Sankhyā : The Indian Journal of Statistics
1988, Volume 50, Series A, Pt. 1, pp. 62–68.

ON WEAK LIMITS OF SEMISTABLE LAWS

By K. RAMA MURTHY
Indian Statistical Institute

SUMMARY. Let (μ_n) be a sequence of semistable laws on a real separable Banach space B converging weakly to a law μ on B. If μ_n has parameters r_n and α_n $(n = 1, 2, ..)$ and $\liminf_n r_n > 0$, then we show that μ is semistable. In general, a weak limit of semistable laws need not be semistable. In fact, we show that every infinitely divisible law on B is the limit of a sequence of semistable laws.

1. Introduction

Let $\{\mu_n\}$ be a sequence of semistable laws on a real separable Banach space B converging weakly to a probability measure (p.m.) μ on B. From the stable case considered by Kumar (1973), one would like to ask if μ is necessarily semistable. We show that if μ_n has parameters r_n and α_n $(n = 1, 2, ...,)$, and if $\liminf_n r_n > 0$, then μ is, indeed, semistable. However, if no condition is imposed on $\{r_n\}$ then μ need not be semistable. In fact, we show that the class of semistable laws on B is dense in the class of all infinitely divisible (i.d.) laws on B, for the topology of weak convergence ! We first state some basic definitions and notations (c.f. Rajput and Rama Murthy (1987)).

We write $\mu_n \to \mu$ to indicate that the p.m.'s μ_n converge weakly to the p.m. μ, as $n \to \infty$. The space of all p.m.'s on the Borel σ-algebra $\mathcal{B}$ of B has a metric under which a sequence $\{\mu_n\}$ converges to μ if and only if $\mu_n \to \mu$.

Let $0 < r < 1$ and μ be a p.m. on $(B, \mathcal{B})$. We say that μ is r-semistable if there exist $\{x_n\} \subseteq B$, $\{a_n\} \subseteq (0, \infty)$, positive integers $k_n (n = 1, 2, ...)$ and a Borel p.m. ν on B such that

$$\frac{k_n}{k_{n+1}} \to r$$

and

$$a_n \,.\, \nu^{*k_n} * \delta_{x_n} \to \mu.$$

(Here, $a.\mu$ is the measure $\mu o T_a^{-1}$ where $T_a : B \to B$ is defined by $T_a x = a x$). Such a μ is *i.d.*. Further, a given *i.d.* law μ is r-semistable if and only if

$$\mu^{r^n} = r^{n/\alpha} . \mu * \delta_{x_n} \ (n = 1, 2, ...)$$

AMS (1980) *subject classification :* 60B11.

Key words and phrases : Banach space, Weak limits, Infinitely divisible laws, Levy measures, Semistable laws.

for some sequence $\{x_n\} \subseteq B$. If this equation holds, we say that μ is r-semistable with index α or r-$SS(\alpha)$ for short. If $\alpha \neq 1$, then the characteristic function (ch.f.) $\hat{\mu}$ of μ is given by

$$\hat{\mu}(y) = \exp. \{i < x_0, y > - \int_{\Delta_0} | < x, y > |^\alpha k_\alpha(< x, y >) \, d\, \Gamma(x)\}$$

for all $y \in B^*$, the topological dual of B, where $< x, y >$ denotes the evaluation of y at x, Γ is the restriction of the Levy measure F of μ to

$$\Delta_0 \equiv \{x \in B : r^{1/\alpha} < \|x\| \leqslant 1\}$$

and k_α is a complex valued function on $R \setminus \{0\}$ with the following properties : $k_\alpha(-t) = \overline{k_\alpha(t)}$, $k_\alpha(e^t)$ is periodic on $(0, \infty)$ with period $-\frac{1}{\alpha} \log r$, k_α is continuous on $R \setminus \{0\}$ and there exist positive constants C_0 and C_1 with

$$C_0 \leqslant \text{Re. } k_\alpha(t) \leqslant |k_\alpha(t)| \leqslant C_1 \text{ for all } t \in R \setminus \{0\}.$$

(R stands for the real line and Re. for the real part of a complex number). If μ is symmetric, then a similar result holds for $\alpha = 1$.

The Lévy measure F of μ can be recovered from its restriction Γ to Δ_0 by the formula :

$$F(A) = \sum_{k=-\infty}^{+\infty} r^k \Gamma(r^{k/\alpha} A \cap \Delta_0).$$

Further, F satisfies the relations :

$$r^{n/\alpha}. F = r^n F \quad (n = 0, \pm 1, \pm 2, \ldots).$$

2. The main theorems

Theorem 1 : *Let $0 < r_n < 1$, $0 < \alpha_n < 2$ and $\liminf_n r_n > 0$. Let μ_n be an $r_n - SS(\alpha_n)$ p.m. on $(B, \mathcal{B})$ for each n and let μ_n be symmetric if $\alpha_n = 1$. If $\mu_n \to \mu$, then μ is semistable.*

Proof : It is well known that weak limits of i.d. laws are i.d.. Hence, μ is i.d.. To show that μ is semistable we begin with the defining relation :

$$\mu_n^{r_n} = r_n^{1/\alpha_n} . \mu_n * \delta_{x(n)} \quad \ldots \ (1)$$

where $\{x(n)\} \subseteq B$. We split the proof into four cases.

Case 1 : $0 < \liminf_n r_n \leqslant \limsup_n r_n < 1$

and $0 < \liminf_n \alpha_n \leqslant \limsup_n \alpha_n < 2$.

In this case we may suppose (by going to a subsequence, if necessary) that $r_n \to r$ and $\alpha_n \to \alpha$ with $0 < r < 1$ and $0 < \alpha < 2$. We claim that

$$\mu^r = r^{1/\alpha}. \mu * \delta_x \quad \text{for some } x \in B.$$

For this, we first note that $r_n^{1/\alpha_n}. \mu_n \to r^{1/\alpha}. \mu$. Indeed, let f be a bounded continuous function : $B \to R$. Given $\epsilon > 0$, there is a compact set K in B with $\mu_n(K) > 1-\epsilon$ and $\mu(K) > 1-\epsilon$ ($n = 1, 2, \ldots$). We may suppose K is absolutely convex. Now, by the uniform continuity of f on K, there exists $\delta > 0$ such that $| f(x)-f(y)| < \epsilon$ whenever $\|x-y\| < \delta$ and $x, y \in K$. Now,

$$\left| \int f\left(r_n^{1/\alpha_n} x\right) d\mu_n(x) - \int f(r^{1/\alpha} x)\, d\, \mu(x)\right|$$

$$\leqslant \left(\sup_{x \in B} |f(x)|\right)(2\epsilon) + \left| \int_K \left\{ f\left(r_n^{1/\alpha_n} x\right) - f(r^{1/\alpha} x)\right\} d\mu_n(x)\right|$$

$$+ \left| \int_K f(r^{1/\alpha} x)\, d\, \mu_n(x) - \int_K f(r^{1/\alpha} x)\, d\mu(x)\right|$$

$$\leqslant \left(\sup_{x \in B} |f(x)|\right)(2\epsilon) + \epsilon\, \mu_n(K) + \left| \int f(r^{1/\alpha} x) d\, \mu_n(x)\right.$$

$$\left. - \int f(r^{1/\alpha} x)\, d\, \mu(x)\right| + (\sup_{x \in B} |f(x)|)(2\epsilon)$$

if n is so large that $\left| r_n^{1/\alpha_n} - r^{1/\alpha}\right| < \frac{\delta}{M}$, where $M = \sup_{x \in K} \|x\|$.

Since $\mu_n(K) \leqslant 1$, $\int f(r^{1/\alpha}x)\, d\, \mu_n(x) \to \int f(r^{1/\alpha} x)\, d\mu(x)$, and ϵ is arbitrary, we have proved that

$$r_n^{1/\alpha_n}. \mu_n \to r^{1/\alpha}. \mu.$$

Next, we show that $\mu_n^{r_n} \to \mu^r$. Since $\{\mu_n\}$ is tight, it follows from the relation

$$\mu_n = \mu_n^{r_n} * \mu_n^{1-r_n} \qquad (n = 1, 2, \ldots)$$

that $\{\mu_n^{r_n}\}$ is shift tight. Further, the ch.f. of $\mu_n^{r_n}$ converges to the ch.f. of μ^r as $n \to \infty$. Hence, $\mu_n^{r_n} \to \mu^r$. Finally, letting $n \to \infty$ in (1) we see that there exists $x \in B$ with $\mu^r = r^{1/\alpha}\ \mu * \delta_x$. This completes the proof of the theorem in case 1.

Case 2 : $\limsup_n r_n = 1$.

In this case, we suppose that $r_n \to 1$ and $\alpha_n \to \alpha$ with $0 \leqslant \alpha \leqslant 2$. Let $t > 0$ and $\{k_n\}$ be a sequence of positive integers with $k_n \to \infty$ and $r_n^{k_n} \to t$ (*e.g* ; $k_n = [(\log t)/(\log r_n)]$). Iteration of (1) leads to

$$\mu_n^{r_n^{k_n}} = r_n^{k_n/\alpha_n} \cdot \mu_n * \delta_{y(n)} \qquad \ldots \quad (2)$$

for some sequence $\{y(n)\} \subseteq B$. As in Case 1 we obtain

$$\mu^t = t^{1/\alpha} \cdot \mu * \delta_y \qquad \ldots \quad (3)$$

for some $y \in B$, if $\alpha \neq 0$. If $\alpha = 0$ then $\mu^t = \delta_y$. Thus, μ is degenerate if $\alpha = 0$ and stable (hence semistable) if $\alpha \neq 0$.

Case 3 : $\liminf_n \alpha_n = 0$ and $0 < \liminf_n r_n \leqslant \limsup_n r_n < 1$.

Assuming that $\alpha_n \to 0$ and $r_n \to r$ $(0 < r < 1)$, we get $\mu^r = \delta_x$. μ is thus degenerate, and hence semistable.

Case 4 : $\limsup_n \alpha_n = 2$ and $0 < \liminf_n r_n \leqslant \limsup_n r_n < 1$

We assume that $\alpha_n \to 2$ and $r_n \to r$ $(0 < r < 1)$. As in the above cases we get $\mu^r = r^{1/2} \cdot \mu * \delta_x$ for some $x \in B$. We show that μ is Gaussian (and hence $r-S\,S(2)$ for any $r \in (0, 1)$). For this, it suffices to show that the Levy measure F of μ vanishes identically. Now, the symmetrization $\bar{F}$ of F, defined by $\bar{F}(A) = F(A) + F(-A)$ $(A \in \mathcal{B})$, satisfies $r\,\bar{F} = r^{1/2} \cdot \bar{F}$ and

$$\int \{1-\cos <x, y>\}\, d\bar{F}(x) = \int_{\Delta_0} \sum_{k=-\infty}^{\infty} r^{-k}\{1-\cos r^{k/2} <x, y>\}\, d\,\bar{F}(x).$$

However, if $<x, y> \neq 0$, then

$$\sum_{k=-\infty}^{\infty} r^{-k}\{1-\cos r^{k/2} <x, y>\} \geqslant \sum_{k=k_0}^{\infty} \frac{r^{-k}}{4}\, r^k <x, y>^2 = \infty,$$

where k_0 is chosen so large that $1-\cos\; r^{k/2} <x, y> \;\geqslant \dfrac{r^k <x, y>^2}{4}$ for $k \geqslant k_0$ and $x \in \Delta_0$. Hence

$$\int \{1-\cos <x, y>\}\, d\,\bar{F}(x) = 0 \text{ or } \infty$$

for all y. However, $\int\{1-\cos <x, y>\}\, d\,\bar{F}(x) < \infty$ for all y, since $\bar{F}$ is a symmetric Levy measure. It follows that $\int \{1-\cos <x, y>\}\, d\bar{F}(x) = 0$ for all $y \in B^*$ and hence $\bar{F} \equiv 0$. Of course, this implies $F \equiv 0$ too.

The proof of Theorem 1 is now complete.

Theorem 2 : *Let μ be an i.d. p.m. on $(B, \mathcal{B})$. Then, there exists a sequence $\{r_n\}$ in $(0, 1)$ converging to 0, a sequence $\{\alpha_n\}$ in $(0, 2)$ converging to 2 and a sequence $\{\mu_n\}$ of p.m.'s on $(B, \mathcal{B})$ such that μ_n is $r_n - SS(\alpha_n)$ for each n, and $\mu_n \to \mu$.*

We prove a slightly stronger statement that if $\{s_n\}$ and $\{\beta_n\}$ are sequences with $0 < s_n < 1$, $0 < \beta_n < 2$ $\lim_n s_n = 0$, $\lim_n \beta_n = 2$, $\lim_n s_n^{2/\beta_n - 1} = 0$, then there is a subsequence $\{n_j\}$ of the integers such that for some sequence $\{\mu_j\}$ of p.m.'s we have $\mu_j \to \mu$ and μ_j is $s_{n_j} - S\,S(\beta_{n_j})$ for each j.

Proof of Theorem 2 : Since μ is i.d., we may write

$\mu = \mu_1 * \mu_2 * \mu_3$, where μ_2 is a centered Gaussian p.m. and the ch. f.'s of μ_1, μ_2 and μ_3 are given by

$$\hat{\mu}_1(y) = \exp.(i < x_0, y >)$$

$$\hat{\mu}_2(y) = \exp.(-\tfrac{1}{2} \int < x, y >^2 d\,\mu_2)$$

$$\hat{\mu}_3(y) = \exp.\{ \int \{e^{i<x,y>} - 1 - i\, I_{||x|| \leq \delta}(x) < x, y > \}\, d\,F(x)\}$$

with $\delta > 0$, $x_0 \in B$ and F the Levy measure of μ. (c.f. Araujo and Gine, 1980, p. 137).

We begin by noting that μ_1 is $r - S\,S(\alpha)$ for any $r \in (0, 1)$ and any $\alpha \in (0, 2)$. Next, we show that whenever $r_n \to 0$ and $\alpha_n \to 2$

$$\exp.\{-\tfrac{1}{2} (\int < x, y >^2 d\,\mu_2)^{\alpha_n/2}\}$$

is the ch.f. of an $r_n - S\,S(\alpha_n)$ p.m. λ_n with $\lambda_n \to \mu_2$. Indeed, there is a p.m. ψ_n on $(0, \infty)$ whole Laplace Transform is $\exp.\{-C_n\, t^{\alpha_n/2}\}$, where $C_n = 2^{\alpha_n/2 - 1}$ (c.f. Feller, 1971, p. 424).

If we define $\Phi_n(A) = \psi_n\ \{a \in (0, \infty) : \sqrt{a} \in A\}$, then $\lambda_n(A) = \int_0^\infty \mu_2(a^{-1} A)\, d\,\Phi_n(a)$ satisfies our requirements. (one verifies directly, from the def. of weak convergence that $\lambda_n \to \mu_2$).

In view of the discussion in the above paragraph, and the fact the convolution of any two $r - S\,S(\alpha)$ p.m.'s is $r - S\,S(\alpha)$, it suffices to consider the case when $\mu = \mu_3$, i.e., when the ch.f. of μ is given by

$$\hat{\mu}(y) = \exp.\left\{ \int \left\{e^{i<x,y>} - 1 - i < x, y > I_{||x|| \leq \delta}(x) \right\} dF(x) \right\}.$$

Now, restricting F to $\{x \in B : \|x\| > 1/n\}$ and using Proposition 2.1 of Araujo and Gine (1980, p. 45), we reduce the proof to the case when F vanishes identically in a neighbourhood of the origin in B. However, in this case F is a finite measure and hence there is a sequence of measures with finite supports (contained in $B\setminus\{0\}$) converging weakly to F. Once again, proposition 2.1 of Araujo and Gine, (1980) can be used to reduce the proof to the case when F itself has finite support. In this case μ is the convolution of a finite number of i.d. p.m.'s each having a degenerate Lévy measure. We may thus take F in the form $a\ \delta_{x_0}$ with $a > 0$ and $x_0 \in B$. However, it is clear that in this case, μ is supported by the 1-dimensional space spanned by x_0. So, we assume that μ is an i.d. law on R with Lévy measure $a\ \delta_{x_0}$ $(a > 0, x_0 \in R)$. Now

$$\hat{\mu}(t) = \exp.\ a\ \{e^{itx_0} - 1\},$$

or

$$\hat{\mu}(t) = \exp.\ a\ \{e^{itx_0} - 1 - i\ t\ x_0\}$$

according as $|x_0| > \delta$ or $|x_0| \leqslant \delta$. However, when $|x_0| > \delta$ we may write $\hat{\mu}(t) = \exp\ (iat\ x_0)\ \exp.\ a(e^{itx_0} - 1 - i\ t\ x_0)$ and $\exp\ (iat\ x_0)$ is the ch.f. of an $r - S\,S\ (\alpha)$ p.m. for any r and α. Thus, we may suppose

$$\hat{\mu}(t) = \exp.\ a\{e^{itx_0} - 1 - i\ t\ x_0\}.$$

We now define

$$\hat{\mu}_n(t) = \exp.\ a\left\{\sum_{k=-\infty}^{\infty} r_n^{-k}\left\{e^{itx_0 r_n^{k/\alpha_n}} - 1 - i\ t\ x_0\ r_n^{k/\alpha_n}\right\}\right.$$

where $r_n \to 0$, $\alpha_n \to 2$, $0 < r_n < 1$, $0 < \alpha_n < 2$ and $r_n^{2/\alpha_n - 1} \to 0$, but $\{r_n\}$ and $\{\alpha_n\}$ are otherwise arbitrary. μ_n is an $r_n - S\,S(\alpha_n)$ p.m. on R with Levy measure $F_n \equiv \sum_{k=-\infty}^{\infty} a\ r_n^{-k}\ \delta_{x_0 r_n^{k/\alpha_n}}$. (It is easily seen that $\int_{|x| > \delta} |x|\,d\ F_n(x) < \infty$ for n so large that $\alpha_n > 1$. Since $\exp.\ a\{\int (itx - itxI_{|x| \leqslant \delta}(x))dF_n(x)\}$ is the ch.f. of a degenerate law, it follows that $\hat{\mu}_n$ is indeed a ch.f.). To complete the proof of Theorem 2, it suffices to show that $\hat{\mu}_n(t) \to \hat{\mu}(t)$ for each t. Now,

$$\left|\sum_{k=1}^{\infty} r_n^{-k}\left\{e^{itx_0 r_n^{k/\alpha_n}} - 1 - i\ t\ x_0\ r_n^{k/\alpha_n}\right\}\right|$$

$$\leqslant \sum_{k=1}^{\infty} r_n^{-k}\ \tfrac{1}{2}\ t^2\ x_0^2\ r_n^{2k/\alpha_n}$$

$$= \frac{t^2\ x_0^2}{2}\ \frac{r_n^{2/\alpha_n - 1}}{1 - r_n^{2/\alpha_n - 1}} \to 0 \text{ as } n \to \infty$$

for each t. Also

$$\left| \sum_{k=-\infty}^{-1} r_n^{-k} \left\{ e^{itx_0 r_n^{k/a_n}} - 1 - itx_0 \; r_n^{k/a_n} \right\} \right|$$

$$\leqslant \sum_{k=-\infty}^{-1} r_n^{-k} \left\{ 2 + |t\, x_0| \, r_n^{k/a_n} \right\}$$

$$= \frac{2\, r_n}{1-r_n} + |t\, x_0| \, \frac{r_n^{1-(1/a_n)}}{1-r_n^{1-(1/a_n)}} \to 0 \text{ as } n \to \infty$$

for each t. Since

$$\exp.\, a\, r_n^{-0} \left(e^{itx_0 r_n^0} - 1 - itx_0 \, r^{0/a_n} \right)$$

$$= \exp.\, a \left(e^{itx_0} - 1 - itx_0 \right),$$

it follows that $\hat{\mu}_n(t) \to \hat{\mu}(t)$ as $n \to \infty$ for each $t \in R$. The proof of Theorem 2 is now complete.

Acknowledgement. The author wishes to thank Dr. S. Natarajan for helpful comments on the organization of the paper.

References

ARAUJO, A. and GINE, E (1980): *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York.

FELLER, W. (1971): *An Introduction to Probability Theory and its Applications*, Vol. II, 2nd ed. Wiley, New York.

KUMAR, A. (1973): A note on the convergence of stable and class L probability measures on Banach spaces. *Ann. Prob.*, **1**, 716-718.

Rajput, B. S. and Rama Murthy, K. (1987): Spectral representation of semistable processes, and semistable laws on Banach spaces. *Jour. Mult. Anal.*, **21**, 139-157.

Paper received : November, 1985.