ON STRONG CONVERGENCE OF REGRESSION RANK STATISTICS

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SUMMARY. For independent but not necessarily identically distributed random variables, almost sure convergence of regression (simple linear) rank statistics is established. For independent and identically distributed random variables, weak as well as almost sure convergence of regression rank statistics to appropriate Wiener processes is studied, and a law of iterated logarithm is derived. These results are also extended to signed linear rank statistics.

1. Introduction

Let $\{X_i, i \geqslant 1\}$ be a sequence of independent random variables (r.v.'s) defined on a measure space $(\Omega, \mathcal{A}, P)$ with continuous distribution functions (d.f.) $\{F_i(x), i \geqslant 1\}$. Let $u(t) = 1$ or 0 according as $t \geqslant$ or < 0, and for a sample $X_1, \ldots, X_n$ of size n, let $R_{ni} = \sum_{j=1}^{n} u(X_i - X_j)$ be the rank of X_i among $X_1, \ldots, X_n$ for $i = 1, \ldots, n$. Let $\{c_i, i \geqslant 1\}$ be a sequence of known regression constants. We define a regression (simple linear) rank statistic T_n (see e.g., Hájek, 1962; 1968) by

$$T_n = \sum_{i=1}^{n} (c_i - \bar{c}_n) J_n(R_{ni}/(n+1)); \quad \bar{c}_n = n^{-1} \sum_{i=1}^{n} c_i, \; n \geqslant 1, \qquad \ldots (1.1)$$

where $c_1, \ldots, c_n$ are assumed to be not all equal, $J_n(i/(n+1)) = EJ(U_{ni}),\ i = 1, \ldots, n$, $U_{n1} \leqslant \ldots \leqslant U_{nn}$ are the ordered r.v.'s in a sample of size n from a rectangular $(0, 1)$ distribution, and $J(u)$, $0 < u < 1$ is a score function satisfying

$$\int_0^1 |J(u)|^r du < \infty \text{ for some } r > 2. \qquad \ldots (1.2)$$

Without any loss of generality, we let $\mu = \int_0^1 J(u)du = 0$ and define

$$A^2 = \int_0^1 J^2(u)du (>0), \quad A_n^2 = n^{-1} \sum_{i=1}^{n} J_n^2(i/(n+1)), \qquad \ldots (1.3)$$

$$C_n^2 = \sum_{i=1}^{n} (c_i - \bar{c}_n)^2 \text{ and } c_{ni}^* = (c_i - \bar{c}_n)/C_n, \quad i = 1, \ldots, n; n \geqslant 1. \qquad \ldots (1.4)$$

Also, we assume that (cf. Hájek, 1968) that

$$\max_{1 \leqslant i \leqslant n} |c_{ni}^*| = O(n^{-\frac{1}{2}}). \qquad \ldots (1.5)$$

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Let then

$$T_n^* = (n^{\frac{1}{2}} A_n C_n)^{-1} T_n = n^{-\frac{1}{2}} A_n^{-1} \sum_{i=1}^{n} c_{ni}^* J_n((n+1)^{-1} R_{ni}); \quad \dots \quad (1.6)$$

$$\bar{F}_n(x) = (n+1)^{-1} \sum_{i=1}^{n} F_i(x),\ F_n^*(x) = n^{-\frac{1}{2}} A_n^{-1} \sum_{i=1}^{n} c_{ni}^* F_i(x); \quad \dots \quad (1.7)$$

$$\tau_n^* = \int_{-\infty}^{\infty} J(\bar{F}_n(x)) dF_n^*(x), \quad n \geqslant 1. \quad \dots \quad (1.8)$$

For every positive integer n, we define $Q(n) = C_n^2$, and by linear interpolation, we let for $x \in (n, n+1)$, $Q(x) = (x-n)Q(n+1)+(n+1-x)Q(n)$, $n \geqslant 0$, $Q(0) = 0$. Then our second assumption on the regression constants is the following :

(i) $Q(x)$ is $\uparrow$ in $x : 0 \leqslant x < \infty$, $\lim_{x\to\infty} Q(x) = \infty$; ... (1.9)

(i) for every sequence $\{a_n\}$ of positive numbers for which $a_n \to 1$ as $n \to \infty$, $Q(na_n)/Q(n) \to 1$ as $n \to \infty$; ... (1.10)

(iii) $\liminf_{n\to\infty} n^{-1}Q(n) \geqslant C_0^2 > 0$, $\limsup_{n\to\infty} n^{-h}Q(n) \leqslant C^* < \infty$, ... (1.11)

where $h (\geqslant 1)$ is some positive number. Finally, let $[s]$ be the largest integer contained in s.

$$Q^{-1}(u) = \inf\{u : Q(x) \geqslant u\}, \quad \dots \quad (1.12)$$

$$V_n = \sum_{i=2}^{n} E[(T_i - T_{i-1})^2 | T_1, \dots, T_{i-1}],\ n \geqslant 2, \quad \dots \quad (1.13)$$

$$\tilde{T}_{V_n} = T_n,\ n = 1, 2, \dots,\quad \tilde{T}_0 = T_0 = 0, \quad \dots \quad (1.14)$$

and by linear interpolation complete the definition of $\tilde{T}_t$ for every $t \in (V_n, V_{n+1})$, $n \geqslant 1$. Then, we have the following theorems.

Theorem 1.1 : *Under the assumptions* (1.2), (1.5) *and*

$$n^{-1} \sum_{i=1}^{n} | J_n(i/(n+1)) - J(i/(n+1)) | = O(n^{-\frac{1}{2}}), \quad \dots \quad (1.15)$$

$$\lim_{n\to\infty} (T_n^* - \tau_n^*) = 0 \text{ a.s.} \quad \dots \quad (1.16)$$

Theorem 1.2 : *If* $J(u)$ *is non-decreasing in* u, (1.2), (1.5) *and* (1.11) *hold*,

$F_i = F$ *for all* $i \geqslant 1$, *and* $|J'(u)| \leqslant K[u(1-u)]^{-\frac{3}{2}+\delta_0}$, $0 < u < 1$, $\delta_0 > 0$,

then (a) *there is a standard Brownian motion* $\xi(t)$ *on* $[0, \infty)$ *such that*

$$\tilde{T}_t = \xi(t) + o(t \log\log t)^{\frac{1}{2}} \text{ a.s., as } t \to \infty, \quad \dots \quad (1.17)$$

and (b)

$$\limsup_{n\to\infty} T_n/(2A_n^2 C_n^2 \log\log C_n^2)^{\frac{1}{2}} = 1 \text{ a.s.}, \quad \dots \quad (1.18)$$

$$\liminf_{n\to\infty} T_n/(2A_n^2 C_n^2 \log\log C_n^2)^{\frac{1}{2}} = -1 \text{ a.s.} \quad \dots \quad (1.19)$$

The above theorems relate to the tail sequence $\{T_k,\ k \geqslant n\}$. For the complementary sequence $\{T_k : k \leqslant n\}$, in the light of the classical Donsker theorem (see Billingsley, 1968, p. 68), we have the following weak convergence theorem.

Consider the space $C[0, 1]$ of real continuous functions on the unit interval $I = \{t : 0 \leqslant t \leqslant 1\}$. For every $n(\geqslant 2)$, define a process $Z_n = \{Z_n(t),\ t \in I\}$ by

$$Z_n(t_{k,n}) = A_n^{-1}C_n^{-1}T_k, \qquad \ldots \quad (1.20)$$

where $t_{k,n} = A_k^2C_k^2/A_n^2C_n^2$ for $k = 0, 1, \ldots, n$; and by linear interpolation, complete the definition of $Z_n(t)$ for $t \in [t_{k,n}, t_{k+1,n}]$, $k = 0, 1, \ldots, n-1$. Also introduce the uniform topology

$$\rho(x, y) = \sup_{t \in I} |x(t)-y(t)|, \text{ where } x, y \in C[0, 1]. \qquad \ldots \quad (1.21)$$

Theorem 1.3 : *Under the assumptions of Theorem 1.2, $Z_n \to W$, in the uniform topology on $C[0, 1]$, where $W = \{W(t), t \in I\}$ is a standard Brownian motion.*

The proofs of the theorems are postponed to the next section. After the present paper was submitted, the authors became aware that Professor Hajek (1971) had an alternative proof of Theorem 1.1 in the particular case of two sample problem assuming J_n to be of bounded variation uniformly in n and $\int_0^1 J(u)\,du < \infty$. Hájek has employed a different kind of truncation than ours. We may also refer to Koul (1970) for a parallel version of Theorem 1.1 under essentially more restrictive assumptions.

The second theorem requires the identity of the F_i and some additional condition on the constants c_i, $i \geqslant 1$. However, it proves a much stronger result than Theorem 1.1. The last section is devoted to signed rank statistics (cf. Hušková, 1970), and results parallel to Theorems 1.1, 1.2 and 1.3 are sketched briefly.

We may mention one application of Theorems 1.1 and 1.2. Specify the d.f.'s $F_1, F_2, \ldots$ by

$$F_i(x) = F(x-\beta_0-\beta c_i), \quad i \geqslant 1, \qquad \ldots \quad (1.22)$$

where β is the regression coefficient, β_0 is a nuisance parameter, and c_i, $i \geqslant 1$, are known regression constants satisfying the assumptions made earlier in this section. Also it is assumed that F is absolutely continuous with respect to Lebesgue measure, having a continuous, positive and finite (a.e.) density $f(x)$ such that

$$\lim_{x \to \pm\infty} (d/dx)J(F(x)) \text{ is finite.} \qquad \ldots \quad (1.23)$$

Finally, assume that $J(u)$ is strictly increasing in $u : 0 < u < 1$, so that

$$B(F) = \int_{-\infty}^{\infty} (d/dx)J(F(x))\,dF(x) > 0. \qquad \ldots \quad (1.24)$$

For testing

$$H_0 : \beta = 0 \text{ against } H : \beta > 0, \qquad \ldots \quad (1.25)$$

a test procedure can be formulated along the lines of Darling and Robbins (1968). Let

$$N = \begin{cases} \text{first positive integer } n(\geqslant n_0) \text{ such that } T_n > d_n, \\ \infty, \quad \text{if no such } n \text{ occurs,} \end{cases} \qquad \ldots \quad (1.26)$$

where $\{d_n\}$ is some suitably chosen sequence of positive constants such that $d_n \to \infty$ as $n \to \infty$. The test procedure consists in rejecting H_0 if $N < \infty$ and accepting H_0 otherwise. Then Theorems 1.1 and 1.2 can be used to show that the test has power one and size arbitrarily small by choosing (i) n_0 sufficiently large and (ii) d_n increasing at a rate not slower than the denominator on the left hand side of (1.18).

2. Proofs of the theorems 1.1 and 1.2.

We first prove Theorem 1.1. Define

$$S_n(x) = (n+1)^{-1} \sum_{i=1}^{n} u(x-X_i),\ S_n^*(x) = n^{-1}A_n^{-1} \sum_{i=1}^{n} c_{ni}^* u(x-X_i), -\infty < x < \infty; \quad \dots \quad (2.1)$$

$$T_n^{**} = \int_{-\infty}^{\infty} J(S_n(x)) dS_n^*(x) = n^{-1}A_n^{-1} \sum_{i=1}^{n} c_{ni}^* J(R_{ni}/(n+1)). \quad \dots \quad (2.2)$$

One obtains under (1.5) and (1.15) (ensuring $A_n \to A (> 0)$ as $n \to \infty$) that

$$|T_n^* - T_n^{**}| \leqslant A_n^{-1} \left[\max_{1 \leqslant i \leqslant n} |c_{ni}^*|\right] \left\{ n^{-1} \sum_{i=1}^{n} \left| J_n\left(\frac{i}{n+1}\right) - J\left(\frac{i}{n+1}\right) \right| \right\} = O(n^{-\frac{1}{2}}). \quad \dots \quad (2.3)$$

Thus, it suffices to work with T_n^{**}. If we let $J_n(u) = J_n(i/(n+1))$ for $(i-1)/n < u \leqslant i/n$, $1 \leqslant i \leqslant n$, as (1.2) holds, $\lim_{n\to\infty} \int_0^1 |J_n(u)|^r du = \int_0^1 |J(u)|^r du$, $r = 1, 2$, and hence, for every $\varepsilon > 0$, there exist a $\delta (> 0)$ and an $n_0(\varepsilon)$, such that for all $n \geqslant n_0(\varepsilon)$,

$$\int_0^{\delta} + \int_{1-\delta}^{1} |J_n(u)|^r du < \varepsilon A^r (\Longrightarrow \int_0^{\delta} + \int_{1-\delta}^{1} |J(u)|^r du < \varepsilon A^r),\ r = 1, 2. \quad \dots \quad (2.4)$$

Further, $J(u)$ is continuous in the open interval (0, 1). Hence, $J(u)$ is uniformly continuous in u in every closed interval $[\eta, 1-\eta]$, $0 < \eta \leqslant \frac{1}{2}$. Hence, for every $\varepsilon > 0$, there exist $\delta_1, \delta_2, (0 < \delta_2 < \frac{1}{2}\delta_1, \delta_1 + \delta_2 = \delta)$, such that

$$\sup_{|v| < \delta_2} \sup_{\delta_1 < u < 1-\delta_1} |J(u+v) - J(u)| < \varepsilon. \quad \dots \quad (2.5)$$

Defining then $a_n = \sup\{x : \bar{F}_n(x) \leqslant \delta_1\}$ and $b_n = \inf\{x : \bar{F}_n(x) \geqslant 1-\delta_1\}$, one can write

$$T_n^{**} - \tau_n^* = I_{n1} + I_{n2} + I_{n3} + I_{n4}; \quad \dots \quad (2.6)$$

$$I_{n1} = -\left(\int_{-\infty}^{a_n} + \int_{b_n}^{\infty}\right) J(\bar{F}_n(x))\, dF_n^*(x); \quad \dots \quad (2.7)$$

$$I_{n2} = \left(\int_{-\infty}^{a_n} + \int_{b_n}^{\infty}\right) J(S_n(x))\, dS_n^*(x); \quad \dots \quad (2.8)$$

$$I_{n3} = \int_{a_n}^{b_n} [J(S_n(x)) - J(\bar{F}_n(x))]\, dS_n^*(x); \quad \dots \quad (2.9)$$

$$I_{n4} = \int_{a_n}^{b_n} J(\bar{F}_n(x)) d[S_n^*(x) - F_n^*(x)]. \quad \dots \quad (2.10)$$

By (1.5) and (1.7), $d|F_n^*(x)| \leq [O(1)]A_n^{-1}d\bar{F}_n(x)$, where $A_n \to A$ as $n \to \infty$. Hence, by (2.4), for large n, for every $\varepsilon(>0)$ there exists an $\varepsilon'(>0)$, such that

$$|I_{n1}| \leq A_n^{-1}[O(1)]\left\{\int_0^{\theta_1} + \int_{1-\theta_1}^{1} |J(u)|du\right\} \leq [O(1)]\varepsilon(A_n/A)^{-1} \leq \varepsilon'. \quad \dots \quad (2.11)$$

Again, noting that $d|S_n^*(x)| \leq [O(1)]A_n^{-1}dS_n(x)$ and $\sup_x |S_n(x)-\bar{F}_n(x)| \to 0$ a.s., as $n \to \infty$, we obtain from (2.1), (2.4) and (2.7) that

$$|I_{n2}| < \varepsilon' \text{ a.s., as } n \to \infty. \quad \dots \quad (2.12)$$

Again, using $\sup_x |S_n(x)-\bar{F}_n(x)| \to 0$ a.s. as $n \to \infty$ along with (2.5) and the definition of (a_n, b_n), we obtain by some straightforward computations that

$$|I_{n3}| < \varepsilon' \text{ a.s., as } n \to \infty. \quad \dots \quad (2.13)$$

Finally, $I_{n4} = n^{-1}\sum_{i=1}^{n}(Z_{ni}-EZ_{ni})$, where, $Z_{ni} = A_n^{-1}n^{\frac{1}{2}}c_{ni}^* J(\bar{F}_n(X_i))u_n(X_i)$, $1 \leq i \leq n$, $u_n(x) = 1$ or 0 according as $x\epsilon[a_n, b_n]$ or not. But, from (1.5), and the fact that $A_n \to A(>0)$ as $n \to \infty$, $A_n^{-1}n^{\frac{1}{2}}|c_{ni}^*| = O(1)$ for all $1 \leq i \leq n$. Now, putting $\gamma = 2+\delta$, $\delta > 0$ in (1.2), one gets,

$$n^{-1}\sum_{i=1}^{n} E|Z_{ni}|^{2+\delta} \leq [O(1)]n^{-1}\sum_{i=1}^{n} E|J(\bar{F}_n(X_i))|^{2+\delta}$$

$$= O(1)\int_0^1 |J(u)|^{2+\delta}du < \infty. \quad \dots \quad (2.14)$$

Hence, applying Theorem 3 of Sen (1970) along with the Borel-Cantelli lemma, it follows that

$$I_{n4} \to 0 \quad \text{a.s., as } n \to \infty. \quad \dots \quad (2.15)$$

Theorem 1.1 now follows from (2.6), (2.11)–(2.13) and (2.15).

Next we prove Theorem 1.2. Let $\mathscr{F}_n$ denote the σ-field generated by $R_n = (R_{n1}, \dots, R_{nn})$; $\mathscr{F}_n$ is obviously $\uparrow$ in $n(n \geq 1)$. We first prove the following.

Lemma 2.1 : *If $F_1 = \dots = F_n = F$ for all $n \geq 1$, then $\{T_n, \mathscr{F}_n; n \geq 1\}$ is a martingale sequence.*

Proof: Note that

$$E(T_{n+1}|\mathscr{F}_n) = (c_{n+1}-\bar{c}_{n+1})E(J_{n+1}(R_{n+1n+1}/(n+2))|\mathscr{F}_n)$$

$$+\sum_{i+1}^{n}(c_i-\bar{c}_{n+1})E(J_{n+1}(R_{n+1i}/(n+2))|\mathscr{F}_n). \quad \dots \quad (2.16)$$

But, $$E[J_{n+1}(R_{n+1n+1}/(n+2))|\mathscr{F}_n] = (n+1)^{-1}\sum_{i=1}^{n+1}J_{n+1}(i/n+2)) = \mu.$$

Also, $$E[J_{n+1}(R_{n+1i}/(n+2))|\mathscr{F}_n] = (n+1)^{-1}R_{ni}J_{n+1}((R_{ni}+1)/(n+2))$$
$$+\{1-(n+1)^{-1}R_{ni}\}J_{n+1}(R_{ni}/(n+2))$$
$$= J_n(R_{ni}/(n+1)),\ 1 \leq i \leq n,$$

from the following well-known recursion relation among the expected values of functions of order statistics :

$$\left(\frac{i}{n+1}\right) J_{n+1}((i+1)/(n+2))+\left(1-\frac{i}{n+1}\right) J_{n+1}\left(\frac{i}{n+2}\right) = J_n\left(\frac{i}{n+1}\right),\ 1 \leq i \leq n.$$

Thus for every $n \geqslant 1$,

$$E(T_{n+1}|\mathscr{F}_n) = (c_{n+1}-\bar{c}_{n+1})\mu + \sum_{i=1}^{n} (c_i - c_{n+1}) J_n(R_{ni}/(n+1))$$

$$= (c_{n+1}-\bar{c}_{n+1})\mu + n(\bar{c}_n - \bar{c}_{n+1})\mu + T_n = T_n. \quad \ldots \quad (2.17)$$

Hence, $\{(T_n, \mathscr{F}_n), (n \geqslant 1)\}$ is a martingale sequence.

To prove the theorem, we need now only verify the conditions of Theorem 4.4 of Strassen (1967) (see also his Corollary 4.5), which gives us access to the law of iterated logarithm via the extension of the Kolmogorov-Petrovski-Erdos criterion for martingales.

Let $\quad Z_1 = T_1 = 0, \quad Z_n = T_n - T_{n-1} = (c_n - \bar{c}_{n-1}) J_n(R_{nn}/(n+1)) + \phi_n,$

$$\phi_n = \sum_{i=1}^{n-1} (c_i - \bar{c}_{n-1}). \quad [J_n(R_{ni}/(n+1)) - J_{n-1}(R_{n-1i}/n)], \quad n \geqslant 2.$$

Hence, $\quad E(Z_2^2) = \frac{1}{4}(c_1-c_2)^2 \left\{ J_2\left(\frac{2}{3}\right) - J_2\left(\frac{1}{3}\right) \right\}^2 > 0$ is $c_1 \neq c_2$;

otherwise show $E(Z_i^2) > 0$, where i is the first positive integer ($\geqslant 2$) for which $c_1, \ldots, c_i$ are not all equal. Also.

$$E(Z_n^2|\mathscr{F}_{n-1}) = (c_i - \bar{c}_{i-1})^2 E(J_i^2(R_{ii}/(i+1))|\mathscr{F}_{i-1}) + E(\phi_i^2|\mathscr{F}_{i-1})$$
$$+ 2(c_i - \bar{c}_{i-1}) E(\phi_i J_i(R_{ii}/(i+1))|\mathscr{F}_{i-1}),$$

and hence from (1.13),

$$V_n = \sum_{i=2}^{n} E(Z_i^2|F_{i-1}) = W_{n1} + W_{n2} + W_{n3} \text{ (say)}, \quad \ldots \quad (2.18)$$

where

$$W_{n1} = \sum_{i=2}^{n} (c_i - \bar{c}_{i-1})^2 E(J_i^2(R_{ii}/(i+1))|\mathscr{F}_{i-1}), \quad \ldots \quad (2.19)$$

$$W_{n2} = 2\sum_{i=2}^{n} (c_i - \bar{c}_{i-1}) E(\phi_i J_i(R_{ii}/(i+1))|\mathscr{F}_{i-1}), \quad \ldots \quad (2.20)$$

$$W_{n3} = \sum_{i=2}^{n} E(\phi_i^2|\mathscr{F}_{i-1}). \quad \ldots \quad (2.21)$$

Now, $\quad E(J_i^2(R_{ii}/(i+1))|\mathscr{F}_{i-1}) = E(J_i^2(R_{ii}/(i+1))) = A_i^2, \quad i \geqslant 1.$

Hence, $\quad W_{n1} = \sum_{i=1}^{n} (c_i - \bar{c}_{i-1})^2 A_i^2.$

Noting that

$$C_n^2 = \sum_{i=1}^{n} (c_i - \bar{c}_n)^2 = \sum_{i=2}^{n} Y_i^2 \text{ and } \sum_{i=2}^{n} (c_i - \bar{c}_{i-1})^2 = \sum_{i=2}^{n} (i/(i-1)) Y_i^2,$$

where $\quad Y_i = (c_1 + \ldots + c_{i-1} - (i-1)c_i)/[i(i-1)]^{\frac{1}{2}}, \; i = 2, 3, \ldots, n,$

one gets $\quad C_n^{-2} \sum_{i=2}^{n} (c_i - \bar{c}_{i-1})^2 A_i^2 \to A^2$ as $n \to \infty$,

i.e., $\quad C_n^{-2} W_{n1} \to A^2$ as $n \to \infty. \quad \ldots \quad (2.22)$

Next, we shall show that there exists an $\eta > 0$, such that

$$C_n^{-2} W_{n2} = O(n^{-\eta}) \text{ a.s., as } n \to \infty. \quad \ldots \quad (2.23)$$

In passing, we may note that $V_n \geqslant W_{n1}+W_{n2}$, and $C_n^2 \to \infty$ as $n \to \infty$, so that by (2.22) and (2.23),

$$\lim_{n\to\infty} V_n = \infty \text{ a.s.} \quad \dots \quad (2.24)$$

To prove (2.23), first note that

$$E(\phi_n J_n(R_{nn}/(n+1)) \mid \mathcal{F}_{n-1}) = \sum_{j=1}^{n-1} (c_j - \bar{c}_{n-1})[n^{-1} \sum_{i=1}^{R_{n-1j}} J_n(i/(n+1))]$$
$$\cdot [J_n(R_{n-1j}+1)/(n+1)) - J_n(R_{n-1j}/(n+1))]. \quad \dots \quad (2.25)$$

Hence, one can write

$$2(c_n - \bar{c}_{n-1}) E(\phi_n J_n(R_{nn}/(n+1)) \mid \mathcal{F}_{n-1}) = 2A_n C_{n-1} n^{-1}(c_n - \bar{c}_{n-1}) \sum_{j=1}^{n-1} d_{n-1j} g_{n-1j}(R_{n-1j}) \quad \dots \quad (2.26)$$

where $\quad d_{n-1j} = n^{\frac{1}{2}} c^{*}_{n-1j}$ and $\max_{1 \leqslant j \leqslant n-1} |d_{n-1j}| = O(1)$, [by (1.5)], $\quad \dots \quad (2.27)$

$$g_{n-1}(j) = A_n^{-1}[n^{-1} \sum_{i=1}^{j} J_n(i/(n+1))][J_n((j+1)/(n+1)) - J_n(j/(n+1))], \quad 1 \leqslant j \leqslant n. \quad \dots \quad (2.28)$$

By (2.27), $\sum_{j=1}^{n-1} d_{n-1j} = 0$, $\sum_{j=1}^{n-1} d^2_{n-1j} = n$, and for every $r > 2$,

$$\sum_{j=1}^{n-1} |d_{n-1j}|^r \leqslant \Big(\max_{1 \leqslant j \leqslant n-1} |d_{n-1j}|^{r-2}\Big) \sum_{j=1}^{n-1} d^2_{n-1j} = O(n). \quad \dots \quad (2.29)$$

Also, taking $r = 2+\delta$, $\delta > 0$, it follows from (1.2) that

$$\int_0^1 J^2(u)[\log(1+|J(u)|)]^{2+\delta} du < \infty \Longrightarrow \int_0^1 |J(u)| \{u(1-u)\}^{-\frac{1}{2}} du < \infty$$

(see proposition 1 of Hoeffding, 1968). Hence, noting that $J(u)$ is $\uparrow$ in $u: 0 < u < 1$, one gets that

$$n^{-1} \sum_{i=1}^{n} \Big| J\Big(\frac{i}{n+1}\Big)\Big| \Big\{\frac{i(n+1-i)}{(n+1)^2}\Big\}^{-1/2} \leqslant K_1 \int_0^1 |J(u)| \{u(1-u)\}^{-\frac{1}{2}} du < \infty,$$

where $K_1 < \infty$. The last equation in turn implies that there exists a finite K such that

$$\left.\begin{aligned} n^{-1} \sum_{i=1}^{j} \Big| J\Big(\frac{i}{n+1}\Big)\Big| &\leqslant K \Big\{\frac{j(n+1-j)}{(n+1)^2}\Big\}^{1/2}, \; 1 \leqslant j \leqslant \Big[\frac{n+1}{2}\Big], \\ n^{-1} \sum_{i=j+1}^{n} \Big| J\Big(\frac{i}{n+1}\Big)\Big| &\leqslant K \Big\{\frac{j(n+1-j)}{(n+1)^2}\Big\}^{1/2}, \; \Big[\frac{n+1}{2}\Big] \leqslant j \leqslant n. \end{aligned}\right\} \quad \dots \quad (2.30)$$

One can also write

$$g_{n-1}(j) = \Big[-n^{-1} \sum_{i=j+1}^{n} J_n(i/(n+1))\Big] \Big[J_n\Big(\frac{j+1}{n+1}\Big) - J_n\Big(\frac{j}{n+1}\Big)\Big], \; 1 \leqslant j \leqslant n-1,$$

Write

$$J_n(i/(n+1)) = J\left(\frac{i}{n+1}\right)+J_n\left(\frac{i}{n+1}\right)-J\left(\frac{i}{n+1}\right) \ (1 \leqslant i \leqslant n).$$

Proceeding as in Section 10.5 of Puri and Sen (1971), one can show that under condition (iv), as $n \to \infty$,

$$\sup_{1 \leqslant i \leqslant n}\left| J_n\left(\frac{i}{n+1}\right)-J\left(\frac{i}{n+1}\right)\right| = O(n^{-\frac{1}{2}-\gamma}), \quad \gamma > 0.$$

Also from (iv) $\left| J\left(\frac{j+1}{n+1}\right)-J\left(\frac{j}{n+1}\right)\right| \leqslant K \int_{j/(n+1)}^{(j+1)/(n+1)} [u(1-u)]^{-\frac{3}{2}+\delta_0} du = O(n^{-\frac{1}{2}-\delta_0})$

for all $1 \leqslant j \leqslant n-1$.

Thus, for large n,

$$g_{n-1}(j) = \left[n^{-1}\sum_{i=1}^{j} J\left(\frac{i}{n+1}\right)\right]\left[J\left(\frac{j+1}{n+1}\right)-J\left(\frac{j}{n+1}\right)\right]+O(n^{-\gamma-\delta_0^*})$$

$$= \left[-n^{-1}\sum_{i=j+1}^{n} J\left(\frac{i}{n+1}\right)\right]\left[J\left(\frac{j+1}{n+1}\right)-J\left(\frac{j}{n+1}\right)\right]+O(n^{-\gamma-\delta_0^*}), \quad \ldots (2.31)$$

where $\delta_0^* = \min(\frac{1}{2}, \delta_0)$. Since (1.2) implies that $|J(u)| \leqslant K[u(1-u)]^{-\frac{1}{2}+\delta^*}$, where δ^* can be so selected that $0 < \delta^* < \delta_0^*$, we have $|J(j/(n+1))| \leqslant K[j(n+1-j)/(n+1)^2]^{-\frac{1}{2}+\delta^*}$, $1 \leqslant j \leqslant n$. Using this along with (2.30) and (2.31) we have

$$\sup_{j \leqslant [n^{2/3}]}\left(\text{or} \sup_{j \geqslant n-[n^{2/3}]}\right) |g_{n-1}(j)| = O(n^{-\gamma_0}), \quad \gamma_0 > 0. \quad \ldots (2.32)$$

Also, for $[n^{2/3}] < j < n-[n^{2/3}]$, by assumption (iv),

$$|J((j+1)/(n+1))-J(j/(n+1))| = O(n^{-\frac{1}{2}(\delta_0)/3}). \quad \ldots (2.33)$$

Thus, from (2.32) and (2.33), we have

$$\sup_{1 \leqslant j \leqslant n-1} |g_{n-1}(j)| = O(n^{-\gamma_0}), \quad \text{for some } \gamma_0 > 0. \quad \ldots (2.34)$$

Now, the vector $R_{n-1} = (R_{n-11}, \ldots, R_{n-1n-1})$ can assume all possible permutations of $(1, \ldots, n-1)$ with the common probability $[(n-1)!]^{-1}$. Also

$$\sum_{j=1}^{n-1} d_{n-1j}\, g_{n-1}(R_{n-1j}) = \sum_{j=1}^{n-1} d_{n-1j}\, g_{n-1}^0(R_{n-1j}) \quad \ldots (2.35)$$

where $g_{n-1}^0(j) = g_{n-1}(j)-(n-1)^{-1}\sum_{i=1}^{n-1} g_{n-1}(i)$, $1 \leqslant j \leqslant n-1$. Hence for any positive integer k,

$$E\left\{\sum_{j=1}^{n-1} d_{n-1j}\, g_{n-1}^0(R_{n-1j})\right\}^{2k} = \sum_{m=1}^{2k} \Sigma_m^* \, (n-1)^{-[m]} D_n(e_1, \ldots, e_m) G_n(e_1, \ldots, e_m), \quad \ldots (2.36)$$

where the summation Σ_m^* extends over all $e_i \geq 1 (1 \leq i \leq m)$ for which $\sum_{i=1}^{m} e_i = 2k$, $(n-1)^{-[m]} = [(n-1)\ldots(n-m)]^{-1} = ((n-1)^{[m]})^{-1}$, and

$$D_n(e_1, \ldots, e_m) = \sum_{1 \leq j_1 \neq \cdots \neq j_m \leq n-1} \cdots \sum d_{n-1j_1}^{e_1} \cdots d_{n-1j_m}^{e_m}. \quad \ldots \quad (2.37)$$

$$G_n(e_1, \ldots, e_m) = \sum_{1 \leq j_1 \neq \cdots \neq j_m \leq n-1} \cdots \sum g_{n-1}^{0 e_1}(j_1) \cdots g_{n-1}^{0 e_m}(j_m). \quad \ldots \quad (2.38)$$

Using (2.27) and (2.29), one can show that $D_n(e_1, \ldots, e_m) = O(n^m)$ for $m \leq k$ and $O(n^k)$ for $m < k$ (see e.g., Puri and Sen, 1971, p. 74). Also, we shall show that

$$\sum_{j=1}^{n-1} |g_{n-1}(j)| = A_n^2 \leq 2A^2 \quad \ldots \quad (2.39)$$

To prove (2.39) note that since $J_n(j/(n+1))$ is ↑ in j, and $\sum_{j=1}^{n} J_n(j/(n+1)) = 0$, $\sum_{j=1}^{r} J_n(j/(n+1)) \leq 0$ for every $r \leq n$.

Thus,
$$\sum_{j=1}^{n-1} |g_{n-1}(j)| = -n^{-1} \sum_{j=1}^{n-1} \sum_{i=1}^{j} J_n\left(\frac{i}{n+1}\right)\left[J_n\left(\frac{j+1}{n+1}\right) - J_n\left(\frac{j}{n+1}\right)\right]$$
$$= -n^{-1} \sum_{i=1}^{n-1} J_n\left(\frac{i}{n+1}\right) \sum_{j=i}^{n-1} \left[J_n\left(\frac{j+1}{n+1}\right) - J_n\left(\frac{j}{n+1}\right)\right]$$
$$= -n^{-1} \sum_{i=1}^{n-1} J_n\left(\frac{i}{n+1}\right)\left[J_n\left(\frac{n}{n+1}\right) - J_n\left(\frac{i}{n+1}\right)\right] = A_n^2 \leq 2A^2.$$

Now, from (2.28), (2.34) and (2.39), one can show that $G_n(e_1, \ldots, e_m) = O(n^{-(2k-m)\gamma_0}) \leq O(n^{-k\gamma_0})$ for all $m \leq k$, and $= O(n^{-k\gamma_0})$ for all $m > k$. Thus, $(n-1)^{-[m]} D_n(e_1, \ldots, e_m) G_n(e_1, \ldots, e_m) = O(n^{-k\gamma_0})$ for all $e_1, \ldots, e_m$ satisfying $\sum_{i=1}^{m} e_i = 2k$, $e_i \geq 1$, $i = 1, \ldots, m$, $1 \leq m \leq 2k$. Now, let $\gamma_0 = \gamma_{01} + \gamma_{02}$ where $\gamma_{0i} > 0 (i = 1, 2)$, and select k such that $k\gamma_{02} = 1 + \gamma'$, $\gamma' > 0$. Then

$$P\left\{\left|\sum_{j=1}^{n-1} d_{n-1j} g_{n-1}(R_{n-1j})\right| > n^{-k\gamma_{01}}\right\} \leq n^{k\gamma_{01}} E\left\{\sum_{j=1}^{n-1} d_{n-1j} g_{n-1}(R_{n-1j})\right\}^{2k}$$
$$= O(n^{-k\gamma_{02}}) = O(n^{-1-\gamma'}), \quad \ldots \quad (2.40)$$

$\gamma' > 0$, and hence, by the Borel-Cantelli lemma

$$\left|\sum_{j=1}^{n-1} d_{n-1j} g_{n-1}(R_{n-1j})\right| = O(n^{-k\gamma_{01}}) \text{ a.s., as } n \to \infty. \quad \ldots \quad (2.41)$$

Also, $\left|\sum_{i=2}^{n} i^{-1-\gamma_0} C_{i-1}(c_i - \bar{c}_{i-1}) A_i\right| \leq \left(\sum_{i=2}^{n} i^{-1-2\gamma_0} C_{i-1}^2\right)^{\frac{1}{2}} \left(\sum_{i=2}^{n} (c_i - \bar{c}_{i-1})^2 A_i^2\right)^{\frac{1}{2}} \leq C_n O(n^{-\gamma_0}) \left\{\sum_{i=2}^{n} (c_i - \bar{c}_{i-1})^2 A_i^2\right\}^{\frac{1}{2}}$, using the fact that C_n is ↑ in n. Hence, (2.23) follows from (2.20), (2.22), (2.39) and the above inequality.

Finally, we prove that as $n\to\infty$, $C_n^{-2}W_{n3}\to 0$ a.s. Note that by definition, $\phi_n^2 = \sum_{1\leqslant i,j\leqslant n-1} (c_i-\bar{c}_{n-1})(c_j-\bar{c}_{n-1})[J_n(R_{ni}/(n+1))-J_{n-1}(R_{n-1i}/n)](J_n(R_{nj}/(n+1))-J_{n-1}(R_{n-1j}/n)]$. Now, $E[\{J_n(R_{ni}/(n+1))-J_{n-1}(R_{n-1i}/n)\}^2|\mathscr{F}_{n-1}] = n^{-1}(n-R_{n-1i})[J_n(R_{n-1i}/(n+1))-J_{n-1}(R_{n-1i}/n)]^2+n^{-1}R_{n-1i}[J_n((R_{n-1i}+1)/(n+1))-J_{n-1}(R_{n-1i}/n)]^2$. Using the identity that $J_{n-1}(i/n) = n^{-1}(n-i)J_n(i/(n+1))+n^{-1}iJ_n((i+1)/(n+1))$, $i = 1, \ldots, n-1$, one gets on simplifications that $E([J_n(R_{ni}/(n+1))-J_{n-1}(R_{n-1i}/n)]^2|\mathscr{F}_{n-1}) = n^{-2}R_{n-1i}(n-R_{n-1i})[J_n((R_{n-1i}+1)/(n+1))-J_n(R_{n-1i}/(n+1))]^2$. Again, for $R_{n-1i} < R_{n-1j}$, $E([J_n(R_{ni}/(n+1))-J_{n-1}(R_{n-1i}/n)][J_n(R_{nj}/(n+1))-J_{n-1}(R_{n-1j}/n)]|\mathscr{F}_{n-1}) = n^{-1}R_{n-1i}[J_n((R_{n-1i}+1)/(n+1))-J_{n-1}(R_{n-1i}/n)][J_n((R_{n-1j}+1)/(n+1))-J_{n-1}(R_{n-1j}/n)]+n^{-1}(R_{n-1j}-R_{n-1i})[J_n(R_{n-1i}/(n+1))-J_{n-1}(R_{n-1i}/n)][J_n((R_{n-1j}+1)/(n+1))-J_{n-1}(R_{n-1j}/n)]+n^{-1}(n-R_{n-1j})[J_n(R_{n-1i}/(n+1))-J_{n-1}(R_{n-1i}/n)][J_n(R_{n-1j}/(n+1))-J_{n-1}(R_{n-1j}/n)]$ which simplifies to $n^{-2}R_{n-1i}(n-R_{n-1j})[J_n((R_{n-1i}+1)/(n+1))-J_n(R_{n-1i}/(n+1))][J_n((R_{n-1j}+1)/(n+1))-J_n(R_{n-1j}/(n+1))]$ (by using the same identity among the expected order statistics). A similar case holds with $R_{n-1i} > R_{n-1j}$. Thus we get

$$\begin{aligned}E([J_n(R_{ni}/(n+1))-J_{n-1}(R_{n-1i}/n)][J_n(R_{nj}/(n+1))-J_{n-1}(R_{n-1j}/n)]|\mathscr{F}_{n-1})\\ = n^{-2}[\min(R_{n-1i}, R_{n-1j})][n-\max(R_{n-1i}, R_{n-1j})]\\ [J_n((R_{n-1i}+1)/(n+1))-J_n(R_{n-1i}/(n+1))]\\ [J_n((R_{n-1j}+1)/(n+1))-J_n(R_{n-1j}/(n+1))].\end{aligned}$$

Hence,
$$E(\phi_n^2|\mathscr{F}_{n-1}) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(c_i-\bar{c}_{n-1})(c_j-\bar{c}_{n-1})g_n^*(R_{n-1i}, R_{n-1j}), \qquad \ldots \quad (2.42)$$

where
$$\begin{aligned}g_n^*(i,j) = n^{-2}[\min(i,j)][n-\max(i,j)][J_n((i+1)/(n+1))\\ -J_n(i/(n+1))][J_n((j+1)(n+1))-J_n(j/(n+1))]. \qquad \ldots \quad (2.43)\end{aligned}$$

Note that by virtue of condition (iv), we have as in after (2.30), $J_n((i+1)/(n+1))-J_n(i/(n+1)) = [J((i+1)/(n+1))-J(i/(n+1))]+\{[J_n((i+1)/(n+1))-J((i+1(n+1))]-[J_n(i/(n+1))-J(i/(n+1))]\} = [J((i+1)/(n+1))-J(i/(n+1))]+O(n^{-1-\gamma})$, for all $1\leqslant i\leqslant n$, where $\gamma>0$. Therefore, by the C_2-inequality,

$$[J_n((i+1)/(n+1))-J(i/(n+1))]^2 \leqslant 2[J((i+1)/(n+1))-J(i/(n+1))]^2+O(n^{-1-2\gamma}),$$

for all $1\leqslant i\leqslant n$. Since $R_{n-1} = (R_{n-11}, \ldots, R_{n-1})'$ takes on each permutation of $(1, \ldots, n-1)$ with the common probability $[(n-1)!]^{-1}$, it is easy to show that $C_{n-1}^{-2}E\{E(\phi_n^2|\mathscr{F}_{n-1})\} = O(n^{-1-2\gamma})$. On the other hand, by using (2.42) and proceeding as in (2.34) through (2.40), it follows that for every (fixed) positive integer k, $C_{n-1}^{-2k}E\{E(\phi_n^2|\mathscr{F}_{n-1})^k\} = O(n^{-k-2k\gamma})$. Consequently, for some $C>0$,

$$P\{C_{n-1}^{-2}E(\phi_n^2|\mathscr{F}_{n-1}) \geqslant C_n^{-1-\gamma}\} \leqslant C^{-k}n^{k+k\gamma}[O(n^{-k-2k\gamma})] = O(n^{-k\gamma}). \qquad \ldots \quad (2.44)$$

So that, if we choose k such that $k\gamma > 1$, we obtain from (2.44) that as $n\to\infty$,

$$C_{n-1}^{-2}E(\phi_n^2|\mathscr{F}_{n-1}) < Cn^{-1-\gamma} \quad \text{a.s.} \qquad \ldots \quad (2.45)$$

Finally, by (1.11) and (2.21),

$$C_n^{-2}W_{n3} = \sum_{l=2}^{n}(\phi_l^2/C_l^2)(C_l^2/C_n^2). \qquad \ldots \quad (2.46)$$

where $C_i^2 \geq C_{i-1}^2$. Since $\sum_{n\geq 1} n^{-1-\gamma} < \infty$, $0 \leq C_i^2/C_n^2 \leq 1$, $\forall\ 2 \leq i \leq n$, and $C_n^2 \to \infty$ as $n \to \infty$, by (2.45) and (2.46), we obtain that

$$C_n^{-2} W_{n2} \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty. \qquad \ldots \quad (2.47)$$

Thus, by (2.18) through (2.23) and (2.47), we obtain that as $n \to \infty$,

$$C_n^{-2} A^{-2} V_n \to 1, \quad \text{a.s.} \quad \text{as } n \to \infty. \qquad \ldots \quad (2.48)$$

Now, for every $t > 1$, we define

$$f(t) = \max\{1, t(\log\log t)^2/(\log t)^4\}, \qquad \ldots \quad (2.49)$$

so that $f(t)$ is $\uparrow$ is t but $t^{-1}f(t)$ is $\downarrow$ in t. It follows from (2.48) that for every $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$, such that for $n \geq n_0(\varepsilon)$,

$$1-\varepsilon \leq C_n^{-2} V_n A^{-2} \leq 1+\varepsilon \quad \text{a.s.} \qquad \ldots \quad (2.50)$$

Thus, from (1.11), (2.49) and (2.50), we obtain that as $n \to \infty$,

$$\begin{aligned} f(V_n) &\geq [(1-\varepsilon)A^2 C_n^2 (\log\log[(1-\varepsilon)A^2 n C_0^2])^2]/[\log(1+\varepsilon)+2\log A+h\log n]^4 \\ &\geq K_\varepsilon\, n(\log\log n)^2/(\log n)^4 \ \text{a.s.}, \end{aligned} \qquad \ldots \quad (2.51)$$

where $K_\varepsilon (> 0)$ depends on ε. On the other hand,

$$\begin{aligned} Z_n &\leq |c_n - \bar{c}_{n-1}|\, |J_n(n/(n+1))| + |\phi_n| \\ &\leq |c_n - \bar{c}_{n-1}|\, |J_n(n/(n+1))| + \max_{1\leq i\leq n-1} |c_i - \bar{c}_{n-1}|\, |J_n(n/(n+1)) - J_n(1/(n+1))|. \end{aligned}$$

But, $|J_n(n/(n+1))| = |EJ(U_{nn})| \leq \{E|J^r(U_{nn})|\}^{\frac{1}{r}} = [n\int_0^1 |J(u)|^r u^{n-1} du]^{1/r} = O(n^{1/r})$, $n \geq 2$, where $r > 2$. A similar bound holds for $|J_n(1/(n+1))|$. Using (1.5), one gets then $|Z_n| \leq C_n[O(n^{-1})][O(n^{1/r})] = C_n[O(n^{-(r-2)/2r})]$. Let then $A_m = \{\omega\epsilon\Omega : V_n(\omega) \geq (1-\varepsilon)A^2C_n^2,\ \forall n \geq m\}$, for every $m \geq 1$. Noting that both V_n and $f(V_n)$ are non-decreasing (a.e.), we obtain from (2.51) and the above bound for $|Z_n|$, that as $n \to \infty$,

$$P\{Z_n^2 > f(V_n) \mid \mathscr{F}_{n-1}(\omega)\} = 0\ \forall\ \omega\, \epsilon A_n. \qquad \ldots \quad (2.52)$$

Thus, if $\omega \epsilon A_n$,

$$\sum_{n\geq 2} [f(V_n)]^{-1} \int_{x^2 > f(V_n)} x^2 dP\{Z_n \leq x \mid \mathscr{F}_{n-1}\} < \infty \quad \text{a.s.} \qquad \ldots \quad (2.53)$$

Hence, the conditions of Theorem 4.4 of Strassen (1967) are all satisfied, and (1.17) follows. Finally, (1.18) and (1.19) are direct consequences of (1.17). Q.E.D.

Consider now the proof of Theorem 1.3. Using Lemma 2.1, and a recent martingale functional central limit theorem by Brown (1971), it suffices to show that

$$V_n/\text{var}(T_n) \to 1 \text{ in probability as } n \to \infty, \qquad \ldots \quad (2.54)$$

and the following condition holds :

$$(\text{var}\, T_n)^{-1} \sum_{j=2}^{n} \int_{x^2 > \varepsilon \,\text{var}\, T_n} x^2 dP\{Z_j \leq x\} \to 0 \text{ as } n \to \infty. \qquad \ldots \quad (2.55)$$

(2.54) is an immediate consequence of (2.50) and the fact that $\text{var}(T_n) = A_n^2 C_n^2$, where $A_n^2 \to A^2$ as $n \to \infty$. Also, by (2.49), (2.52) and the fact that var $(T_n) \to \infty$ as $n \to \infty$, (2.55) follows.

3. Signed rank regression statistics

Let $\{X_i, i \geqslant 1\}$ be a sequence of independent r.v.'s with d.f.'s $\{F_i(x), i \geqslant 1\}$ with each F_i absolutely continuous with respect to Lebesgue measure and symmetric about 0 i.e., $F_i(x)+F_i(-x) = 1$ for all real x and $i \geqslant 1$. Let $R_{ni}^+ = \sum_{j=1}^{n} u(|X_i|-|X_j|)$, $1 \leqslant i \leqslant n (\geqslant 1)$ and let $\widetilde{T}_n = \sum_{i=1}^{n} c_i J_n^*(R_{ni}^+/(n+1))$ sgn X_i, where sgn $u = 1$, 0 or -1 according as u is $<$, $=$ or <0; $J_n^*(i/(n+1)) = EJ^*(U_{ni})$, $U_{n1} \leqslant \ldots \leqslant U_{nn}$ as defined in Section 1, and $J^*(u) = J((1+u)/2)$, $0 < u < 1$. It is assumed that $J(u)+J(1-u) = 0$ for all $0 < u < 1$. Also, let $A_n^{*2} = n^{-1} \sum_{i=1}^{n} [J_n^*(i/(n+1))]^2$, $\bar{c}_{ni} = \left(\sum_{i=1}^{n} c_i^2\right)^{-\frac{1}{2}} c_i$, $i = 1, \ldots, n$ and $\widetilde{T}_n^* = n^{-\frac{1}{2}} A_n^{*-1} \sum_{i=1}^{n} \bar{c}_{ni}$ sgn $X_i J_n^*(R_{ni}^+/(n+1))$. Then, we have the following two theorems whose proofs are omitted because of their essential similarity with the proofs of Theorems 1.1 and 1.2.

Theorem 3.1 : *If* $J \epsilon L_r$ *for some* $r > 2$ *and* $\max_{1 \leqslant i \leqslant n} |\bar{c}_{ni}| = O(n^{-\frac{1}{2}})$, *then*

$$\lim_{n \to \infty} (\widetilde{T}_n^* - \tau_n^*) = 0 \quad a.s., \qquad \ldots (3.1)$$

where $\tau_n^* = n^{-\frac{1}{2}} A_n^{*-1} \sum_{i=1}^{n} c_{ni} \int_{-\infty}^{\infty} sgn\, x J^*(\bar{H}_n(|x|) dF_i(x)$, $H_n(|x|) = n^{-1} \sum_{i=1}^{n} H_i(|x|)$, *and* $\bar{H}_i(|x|) = F_i(|x|) - F_i(-|x|) = 2F_i(|x|)-1$, $i \geqslant 1$, *x real.*

Theorem 3.2 : *If* (i) $J^* \epsilon L_r$ *for some* $r > 2$, (ii) *for every* $0 < u < 1$. $|J'(u)| \leqslant K[u(1-u)]^{-3/2+\delta_0}$ *for some* $\delta_0 > 0$, (iii) $F_1 = \ldots = F_n = F$, $\forall\, n > 1$, (iv) $\max_{1 \leqslant i \leqslant n} |\bar{c}_{ni}| = O(n^{-\frac{1}{2}})$, *and* (v) $\liminf_{n \to \infty} n^{-1} \sum_{i=1}^{n} c_i^2 > C_0 > 0$, $\limsup_{n \to \infty} n^{-\delta} \sum_{i=1}^{n} c_i^4 \leqslant C^* < \infty$, *then* (a) *there is a Brownian motion* $\xi(t)$ *on* $[0, \infty)$ *such that*

$$\widetilde{T}_t^* = \xi(t) + o[(t \log \log t)^{\frac{1}{2}}] \; a.s. \; as \; t \to \infty, \qquad \ldots (3.2)$$

where

$$\tilde{T}^*_{V_n^*} = \tilde{T}_n, \tilde{T}_0 = 0 \; ; \qquad \ldots (3.3)$$

$$V_n^* = \sum_{i=1}^{n} E[(\tilde{T}_i - \tilde{T}_{i-1})^2 | \tilde{T}_1, \ldots, \tilde{T}_{i-1}] \qquad \ldots (3.4)$$

and (b)

$$\limsup_{n \to \infty} \tilde{T}_n [2(A_n^* C_n^*)^2 \log \log (A_n^* C_n^*)^2]^{-\frac{1}{2}} = 1 \; a.s., \qquad \ldots (3.5)$$

$$\liminf_{n \to \infty} \tilde{T}_n [2(A_n^* C_n^*)^2 \log \log (A_n^* C_n^*)^2]^{-\frac{1}{2}} = -1 \; a.s. \qquad \ldots (3.6)$$

where $$(C_n^*)^2 = \sum_{i=1}^{n} c_i^2. \qquad \ldots \quad (3.7)$$

It is possible to replace $(A_n^* C_n^*)^2$ *by* V_n^*.

Consider now another process $Z_n^* = \{Z_n^*(t) : t \in I\}$, where

$$Z_n^*(t_{k,n}) = (A_n^* C_n^*)^{-1} T_k^*, \qquad \ldots \quad (3.8)$$

$t_{k,n} = (C_k^* A_k^*)^2/(C_n^* A_n^*)^2$, $k = 0, 1, \ldots, n$; and by linear interpolation, complete the definition of $Z_n^*(t)$ for $t \in [t_{k,n}, t_{k+1,n}]$, $k = 0, 1, \ldots, n-1$. Then, analogous to Theorem 1.3, we have the following theorem.

Theorem 3.3 : *Under the assumptions of Theorem* 3.2 $Z_n^* \to W$, *in the uniform topology on* $C[0, 1]$, *where* $W = \{W(t) : t \in I\}$ *is a standard Brownian motion.*

Remark : In Theorems 1.2 and 3.2 if we are only interested in (1.18), (1.19), (3.4) and (3.5), without the a.s. convergence to appropriate Wiener processes, and we consider the weaker forms where we replace $A_n^2 C_n^2$(or $A_n^{*2} C_n^{*2}$) by their stochastic counterparts V_n (or V_n^*), then one could have used a more recent paper of Stout (1970) whose conditions are easier to verify. However, in view of our stronger results in (1.17) and (3.2), we have preferred to use (under essentially no stronger regularity conditions) Theorem 4.4 of Strassen (1967).

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