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ON THE DISTRIBUTION OF ADDITIVE ARITHMETICAL FUNCTIONS OF INTEGRAL POLYNOMIALS

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SUMMARY. Let $f_1, f_2, ..., f_s$ be real valued additive arithmetic functions and let $F_1, ..., F_s$ be polynomials with integral coefficients, which are not divisible by square of any irreducible polynomial and $F_i(m) > 0$ ($m = 1, 2, ...;$ $i = 1, ..., s$).

In this paper a condition is given which ensures that $\{f_1(F_1(m)), ..., f_s(F_s(m))\}$ has a distribution. This condition turns out to be necessary also. Our techniques are probabilistic in nature. A stronger sufficient condition was found by Katai (1969) using sharp versions of the sieve method.

1. INTRODUCTION

Katai (1969) has given sufficient conditions which ensure the existence of distribution of $\{f_1(F_1(m)), ..., f_s(F_s(m))\}$, where $f_1(n), ..., f_s(n)$ are real valued additive arithmetic functions and $F_1, ..., F_s$ belong to $\mathcal{P}$, where $\mathcal{P}$ denotes the set of all polynomials F with integer coefficients satisfying the following conditions :

(P1) $F(m) > 0$ for $m = 1, 2, 3, ...$

(P2) F is not divisible by square of any irreducible polynomial.

He used sharp versions of sieve theorems to prove the above result under the additional condition that $F_i(m)$ and $F_j(m)$ are relatively prime polynomials if $i \neq j$.

Here we give a probabilistic proof of a much stronger version of Katai's result, where it is not assumed that $F_i(m)$ and $F_j(m)$ are relatively prime. We also show that these weaker conditions are not only sufficient but also necessary for the existence of the distribution of $\{f_1(F_1(m)), ..., f_s(F_s(m))\}$ where $F_1, ..., F_s \in \mathcal{P}$.

After sending this paper for press the author came to know about a paper due to J. Galambos (Distribution of additive and multiplicative functions ; The theory of arithmetic functions. *Proceedings of the Conference on Arithmetic Functions* held at Michigan University, 1971) where he proved sufficiency part of Theorem 2 for strongly additive functions using probabilistic methods.

2. Notations and definitions

For F in $\mathcal{P}$ let D_F denote the degree of the polynomial F. For any positive integer d, let $r(F, d)$ denote the number of incongruent solutions in integers of the congruence relation $F(m) \equiv 0 \pmod d$.

$p, q, \ldots$ denote prime numbers.

$p_1, p_2, \ldots$ is the sequence of all prime numbers in the increasing order of their magnitude.

Let $f(n), f_1(n), \ldots, f_s(n)$ denote real valued additive arithmetic functions. Let $N_n\{\ldots\}$ denote the number of positive integers less than or equal to n having the property indicated in $\{\ldots\}$. For any subset A of the natural numbers, let $\overline{D}(A)$ and $\underline{D}(A)$ denote the upper and lower natural density of A respectively :

i.e.
$$\overline{D}(A) = \limsup_{n \to \infty} \frac{1}{n} N_n(m \in A)$$

and
$$\underline{D}(A) = \liminf_{n \to \infty} \frac{1}{n} N_n(m \in A).$$

We denote by $D(A)$, the common value of $\overline{D}(A)$ and $\underline{D}(A)$ whenever they coincide.

Put
$$f'(p^k) = \begin{cases} f(p^k) & \text{if} \quad |f(p^k)| < 1 \\ 1 & \text{if} \quad |f(p^k)| \geqslant 1 \end{cases}$$

We define,

$$A(v, n, f, F) = \sum_{v \leqslant p \leqslant n} \frac{f'(p)\, r(F, p)}{p}$$

$$B(v, n, f, F) = \sum_{v \leqslant p \leqslant n} \left[\frac{(f'(p))^2\, r(F, p)}{p} \right]^{1/2}$$

$$A(n, f, F) = A(0, n, f, F)$$

$$B(n, f, F) = B(0, n, f, F).$$

We say that the s-tuple $\{h_1(n), \ldots, h_s(n)\}$ of real arithmetical functions have a distribution, if

$$\frac{1}{n} N_n\{h_1(m) < c_1, \ldots, h_s(m) < c_s\}$$

tends to a s-dimensional probability distribution function $Q(c_1, \ldots, c_s)$ as $n \to \infty$, for all its continuity points.

3. Results

Theorem 1 : *Let $F\epsilon\mathcal{P}$ and $f(m)$ be any real valued additive arithmetical function. Suppose*

$$D_F \geqslant 2 \text{ and } f(p^k)\, r(F, p^k) \to 0 \text{ as } p \to \infty \cdot \text{for } k = 1, \ldots, D_F - 1. \quad \ldots \quad (3.1)$$

Then the distribution of $f(F(m))$ exists if, and only if,

$$\sum_p \frac{f'(p)\, r(F, p)}{p} \text{ converges} \quad \ldots \quad (3.2)$$

$$\sum_p \frac{(f'(p))^2\, r(F, p)}{p} \text{ converges.} \quad \ldots \quad (3.3)$$

Remark : The statement of the theorem holds if $D_F = 1$ without any further assumptions.

Theorem 2 : *Let $f_1(n), \ldots, f_s(n)$ be real valued additive arithmetical functions and $F_1, \ldots, F_s$ belong to $\mathcal{P}$. Suppose*

$$f_i(p^k)\, r(F_i, p^k) \to 0 \text{ as } p \to \infty \text{ whenever } D_{F_i} \geqslant 2 \text{ for } k = 1, \ldots, D_{F_i} - 1. \quad \ldots \quad (3.4)$$

Then the s-tuple $\{f_1F_1(m), \ldots, f_s(F_s(m))\}$ have a distribution if, and only if

$$\sum_p \frac{f_i(p)\, r(F_i, p)}{p} \quad \text{converges for } i = 1, \ldots, s \quad \ldots \quad (3.5)$$

$$\sum_p \frac{\{f_i(p)\}^2\, r(F_i, p)}{p} \quad \text{converges for } i = 1, \ldots, s. \quad \ldots \quad (3.6)$$

Remark : In Theorem 1 if F is a product of linear polynomials we can omit the condition (3.1). Similar remark holds in Theorem 2 also.

Theorem 3 : *Let $f(n)$ be a non-negative additive arithmetic function and $F\epsilon\mathcal{P}$. Suppose for some $c > 0$.*

$$\overline{D}\{f(F(m)) < c\} > 0. \quad \ldots \quad (3.7)$$

Then

$$\sum_p \frac{f'(p)\, r(F, p)}{p} < \infty.$$

4. Outline of Novoselov's method

Here we give a brief outline of Novoselov's (1966) method because our proof mainly depends on the probability space constructed by him.

Let Z denote the set of all integers. Consider the topology induced on Z by taking as a neighbourhood basis at the point a the set of all residue classes with respect to non-zero moduli that contains a.

In this manner Z becomes a topological ring S with the usual addition and multiplication and with a non-discrete topology. Note that S is totally disconnected, totally bounded. Completing S we get a compact ring $\mathcal{S}$, whose elements will be called polyadic numbers. On $\mathcal{S}$, as a compact additive group, there exists a normalized Haar measure μ. This measure is not complete. Its completion is denoted by P. We denote the elements of $\mathcal{S}$ by $x, y, \ldots$.

Basic concepts and notations. Let N_k be a sequence of natural numbers converging to zero in $\mathcal{S}$ satisfying the following conditions :

$$N_{k+1} > N_k, \frac{N_{k+1}}{N_k} \to 1 \text{ as } k \to \infty.$$

This sequence is fixed throughout this paper.

Let $R_k(x)$ be the smallest positive residue of x modulo N_k.

Let S^0 be the class of all complex valued functions on $\mathcal{S}$ such that $f(R_k(x)) \xrightarrow{P} f(x)$ as $k \to \infty$, where $\xrightarrow{P}$ denote the convergence in P-measure. We say that an arithmetic function $f(n) \epsilon S^0$ if there is an extension $f(x)$ of $f(n)$ to $\mathcal{S}$ such that $f(x) \epsilon S^0$.

$p^k \| x$ means the highest power of p that divides x is equal to k if k is a positive integer, and $p^\infty \| x$ means $p^k | x$ for every $k > 0$ ($l | x$ means l divides x).

Some results of Novoselov.

Lemma 1 : $\overline{D}\{h(m) \epsilon A\} = \limsup_{k \to \infty} P\{x : h(R_k(x)) \epsilon A\}$

$$\underline{D}\{h(m) \epsilon A\} = \liminf_{k \to \infty} P\{x : h(R_k(x)) \epsilon A\}$$

for any set A and any complex valued function h on the set of positive integers.

Lemma 2 : *If $h_n(x) \epsilon S^0$, then the validity of any two of the following conditions*

$$h_n(x) \xrightarrow{P} h(x), \lim_{n \to \infty} \overline{D}\{|h(m) - h_n(m)| \geqslant \sigma\} = 0 \quad \textit{for all } \sigma > 0,$$

$h(x) \epsilon S^0$, implies the third.

Lemma 3 : *If $h_1(x) \epsilon S^0$ and $h_2(x) \epsilon S^0$, then*

(1) *$a h_1(x) + b h_2(x) \epsilon S^0$ for any complex numbers a and b,*

(2) *$h_1(x) \cdot h_2(x) \epsilon S^0$.*

If $h(m) \epsilon S^0$ then $h(m)$ has a distribution.

Proofs of all these lemmas are easy. See Novoselov (1966).

5. Lemmas

Lemma 4 : *Let $F \epsilon \mathcal{P}$. Then there exists a p_0 such that $p > p_0 \Longrightarrow r(F, p^k) = r(F, p)$ for any positive integer k. Also $r(F, a \cdot b) = r(F, a) \cdot r(F, b)$ if $(a, b) = 1$ and $r(F, p^k) \leqslant c$ for some constant c depending only on F.*

For proof see Tanaka (1955).

Lemma 5 : *Let $F \epsilon \mathcal{P}$ with $D_F \geqslant 2$. Then for each $\varepsilon > 0$, there exists $v_0 = v_0(\varepsilon)$ and $k = k(\varepsilon)$ such that $v > v_0 \Longrightarrow N_n\{p^{D_F} | F(m)$ for some $p > v$ or $q^k | F(m)$ for some $q\} < \varepsilon n + o(n)$ as $n \to \infty$.*

Proof: Choose $k > D_F$ and v_0 such that

$$\sum_{v_0 \leq p} \frac{r(F, p^{D_F})}{p^{D_F}} < \frac{\varepsilon}{2} \text{ and } \sum_{p} \frac{r(F, p^k)}{p^k} < \frac{\varepsilon}{2}.$$

Let $M > 0$ be such that $Mm^{D_F} \geqslant F(m)$ for every $m \geqslant 2$.

If $v > v_0$,

then

$$N_n\{p^{D_F} \mid F(m) \text{ for some } p > v \text{ or } q^k \mid F(m) \text{ for some } q\}$$

$$\leqslant n \sum_{p \geqslant v} \frac{r(F, p^{D_F})}{p^{D_F}} + \sum_{Mn > p \geqslant v} r(F, p^{D_F}) + n \sum_{p} \frac{r(F, p^k)}{p^k} + \sum_{p < Mn} r(F, p^k)$$

$$< n\varepsilon + O\left(\frac{n}{\log n}\right) = n\varepsilon + o(n).$$

Lemma 6 : *Let U and V be two probability distributions neither of which is concentrated at one point. If for a sequence $\{F_n\}$ of probability distributions and constants $a_n > 0$ and $c_n > 0$*

$$F_n(a_n x + b_n) \to U(x),$$

$$F_n(c_n x + d_n) \to V(x) \text{ at all points of continuity,} \qquad \ldots \quad (5.1)$$

then $$\frac{c_n}{a_n} \to A \neq 0, \quad \frac{d_n - b_n}{a_n} \to B.$$

For proof see Feller (1966, p.246).

Lemma 7 : *Let $F \epsilon \mathcal{P}$. Let f be any additive arithmetical function such that $B(n, f, F) \to \infty$ as $n \to \infty$, and*

$$f(p)\, r(p) = o(B(p, F))$$

$$f(p^\alpha)\, r(p^\alpha) \to 0 \text{ as } p \to \infty \text{ for } \alpha = 2, \ldots, D_P - 1 \text{ if } D_P \geqslant 2.$$

Then

$$\frac{1}{n} N_n\{f(F(m)) < A(n, f, F) + xB(n, f, F)\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

as $n \to \infty$, for all real numbers x.

For proof see Halberstam (1956).

Let $f(n)$ be any additive arithmetic function and $F \epsilon \mathcal{P}$. Suppose $F(m) = a_t m^t + \ldots + a_0$. Define, $F(x) = a_t x^t + \ldots + a_0$, $x \epsilon \mathcal{S}$. Clearly $F(x)$ is uniformly continuous on $\mathcal{S}$ into $\mathcal{S}$.

Define $$f_p(x, F) = \sum_{k=1}^{\infty} f(p^k)\, w(F, p^k, x)$$

where $$w(F, p^k, x) = \begin{cases} 1 & \text{if } p^k \| F(x) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $w(F, p^k, .)$ is measurable and

$$P(x : w(F, p^k, x) = 1) = \frac{r(F, p^k)}{p^k} - \frac{r(F, p^{k+1})}{p^{k+1}}; \text{ see Novoselov (1966).}$$

Lemma 8 : *$f(n)$ be any additive arithmetic function. Let $F \epsilon \mathcal{P}$. Suppose*

$$D_F > 2 \quad \text{and} \quad f(p^k)\ r(F, p^k) \to 0 \text{ as } p \to \infty,\ k = 1, \ldots, D_F - 1. \quad \ldots \quad (5.2)$$

Then given any $\sigma > 0$, there exists $v_0 = v_0(\varepsilon)$ such that $v > v_0 \Longrightarrow$

$$\sum_{m=1}^{n} \Big(\sum_{p > v} \bar{f}_p(m, F) - A(v, n, f, F) \Big)^2 \leqslant CnB^2(v, n, f, F) + \varepsilon n$$

where

$$\bar{f}(p^k) = \begin{cases} f(p^k) & \text{if } k \leqslant D_F - 1 \\ 0 & \text{otherwise} \end{cases}$$

and C depends only on F.

Remark : If F is product of linear polynomials we may omit the condition (5.2).

Proof of this lemma is similar to Turan-Kubilius inequality (Kubilius, 1964, Lemma 3.1, p. 31).

6. Proofs of the theorems

Proof of Theorem 1 : It is easy to show $f_p(x, F)$ is continuous almost everywhere. (See Novoselov, 1966).

Hence for any n, $\sum_{p \leqslant n} f_p(x, F) \,\epsilon\, S^0$.

Let p_0 be such that $p > p_0 \Longrightarrow r(F, p^k) = r(F, p)\ \forall\ k$.

Observe that $$E[w(F, p^k, x)] = \frac{r(F, p)}{p^k} \left(1 - \frac{1}{p}\right) \text{ if } p > p_0 .$$

and, $$E[w(F, p^k, x) w(F, p^t, x)] = 0 \text{ if } k \neq t \text{ and } p > p_0.$$

Since $r(F, d)$ is multiplicative function (Lemma 4), $(f_p(x, F))_{p > p_0}$ are all mutually independent random variables. See Novoselov (1966, p. 244).

Now suppose that (3.2) and (3.3) hold. By Kolmogorov's 3-series theorem it follows that $\sum_{p > p_0} f_p(x, F)$ converges almost everywhere. Hence, $\sum_p f_p(x, F)$ converges almost everywhere.

$$\text{Define} \quad f^*(x, F) = \begin{cases} \sum_p f_p(x, F) & \text{whenever it converges} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f^*(m, F) = f(F(m))$ for natural number m.

To show that $f^*(x, F)\epsilon S^0$ it is enough to show, in view of Lemma 2, that

$$\lim_{r\to\infty} \bar{D}\left\{\left|\sum_{p>r} f_p(m, F)\right| > \sigma\right\} = 0, \quad \text{for every } \sigma > 0$$

which follows from (3.1) and Lemmas 5 and 8. Hence $f(F(m))$ has a distribution.

Conversely, let $U(x)$ be the distribution of $f(F(m))$. If $U(x)$ is degenerate, then choose p_0^k, $k > 1$ such that $r(F, p_0^k) \neq 0$.

Put
$$f^*(p_0^k) = f(p_0^k)+1$$
$$f^*(p^\beta) = f(p^\beta) \quad \text{if} \quad p^\beta \neq p_0^k.$$

Now it is easy to see that, if f^* is the new additive arithmetic function defined above, the distribution of $f^*(F(m))$ exists and is nondegenerate. So without loss of generality we may assume $U(x)$ is nondegenerate probability distribution. From Lemmas 6 and 7 it follows that

$$\lim_{n\to\infty} B(n, f, F) < \infty.$$

By Kolmogorov's 3-series theorem, we have

$$\sum_p \left\{f_p(x, F) - \frac{f'(p)\ r(F, p)}{p}\right\} \text{ converges almost everywhere.}$$

Define

$$g(x) = \begin{cases} \sum_p \left\{f_p(x, F) - \dfrac{f'(p)\, r(F, p)}{p}\right\} & \text{if it converges} \\ 0 & \text{otherwise.} \end{cases}$$

Let
$$Q(c) = P\{x; g(x) < c\}.$$

By Lemma 8 and (3.1) it is easy to see that $\frac{1}{n} N_n\{[f(F(m))-A(n, f, F)] < c\}$ $\to Q(c)$ as $n\to\infty$ at all continuity points c of Q. If Q is degenerate, it follows that $A(n, f, F)$ are bounded, since $\frac{1}{n} N_n\{[f(F(m))-A(n, f, F)] < c\}$ are discrete distributions. Hence there exists a subsequence $\{n_r\}$ of natural numbers such that $A(n_r, f, F)\to b$ as $r\to\infty$ for some b. Hence we conclude that $U(c+b) = Q(c)$ which gives a contradiction, since we assumed that U is nondegenerate. Hence Q is nondegenerate. By Lemma 6, it follows that

$$\sum_p \frac{f'(p)\, r(F, p)}{p} \quad \text{converges.}$$

This proves Theorem 1.

Proof of Theorem 2 : By Theorem 1, (3.4), (3.5) and (3.6) we have,

$$f_1(F_1(m))\epsilon S^0, \ldots, f_s(F_s(m))\, \epsilon S^0.$$

By Lemma 3, for every s-tuple $(t_1, \ldots, t_s)$ of real number $t_1 f_1(F_1(m))+\ldots+t_s f_s(F_s(m))\epsilon S^0$. Hence by Cramer-Wold device (Feller, p. 405), the distribution of $\{f_1(F_1(m)), \ldots, f_s(F_s(m))\}$ exists.

The converse part follows from Theorem 1.

Proof of Theorem 3 : Let p_0 be such that $r(F, p^k) = r(F, p)$ for all $k \geqslant 1$, whenever $p > p_0$.

Suppose that $\sum_p \frac{f'(p)\, r(F, p)}{p} = +\infty$, then by Kolmogorov's 3-series theorem $\sum_{p>p_0} f_p(x, F)$ diverges to $+\infty$ almost everywhere.

Let $h_n(x) = \sum_{p_0<p\leqslant n} f_p(x, F)$. Note that the distribution of $h_n(x)$ is discrete.

Suppose (3.7) holds for some $c > 0$. Let $\beta > c$ be any common continuity point of the distributions of the functions $h_{p_0}, h_{p_0+1}, \ldots$. As $n \to \infty$, the natural density of $h_n^{-1}[0, \beta] \to 0$.

Let $\{N_{k_r}\}$ be a subsequence of $\{N_k\}$ such that

$$\lim_{r\to\infty} \frac{1}{N_{k_r}} N_{N_{k_r}}\{f(F(m)) < \beta\} = \bar{D}\{m : f(F(m)) < \beta\} = Z > 0 \text{ (say)}.$$

By using Lemma 5, choose $t > D_F$ such that $N_n\{p^t \mid F(m) \text{ for some } p\} < \frac{Z}{4}\, n + o(n)$ for all n. Let $\{N_{k(n)}\}$ be an increasing subsequence of $\{N_{k_r}\}$*s. t.* $k > k(n) \Longrightarrow p^t \mid N_k$ if $p \leqslant n$.

So
$$\frac{1}{N_{k(n)}} N_{N_{k(n)}}\{f(F(m)) < \beta\} \leqslant P\Big\{x : \sum_{p_0<p\leqslant n} f_p(R_{k(n)}(x), F) < \beta\Big\}$$

$$P\Big\{x : \sum_{p<p\leqslant n.} f_p(x, F) < \beta\Big\} + \frac{Z}{2} + o(1) \text{ as } n \to \infty.$$

Hence, as $n \to \infty$ the left hand side term converges to Z and the right hand side term converges to $Z/2$ which is a contradiction.

Therefore, $$\sum_p \frac{f'(p)\, r(F, p)}{p} < \infty.$$ (Q.E.D.)

Theorem 4 : *Let $g(m)$ be a multiplicative function and $F \epsilon \mathcal{P}$. Suppose that $D_F \geqslant 2$ and $(g(p^k)-1)\, r(F, p^k) \to 0$ as $p \to \infty$, for $k = 1, \ldots, D_F - 1$, $g(F(m)) \epsilon S^0$ if*

$$\sum_{|g(p)-1|<1} \frac{(g(p)-1)\, r(F, p)}{p}, \quad \sum_{|g(p)-1|<1} \frac{|g(p)-1|^2\, r(F, p)}{p} \quad \sum_{\substack{|g(p)-1|\geqslant 1 \\ g(p)\neq 0}} \frac{1}{p} \quad \ldots \quad (6.1)$$

converge. For a positive $g(F(m)) \epsilon S^0$ this condition is also necessary, if the distribution function $Q(c)$ of the function $g(F(m))$ is continuous for $c = 0$.

Proof: Use Theorem 1 and arguments similar to the proof of proposition 51 of Novoselov (1966, p. 251).

Remark: Most of the results in the monograph of Kubilius can be easily extended to the arithmetical functions whose domain of definition is $\{F(m), m = 1, 2, \ldots\}$ $(F \epsilon \mathcal{P})$, instead of the whole sequence of natural numbers. Using Kubilius methods, we can prove the following two theorems. We do not know whether they have been obtained previously by any one or not.

Theorem 5: *Let $f(m)$ be a strongly additive arithmetic function. Let*

$$A(n) = \sum_{k \leq n} \frac{f(p)}{p} \text{ and } B^2(n) = \sum_{p \leq n} \frac{f^2(p)}{p}.$$

If $B(n) \to \infty$ and if $\forall \varepsilon > 0$,

$$\frac{1}{B^2(n)} \sum_{\substack{p \leq n \\ |f(p)| > \varepsilon B(n)}} \frac{f^2(p)}{p} \to 0 \text{ as } n \to \infty,$$

then

$$\lim_{n \to \infty} \frac{1}{n} N_n \left\{ \max_{k \leq n} \frac{f_k(m) - A(k)}{B(n)} < x \right\} = \frac{2}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt, \; x > 0.$$

$$\lim_{n \to \infty} \frac{1}{n} N_n \left\{ \max_{k \leq n} \left| \frac{f_k(m) - A(k)}{B(n)} \right| < x \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{(u-2kx)^2}{2}\right) du, \; x > 0$$

where

$$f_k(m) = \sum_{\substack{p \leq k \\ p \mid m}} f(p).$$

Theorem 6: *Let $f(m)$ be a strongly additive arithmetical function and F be an integral polynomial with $F(m) > 0$ for $m = 1, 2, \ldots$. Let,*

$$A(n, F) = \sum_{p \leq n} \frac{f(p)\, r(F, p)}{p} \text{ and } B^2(n, F) = \sum_{p \leq n} \frac{\{f(p)\}^2 r(F, p)}{p}.$$

Suppose $B(n, F) \to \infty$ as $n \to \infty$ and $f(p) = o(B(p, F))$ as $p \to \infty$, then

$$\lim_{n \to \infty} \frac{1}{n} N_n \left\{ \max_{k \leq n} \frac{f_k(F(m)) - A(k, F)}{B(n, F)} < x \right\} = \frac{2}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt, \; x > 0.$$

and

$$\lim_{n \to \infty} \frac{1}{n} N_n \left\{ \max_{k \leq n} \left| \frac{f_k(F(m)) - A(k, F)}{B(n, F)} \right| < x \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left\{-\frac{(u-2kx)^2}{2}\right\} du, \; x > 0.$$

We need the following lemmas to prove the Theorem 5.

Notation:
$$f_p(m) = \begin{cases} f(p^k) & \text{if } p^k \parallel m \text{ for some } k \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

for $m \geq 1$.

Lemma 9 : *Let f be an additive arithmetic function such that $B(n) \to \infty$ as $n \to \infty$ and for each $\varepsilon > 0$,*

$$\frac{1}{B(n)^2} \sum_{|f(p)| > \varepsilon B(n)} \frac{f^2(p)}{p} \to 0 \text{ as } n \to \infty.$$

Then there exists a function $r = r(n)$ such that

$$\frac{\log r(n)}{\log n} \to 0, \quad \frac{B(r(n))}{B(n)} \to 1 \quad \text{and for each } \varepsilon > 0$$

$$\frac{1}{n} N_n \left\{ \max_{r \leqslant k \leqslant n} \left| \sum_{r < p \leqslant k} \left(f_p(m) - \frac{f(p)}{p} \right) \right| > \varepsilon B(n) \right\} \to 0$$

as $n \to \infty$.

Proof: Let $\delta_n = \dfrac{1}{n} N_n \left\{ \max\limits_{r \leqslant k \leqslant n} \left| \sum\limits_{r < p \leqslant k} \left(f_p(m) - \dfrac{f(p)}{p} \right) \right| > \varepsilon B(n) \right\}$.

Now, $$\delta_n \leqslant \frac{1}{\varepsilon^2 B(n)^2} \frac{1}{n} \sum_{m=1}^{n} \left(\sum_{r < p \leqslant n} \left| f_p(m) - \frac{f(p)}{p} \right| \right)^2$$

$$\leqslant \frac{2}{n\varepsilon^2 B(n)^2} \left(\sum_{m=1}^{n} \left(\sum_{r < p \leqslant n} |f_p(m)| \right)^2 + 2n \left(\sum_{r < p \leqslant n} \frac{|f(p)|}{p} \right)^2 \right).$$

But,

$$\sum_{m=1}^{n} \left(\sum_{r < p \leqslant n} |f_p(m)| \right)^2 = \sum_{m=1}^{n} \sum_{r < p \leqslant n} |f_p(m)|^2 + \sum_{\substack{p \neq q \\ r < p, q \leqslant n}} |f_p(m) f_q(m)|$$

$$\leqslant n \sum_{r < p \leqslant n} \frac{f(p)^2}{p} + n \left(\sum_{r < p \leqslant n} \frac{|f(p)|}{p} \right)^2.$$

Now, by hypothesis there exists $\varepsilon(n)$ s.t $\varepsilon(n) > 0$ for all n, $\varepsilon(n) \to 0$ as $n \to \infty$ and

$$\frac{1}{B^2(n)} \sum_{|f(p)| > \varepsilon(n) B(n)} \frac{f^2(p)}{p} < \varepsilon(n).$$

Put $r(n) = n^{\varepsilon(n)}$; it is easy to see that

$$\frac{B(r(n))}{B(n)} \to 1 \text{ as } n \to \infty.$$

Also we have

$$\frac{1}{B^2(n)} \left(\sum_{r < p \leqslant n} \frac{|f(p)|}{p} \right)^2 \leqslant \frac{2}{B^2(n)} \left(\sum_{\substack{r < p \leqslant n \\ |f(p)| \leqslant \varepsilon(n) B(n)}} \frac{|f(p)|}{p} \right)^2 + \frac{2}{B^2(n)} \left(\sum_{\substack{r < p \leqslant n \\ |f(p)| > \varepsilon(n) B(n)}} \frac{|f(p)|}{p} \right)^2$$

$$\leqslant 2\varepsilon(n)^2 (\log \varepsilon(n) + O(1))^2 + \frac{2}{B^2(n)} \left(\sum_{r < p \leqslant n} \frac{1}{p} \right) \sum_{\substack{r < p \leqslant n \\ |f(p)| > \varepsilon(n) B(n)}} \frac{f^2(p)}{p}$$

$$\leqslant 2\varepsilon(n)^2 (\log \varepsilon(n) + O(1))^2 + 2\varepsilon(n) \log \frac{1}{\varepsilon(n)} \to 0 \text{ as } n \to \infty, \text{ since } \varepsilon(n) \to 0.$$

Hence $\delta_n \to 0$ as $n \to \infty$. This completes the proof of Lemma 9.

Lemma 10 (Corollary 5.4, Parthasarathy, 1967, p. 230): *Let* $\{\xi_{ni}\}$, $i = 1, 2, \ldots, k_n$ *be a triangular sequence of real valued random variables satisfying the following conditions*

(i) *for each fixed* n, $\{\xi_{ni}\}$ *are independent,*

(ii) $M(\xi_{ni}) = 0, \quad V(\xi_{ni}) = f_{ni}, \sum_{i=1}^{k_n} f_{ni} = 1,$

where M *and* V *denote mean-value and variance respectively,*

(iii) *for each* $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} \int_{|u|>\varepsilon} u^2\, dF_{ni}(u) = 0,$$

where F_{ni} *is the distribution function of* ξ_{ni}.

Then,

$$\lim_{n\to\infty} P\left\{ \max_{1\leqslant k\leqslant k_n} |\xi_{n1}+\ldots+\xi_{nk}| < a \right\} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left[-\frac{(u-2ka)^2}{2}\right] du,$$

and

$$\lim_{n\to\infty} P\left\{ \max_{1\leqslant k\leqslant k_n} (\xi_{n1}+\ldots+\xi_{nk}) < a \right\} = \frac{2}{\sqrt{2\pi}} \int_{0}^{a} \exp\,(-u^2/2)du,\ a > 0.$$

Proof of Theorem 5: Let $r = r(n)$ be the function appearing in the proof of Lemma 9. We now consider the independent and discrete random variables $\xi_p (p \leqslant r)$, where ξ_p assumes the values $f(p)$ and 0 with probabilities $\frac{1}{p}$ and $1-\frac{1}{p}$ respectively.

It is easy to check that the triangular sequence

$$\frac{\xi_p - \dfrac{f(p)}{p}}{\sqrt{\sum\limits_{p\leqslant r(n)} \dfrac{f^2(p)}{p}\left(1-\dfrac{1}{p}\right)}},\quad p \leqslant r(n),\ n = 1, 2, \ldots$$

of random variables satisfies the conditions (i), (ii) and (iii) of Lemma 10. So for any $a > 0$

$$\lim_{n\to\infty} P\left\{ \max_{1\leqslant k\leqslant r(n)} \left| \frac{\sum\limits_{p<k} \xi_p - A(k)}{\sqrt{\sum\limits_{p\leqslant r(n)} \dfrac{f(p)^2}{p}\left(1-\dfrac{1}{p}\right)}} \right| < a \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left[-\frac{(u-2ka)^2}{2}\right] du = \eta(1, a), \quad \text{(say)}$$

and
$$\lim_{n\to\infty} P\left\{\max_{1\leqslant k\leqslant r(n)}\left(\frac{\sum_{p\leqslant k}\xi_p - A(k)}{\sum_{p\leqslant r(n)}\frac{f^2(p)}{p}\left(1-\frac{1}{p}\right)}\right) < a\right\}$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^a \exp(-u^2/2)du = \eta(2, a), \quad \text{(say)}$$

where
$$A(k) = \sum_{p\leqslant k}\frac{f(p)}{p}.$$

By using the results of chapter II of Kubilius (1964) and the facts $\frac{B(r(n))}{B(n)} \to 1$ and $\sum_{p\leqslant r}\frac{f^2(p)}{p^2} = o(B(r)^2)$, we get

$$\lim_{n\to\infty}\frac{1}{n}N_n\left\{\max_{k\leqslant r(n)}\left|\frac{f_k(m)-A(k)}{B(n)}\right| < a\right\} = \eta(1, a)$$

and
$$\lim_{n\to\infty}\frac{1}{n}\; N_n\left\{\max_{k\leqslant r(n)}(f_k(m)-A(k)) < aB(n)\right\} = \eta(2, a).$$

Now the theorem follows easily from Lemma 9.

Proof of Theorem 6 is same as above.

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