i=1}^{2\nu-1} |I+X_i| = 1 \\ &\implies X_i = 0, \quad i = 1, \ldots, (2\nu-1)\end{aligned}$$

since X_i's are semisimple with nonnegative eigen values.

$X_1 = 0 \implies J = 0$. $X_2 = 0 \implies F = 0$. Hence the 'if' part of Theorem 2.1(i) is established. The 'if' part of (ii) is obtained in a similar manner.

3. Some results on A_l^- and A_m^-

We prove

Theorem 3.1 : *Let* $G = A^-$. *Then*

(a) $G = A_l^-$ *iff* $GG^* = (A^*A)^-$

(b) $G = A_{lr}^-$ *iff* $GG^* = (A^*A)_r^-$

(c) $G = A_{lm}^-$ *iff* $GG^* = (A^*A)_{lm}^-$

(d) $G = A^+$ *iff* $GG^* = (A^*A)^+$.

Proof of (a) : Theorem 3.1(a) is due to Rao and Mitra (1971). We however reproduce their proof for completeness.

'Only if' part of (a) follows trivially once we observe that $G = A_l^- \Longrightarrow A^*AG = A^*$. To prove the 'if' part of (a) we proceed as follows. Let $G = A^-$. Then

$$GG^* = (A^*A)^- \Longrightarrow (A^*AG-A^*)(A^*AG-A^*)^*$$
$$= A^*AGG^*A^*A-A^*AGA-A^*G^*A^*A+A^*A = 0$$
$$\Longrightarrow A^*AG = A^*.$$

This completes the proof of (a).

Proof of (b) : Proof of (b) is complete in the light of (a) once we observe that $R(G) = R(A) \Longleftrightarrow R(GG^*) = R(A^*A)$.

Proof of (c) : For the 'only is' part of (c) observe that

$$G = A_{lm}^- \Longrightarrow A^*AA^*AGG^* = A^*AA^*G^* = A^*A$$
$$\Longrightarrow GG^* = (A^*A)_l^-$$
$$\Longrightarrow GG^* = (A^*A)_m^-.$$

For the if part of (c), we first observe that $G = A^-$ and $GG^* = (A^*A)_l^- \Longrightarrow G = A_l^-$ which follows from (a).

Further,

$$GG^* = (A^*A)_l^- \Longrightarrow A^*AA^*AGG^* = A^*A$$
$$\Longrightarrow A^*AA^*G^* = A^*A$$
$$\Longrightarrow G = A_m^-.$$

This completes the proof of (c).

Proof of (d) : To prove (d), first observe that

$$GG^* = (A^*A)^- \Longleftrightarrow GG^* = (A^*A)_m^-.$$

Now (d) follows from (b) and (c).

Let A be an $m \times n$ matrix. Let M and N be positive definite matrices of order $m \times m$ and $n \times n$ respectively. Define inner products using M and N as in Section 1, We have the following corollaries.

Corollary 3.1.1. : *Let* $G = A^-$. *Then*

(a) $G = A^-_{l(M)}$ *iff* $GM^{-1}G^* = (A^*MA)^-$

(b) $G = A^-_{lMr}$ *iff* $GM^{-1}G^* = (A^*MA)^-_r$

(b) "$G = A^-_{l(M)}$ *and* $G = A^-_{m(N)}$" *iff* $GM^{-1}G^* = (A^*MA)^-_{l(N^{-1})}$

(d) $G = A^+_{MN}$ *iff* $GM^{-1}G^* = (A^*MA)^+_{N^{-1}N}$.

Corollary 3.1.2 : *Let* $G = A^-$. *Then*

(a) $G = A^-_m$ *iff* $G^{\#}G = (AA^{\#})^-$

(b) $G = A^-_{mr}$ *iff* $G^{\#}G = (AA^{\#})^-_r$

(c) $G = A^-_{lm}$ *iff* $G^{\#}G = (AA^{\#})^-_{lm}$

(d) $G = A^{\#}$ *iff* $G^{\#}G = (AA^{\#})^+$.

Corollaries 3.1.1 and 3.1.2 are easy to prove.

Theorem 3.2 :

(a) $G = A^-_{lm} \Longrightarrow G^{\#}(GG^{\#})^\nu = [(A^{\#}A)^\nu A^{\#}]^-_l$ *for all positive integers* ν.

(b) $G^{\#}(GG^{\#})^\nu = [(A^{\#}A)^\nu A^{\#}]^-$ *for some positive integer* ν *and* $G^{\#}G = (AA^{\#})^-_{lm}$ $\Longrightarrow G = A^-$.

Proof of (a) : Proof of (a) is computational.

Proof of (b) : Observe that

$$G^{\#}G = (AA^{\#})^-_{lm} \Longrightarrow (G^{\#}G)^\nu = [(AA^{\#})]^-_{lm} \Longrightarrow (AA^{\#})^\nu (G^{\#}G)^\nu$$
$$= (G^{\#}G)^\nu (AA^{\#})^\nu \Longrightarrow (A^{\#}A)^\nu A^{\#}(G^{\#}G)^\nu = A^{\#}.$$

Hence

$$(G^{\#}G) = (AA^{\#})^-_{lm} \text{ together with } G^{\#}(GG^{\#})^\nu = [(A^{\#}A)^\nu A^{\#}]^-$$
$$\Longleftrightarrow (A^{\#}A)^\nu A^{\#} = (A^{\#}A)^\nu A^{\#}G^{\#}(GG^{\#})^\nu (A^{\#}A)^\nu A^{\#} = A^{\#}G^{\#}(A^{\#}A)^\nu A^{\#}$$
$$\Longrightarrow G = A^-.$$

An appeal to Theorem 3.1(c) now completes the proof of (b).

Note : The condition $G^{\#}G = (AA^{\#})^-_{lm}$ in Theorem 3.2(b) could be replaced by the condition $GG^{\#} = (A^{\#}A)^-_{lm}$.

4. On g-inverses with the power property

Mitra (1968) raises the following question.

"Let G be a g-inverse of A such that $A^mG^mA^m = A^m$ and $G^mA^mG^m = G^m$ for all positive integers m. Does it follow that either $\mathcal{M}(G) \subset \mathcal{M}(A)$ or $\mathcal{M}(G^*) \subset \mathcal{M}(A)$!"

The answer in general is clearly in the negative. If $R(A) \neq R(A^2)$, we know (Mitra, 1968) that there does not exist a g-inverse G of A such that either $\mathcal{M}(G) \subset \mathcal{M}(A)$ or $\mathcal{M}(G^*) \subset \mathcal{M}(A^*)$. However the Scroggs-Odell pseudoinverse, G does exist and it possesses the property that

$$A^mG^mA^m = A^m \text{ and } G^mA^mG^m = G^m \text{ for all positive integers } m.$$

We shall now give a counter example even in the case where $R(A) = R(A^2)$. Before we do this, we shall first determine the class of all g-inverses of A with power property (i.e. $A^mG^mA^m = A^m$ an $G^mA^mG^m = G^m \forall m$) in the case $R(A) = R(A^2)$.

Let $R(A) = R(A^2)$. Then the Jordan Form of A is $A = L\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}L^{-1}$ where C is nonsingular. We now prove the following.

Theorem 4.1 : *Let $R(A) = R(A^2)$. Then G is a g-inverse of A with power property iff*

$$G = L\begin{pmatrix} C^{-1} & J \\ F & FCJ \end{pmatrix}L^{-1}$$

where F and J are arbitrary subject to the condition that $JF = 0$, C is nonsingular and

$$A = L\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}L^{-1}$$

(as for example in the Jordan representation of A).

Proof : 'If' part follows by straightforward verification.

For the 'only if' part we observe that any g-inverse G of A can be written as

$$G = L\begin{pmatrix} C^{-1} & J \\ F & H \end{pmatrix}L^{-1} \text{ where } F, J \text{ and } H \text{ are arbitrary.}$$

$$GAG = G \Longrightarrow L\begin{pmatrix} C^{-1} & J \\ F & H \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C^{-1} & J \\ F & H \end{pmatrix}L^{-1}$$

$$= L\begin{pmatrix} C^{-1} & J \\ F & FCJ \end{pmatrix}L^{-1} = L\begin{pmatrix} C^{-1} & J \\ F & H \end{pmatrix}L^{-1}$$

$$\Longrightarrow H = FCJ.$$

Further

$$A^2G^2A^2 = A^2 \Longleftrightarrow L\begin{pmatrix} C^2 + C^2JFC^2 & 0 \\ 0 & 0 \end{pmatrix}L^{-1}$$

$$= L\begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}L^{-1}$$

$$\Longrightarrow JF = 0.$$

The following corrolaries are easily established.

Corollary 4.1.1 : *Let A be a square matrix such that $R(A) = R(A^2)$. Then G is g-inverse or A with power property if and only if $G = A_r^-$ and $G^2 = (A^2)^-$.*

Corollary 4.1.2 : *Let A be an $n\times n$ matrix. If $R(A) = R(A^2) = n-1$ and $A = L\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}L^{-1}$ Then every g-inverse G of A with power property can be written as $L\begin{pmatrix} C^{-1} & J \\ F & 0 \end{pmatrix}L^{-1}$ where either J or F or both are null.*

Form Theorem 4.1 and Corollary 4.1.2 it clearly follows that the class of g-inverses G such that $\mathcal{M}(G) \subset \mathcal{M}(A)$ or $\mathcal{M}(G^*) \subset \mathcal{M}(A^*)$ is a subclass of the class of all g-inverses with the power property in the case where $R(A) = R(A^2)$. Further if $R(A) = R(A^2) = n-1$ then by Corollary 4.1.2 these two classes are identical. Thus the answer to Mitra's query is in the affirmative in the case where $R(A) = R(A^2) = n-1$.

5. Concluding remarks

It is interesting to observe that the condition that G is A_r^- in Theorem 2.1 may not be replaced by a weaker condition. In fact $G = A^-$ and $G^*GG^* = (A^*AA^*)^-$ need not imply that $G = A_l^-$ or $G = A_m^-$. This is demonstrated in the following example. Take $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $G = \begin{pmatrix} 1 & 1 \\ 0 & 3/2 \end{pmatrix}$, $M = N = I$. The condition that $G^*G = (AA^*)_{lm}^-$ in Theorem 3.2 may not be replaced by a weaker condition. In fact, $G^*G = (AA^*)^-$ and $G^*GG^* = (A^*AA^*)^-$ need not imply that $G = A^-$. This is demonstrated in the following example. Take $A = \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$, $M = N = I$.

$$G = \begin{bmatrix} \frac{1}{\sqrt{5}} & 1 \\ 0 & \frac{3}{5} - \frac{1}{2\sqrt{5}} \end{bmatrix}$$

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