ON THE REPRESENTATION OF LINEAR FUNCTIONS OF ORDER STATISTICS

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SUMMARY. The usual technique of studying the asymptotic distribution of linear functions of n order statistics is the decomposition of such functions into mean of n independent and identically distributed random variables plus a remainder term, say, R_n such that $\sqrt{n}R_n$ converges to zero in probability as $n \to \infty$. In this note, we have studied how fast R_n converges to zero almost surely as $n \to \infty$. In this context, an interesting inequality concerning the fluctuations of the empirical distribution function from the theoretical distribution function is also derived.

1. Introduction

Let $X_1, \ldots, X_n$ be n independent and identically distributed random variables (i.i.d.r.v.) each having a continuous distribution function (d.f.) $F(x)$. Let $X_{n:1} \leqslant X_{n:2} \leqslant \ldots \leqslant X_{n:n}$ denote the ordered X's. Consider the statistics

$$T_n = n^{-1} \sum_{i=1}^{n} J\left(\frac{i}{n}\right) X_{n:i} = \int_{-\infty}^{\infty} x\, J(F_n(x))\, dF_n(x), \ (n \geqslant 1), \qquad \ldots \quad (1.1)$$

where $F_n(x)$ denotes the empirical d.f. of $X_1, \ldots, X_n$ $(n \geqslant 1)$. It is proved (see e.g. Chernoff, Gastwirth and Johns, 1967; Govindarajulu, 1965; Moore, 1968; Shorack, 1969 and Stigler, 1969) that under suitable regularity conditions on J and F, $\sqrt{n}$ $(T_n - \mu)$ converges in law to a normal $(0, \sigma^2)$ distribution as $n \to \infty$, where $\mu = \int_{-\infty}^{\infty} x\, J(F(x))\, dF(x)$, and $\sigma^2 = 2 \int\int_{-\infty<s<t<\infty} J(F(s))\, J(F(t))\, F(s)\,(1-F(t))\, ds\, dt$. The basic technique used in all these papers is the representation of $\sqrt{n}(T_n - \mu)$ as $n^{-1/2} S_n + R_n$, where S_n is the sum of n i.i.d.r.v. with zero mean and variance σ^2, while R_n, the remainder term converges to zero in probability.

To obtain S_n and R_n explicitly, we introduce the following notations. Let $U_n(u)$ denote the empirical d.f. of $F(X_1), \ldots, F(X_n)$ $(n \geqslant 1)$ which are i.i.d. uniform (0, 1) variables, and G any inverse of F. Then one can write

$$T_n = \int_0^1 G(u)\, J(U_n(u))\, d\, U_n(u) \ (n \geqslant 1); \qquad \ldots \quad (1.2)$$

$$\mu = \int_0^1 G(u)\, J(u)\, du, \ \sigma^2 = 2 \int\int_{0<u<v<1} J(u)\, J(v)\, u(1-v)\, dG(u)\, dG(v). \qquad \ldots \quad (1.3)$$

Assuming that $J'(u)$ exists for all $u \in [0, 1]$, $T_n - \mu$ can be represented as

$$T_n - \mu = I_{1n} + I_{2n} + I_{3n}, \quad \dots \quad (1.4)$$

where,

$$I_{1n} = \int_0^1 G(u)\, J'(u)\, (U_n(u) - u)\, du + \int_0^1 G(u)\, J(u)\, d(U_n(u) - u); \quad \dots \quad (1.5)$$

$$I_{2n} = \int_0^1 G(u)[J(U_n(u)) - J(u) - (U_n(u) - u) J'(u)]\, d\, U_n(u); \quad \dots \quad (1.6)$$

$$I_{3n} = \int_0^1 G(u) J'(u)(U_n(u) - u)\, d(U_n(u) - u). \quad \dots \quad (1.7)$$

The above representation is due to Moore (1968). It can be shown that if $J(u)$ is bounded on $[0, 1]$ and $E(|X_1|) = \int_0^1 |G(u)|\, du < \infty (\Longrightarrow \lim_{u \downarrow 0} uG(u) = \lim_{u \uparrow 1} (1-u)G(u) = 0)$, then after integration by parts, with probability 1, $I_{1n} = -\int_0^1 J(u)(U_n(u) - u)\, dG(u) = n^{-1} \sum_1^n Z_i$, where $Z_i = -\int_0^1 (c(u - U_i) - u)\, J(u)\, dG(u)$ $(i = 1, 2, \dots, n)$, $c(t) = 1$ or 0 as $t \geqslant$ or < 0. Z_i's are i.i.d.r.v. with zero mean and variance σ^2. Also, it is shown that $R_n = I_{2n} + I_{3n} \xrightarrow[P]{} 0$ as $n \to \infty$, under suitable regularity conditions on J and F.

In the present note, we have examined the almost sure (a.s.) rate of convergence of R_n to zero as $n \to \infty$. The following theorem is proved.

Theorem : *If* (i) $J''(u)$ *is bounded on* $[0, 1]$, (ii) $\int [u(1-u)]^{\frac{1}{2+\delta}} d|G(u)| < \infty$, *then* $R_n = O(n^{-1}(\log n)^2)$ *a.s. as* $n \to \infty$.

The proof of the theorem is postponed to the following section. One may note that if $0 < \sigma^2 < \infty$, a law of iterated logarithm (LIL) for T_n follows as an immediate corollary to our theorem. This is because the Z_i's are i.i.d.r.v. with zero mean, and non-zero and finite variance σ^2. Hence, verifying the classical Kolmogorov condition for LIL (see Wintner and Hartman, 1941) one gets, $\limsup_{n \to \infty} (2n\sigma^2 \log \log n)^{-1/2} \sum_{i=1}^n Z_i = 1$ a.s. Also, from our theorem, $(2\sigma^2 \log \log n)^{-1/2} \sqrt{n}\, R_n = O(n^{-1/2}(\log n)^2 (\log \log n)^{-1/2})$ a.s. as $n \to \infty$. The LIL for T_n now follows by writing

$$(2\sigma^2 \log \log n)^{-1/2} \sqrt{n}(T_n - \mu) = (2n\sigma^2 \log \log n)^{-1/2} \sum_1^n Z_i + \sqrt{n} R_n.$$

An alternative representation of T_n is possible using the results of Kiefer (1970). But then, the resulting remainder term $= O(n^{-3/4}(\log n)^{1/2}(\log \log n)^{-1/4})$ a.s. as $n \to \infty$.

2. Proof of the theorem

The following is the basic lemma in proving the theorem.

Lemma : *For every $\delta > 0$, there exist $K(>0)$, $\epsilon > 0$, and n_0 (all depending on δ) such that for $n > n_0$,*

$$P\{\sup_{0<u<1}(u(1-u))^{-1/2+\epsilon}|U_n(u)-u| > Kn^{-1/2}\log n\} < 2n^{-1-\delta} \quad \dots \quad (2.1)$$

Proof : The proof of the lemma is completed in several steps. First we show that for every $\delta > 0$, there exist $K_1(>0)$ and n_1 (positive integer) (both depending on δ) such that for $n > n_1$,

$$P\left\{\sup_{n^{-1}<u<1-n^{-1}}(u(1-u))^{-1/2}|U_n(u)-u| > K_1 n^{-1/2}\log n\right\} < 2n^{-1-\delta}. \quad \dots \quad (2.2)$$

Next we show that for every $\delta > 0$,

$$\sup_{0<u<n^{-2-\delta}}(u(1-u))^{-1/2}|U_n(u)-u| = O(n^{-1-\delta/2}), \quad \dots \quad (2.3)$$

$$\sup_{1-n^{-2-\delta}<u<1}(u(1-u))^{-1/2}|U_n(u)-u| = O(n^{-1-\delta/2}), \quad \dots \quad (2.4)$$

each with probability $\geq 1-n^{-1-\delta}$. Finally we show that for every $\delta > 0$, there exist K_2, n_2 and ϵ (all dependent on δ) such that for $n > n_2$,

$$P\left\{\sup_{u\epsilon[n^{-2-\delta},\, n^{-1})U(1-n^{-1},\, 1-n^{-2-\delta}]}(u(1-u))^{-1/2+\epsilon}|U_n(u)-u| > K_2 n^{-1/2}\log n\right\} < 4n^{-1-\delta} \quad \dots \quad (2.5)$$

Step 1 : To prove (2.1), let $\eta_{r,n} = r/n$, $r = 1, 2, \dots, n-1$. Then, for $u\epsilon[\eta_{r-1,n}, \eta_{r,n}]$, $r = 2, 3, \dots, n-1$,

$$(\eta_{r,n}(1-\eta_{r-1,n}))^{-1/2}(U_n(\eta_{r-1,n})-\eta_{r,n})$$
$$\leq (u(1-u))^{-1/2}(U_n(u)-u) \leq (\eta_{r-1,n}(1-\eta_{r,n}))^{-1/2}(U_n(\eta_{r,n})-\eta_{r-1,n}) \quad \dots \quad (2.6)$$

The upper bound in (2.6) can be expressed as

$$(\eta_{r,n}(1-\eta_{r,n}))^{-1/2}(\eta_{r,n}/\eta_{r-1,n})^{1/2}(U_n(\eta_{r,n})-\eta_{r,n})+(\eta_{r-1,n}(1-\eta_{r,n}))^{-1/2}(\eta_{r,n}-\eta_{r-1,n})$$
$$= (r/(r-1))^{1/2}(\eta_{r,n}(1-\eta_{r,n}))^{-1/2}(U_n(\eta_{r,n})-\eta_{r,n})+[(r-1)(n-r)]^{-1/2}.$$

Similarly, the lower bound in (2.6) can be expressed as

$$((r-1)/r)^{1/2}(\eta_{r-1,n}(1-\eta_{r-1,n}))^{-1/2}(U_n(\eta_{r-1,n})-\eta_{r-1,n})-\{r(n-r+1)\}^{-1/2}.$$

Note that

$$\{(r-1)(n-r)\}^{-1/2} \leq (n-2)^{-1/2},\ (r(n-r+1))^{-1/2} \leq (2(n-1))^{-1/2} \text{ and } (r/(r-1))^{1/2} \leq \sqrt{2}$$

Thus, for $u \in [\eta_{r-1,n}, \eta_{r,n}]$,

$$(u(1-u))^{-1/2} | U_n(u) - u |$$
$$\leqslant \sqrt{2} \max_{j=r-1,r} \{[\eta_{j,n}(1-\eta_{j,n})]^{-\frac{1}{2}} | U_n(\eta_{j,n}) - \eta_{j,n} |\} + O(n^{-1/2}),\ r = 2, 3, \ldots, n-1.$$

Thus

$$\sup_{n^{-1} \leqslant u \leqslant 1-n^{-1}} (u(1-u))^{-1/2} | U_n(u) - u | \leqslant \sqrt{2} \max_{j=1,\ldots,n-1} [\eta_{j,n}(1-\eta_{j,n})]^{-1/2} | U_n(\eta_{j,n}) - \eta_{j,n} | O(n^{-1/2}).$$

Hence, for proving (2.2) is sufficient to show that for every $\delta > 0$, there exist $K_1 (> 0)$ and n_1 such that for $n \geqslant n_1$,

$$P\left\{ \max_{1 \leqslant j \leqslant n-1} [(\eta_{j,n}(1-\eta_{j,n}))^{-1/2} | U_n(\eta_{j,n}) - \eta_{j,n} |] \geqslant \frac{K_1}{\sqrt{2}} n^{-1/2} \log n \right\} \leqslant 2n^{-1-\delta} \quad \ldots \quad (2.7)$$

But L.H.S. of (2.7) is bounded above by

$$\sum_{j=1}^{n-1} P\{| n\, U_n(\eta_{j,n}) - n\eta_{j,n} | \geqslant t_{j,n}\}, \quad \ldots \quad (2.8)$$

where $$t_{j,n} = \frac{K_1}{\sqrt{2}} n^{1/2} (\eta_{j,n} (1-\eta_{j,n}))^{1/2} \log n (j = 1, 2, \ldots, n-1)$$

But $nU_n(\eta_{j,n})$ has a binomial distribution with parameters n and $\eta_{j,n}$. Hence applying Bernstein inequality (see Uspensky (1937), pp. 204-205) and (2.8), one gets L.H.S. of (2.7) bounded above by $2 \sum_{j=1}^{n-1} \exp(-h_{j,n})$, where,

$$h_{j,n} = \frac{1}{2} t_{j,n}^2 / [n\eta_{j,n}(1-\eta_{j,n}) + \frac{1}{3} t_{j,n} \max(\eta_{j,n}, 1-\eta_{j,n})]$$

$$\geqslant \frac{1}{2} t_{j,n}^2 / [n\eta_{j,n}(1-\eta_{j,n}) + t_{j,n}]$$

$$= \frac{1}{2} (K_1^2/2) n\eta_{j,n}(1-\eta_{j,n}) (\log n)^2 / [n\eta_{j,n}(1-\eta_{j,n})$$

$$+ \frac{1}{\sqrt{2}} K_1 n^{1/2} \eta_{j,n}^{1/2} (1-\eta_{j,n})^{1/2} \log n]$$

$$= \frac{1}{4} K_1^2 (\log n)^2 / \left[1 + \frac{1}{\sqrt{2}} K_1 n^{-1/2} \eta_{j,n}^{-1/2} (1-\eta_{j,n})^{-1/2} \log n \right], \quad \ldots \quad (2.9)$$

$j = 1, \ldots, n-1$. But, for all $j = 1, \ldots, n-1$, $\eta_{j,n}^{-1/2}(1-\eta_{j,n})^{-\frac{1}{2}} \leqslant n^{-1}(1-n^{-1})^{-\frac{1}{2}}$. Hence the denominator of the last expression in (2.9) is bounded above by

$$\left(1 + \frac{1}{\sqrt{2}} K_1 \left(\frac{n}{n-1}\right)^{1/2} \log n \right)^{-1}$$

Noting that

$$(n^{-1}(1-n^{-1}))^{-1/2} = \left(\frac{n}{n-1}\right)^{1/2} n^{1/2} \leqslant \sqrt{2}n^{1/2},$$

one gets from (2.9),

$$h_{j,n} \geqslant \frac{1}{4} K_1^2(\log n)^2(1+K_1 \log n)^{-1} \geqslant \frac{1}{8} K_1 \log n$$

for $n \geqslant n_1$, where n_1 depends on K_1. Hence, $2 \sum_{j=1}^{n-1} \exp(-h_{j,n}) \leqslant 2n^{1-\frac{K_1}{8}} \leqslant 2 n^{-1-\delta}$ for $K_1 \geqslant 8(2+\delta)$.

Step 2 : We prove only (2.3) as (2.4) follows analogously. First, note that

$$P\left\{\sup_{0<u<n^{-2-\delta}} U_n(n) = 0\right\} \geqslant P\{U_n(n^{-2-\delta}) = 0\} = P\{F(X_{(1)}) > n^{-2-\delta}\}$$

$$= (1-n^{-2-\delta})^n \geqslant 1-n^{-1-\delta}.$$

Then, with probability $\geqslant 1-n^{-1-\delta}$

$$\sup_{0<u<n^{-1-\delta}} (u(1-u))^{-1/2}|U_n(u)-u| \leqslant \sup_{0<u<n^{-2-\delta}} u^{1/2}(1-u)^{-1/2}$$

$$\leqslant n^{-1-\delta/2}(1-n^{-2-\delta})^{-1/2} = O(n^{-1-\delta/2}).$$

Step 3 : Write $I_{1n} = [n^{-2-\delta}, n^{-1}), I_{2n} = (1-n^{-1}, 1-n^{-2-\delta}]$. To prove (2.5) it is sufficient to show that for every $\delta > 0$, there exist K_2', K_2'', n_2', n_2'' and ε such that

$$P\left\{\sup_{u \epsilon I_{1n}} (u(1-u))^{-1/2+\varepsilon}|U_n(u)-u| < K_2' n^{-1/2} \log n\right\} \leqslant 2n^{-1-\delta} \text{ for } n \geqslant n_2'; \quad \ldots \quad (2.10)$$

$$P\left\{\sup_{u \epsilon I_{2n}} (u(1-u))^{-1/2+\varepsilon}|U_n(u)-u)| \geqslant K_2'' n^{-1/2} \log n\right\} \leqslant 2n^{-1-\delta} \text{ for } n \geqslant n_2''. \quad \ldots \quad (2.11)$$

We prove only (2.9) as (2.11) follows analogously. To prove (2.10), let $\xi_{r,n} = r/n^{2+\delta}$, $r = 1, 2, \ldots, c_n$, $c_n = [n^{1+\delta}]$, the largest integer contained in $n^{1+\delta}$. Arguing similarly, as in step 1, one gets, for $u \epsilon [\xi_{r-1,n}, \xi_{r,n}]$,

$$(u(1-u))^{-1/2+\varepsilon}|U_n(u)-u| \leqslant \sqrt{2} \max_{j=r-1,r} [\xi_{j,n}(1-\xi_{j,n})]^{-1/2+\varepsilon}|U_n(\xi_{j,n})-\xi_{j,n}|$$
$$+O(n^{-2-\delta+(2+\delta)(1/2-\varepsilon)}).$$

Hence,

$$\sup_{u \epsilon I_{1n}} (u(1-u))^{-1/2+\varepsilon}|U_n(u)-u| \leqslant \sqrt{2} \max_{j=1,2,\ldots,c_n} [(\xi_{j,n}(1-\xi_{j,n}))^{-1/2+\varepsilon}$$
$$|U_n(\xi_{j,n})-\xi_{j,n}|]+O(n^{-1-\delta/2-(2+\delta)\varepsilon}).$$

Hence, to prove (2.10), it is sufficient to show that

$$P\left\{\max_{1\leqslant j\leqslant c_n}(\xi_{j,n}(1-\xi_{j,n}))^{-1/2+\epsilon}|U_n(\xi_{j,n})-\xi_{j,n}| > \frac{1}{\sqrt{2}}K_2'n^{-1/2}\log n\right\} \quad \ldots \quad (2.12)$$

$< 2n^{-1-\delta}$ for $n > n_2'$. But LHS of the inequality in (2.12) is bounded above by

$$\sum_{j=1}^{c_n} P\{|nU_n(\xi_{j,n})-n\xi_{j,n}| > t_{j,n}'\}, \quad \ldots \quad (2.13)$$

where $t_{j,n}' = \frac{1}{\sqrt{2}}K_2'n^{1/2}\log n(\xi_{j,n}(1-\xi_{j,n}))^{1/2-\epsilon} \qquad (j = 1, 2, \ldots c_n).$

Using Bernstein inequality once again, the expression in (2.13)

$$\leqslant 2\sum_{j=1}^{c_n}\exp(-g_{j,n}), \text{ where, } \quad g_{j,n} = \frac{t_{j,n}'^2}{2[n\xi_{j,n}(1-\xi_{j,n})+t_{j,n}']} \quad 1 < j < c_n.$$

We can write

$$g_{j,n} = \frac{1}{2\sqrt{2}}K_2'n^{1/2}\log n(\xi_{j,n}(1-\xi_{j,n}))^{1/2-\epsilon}/[1+K_2'^{-1}n^{1/2}(\log n)^{-1}(\xi_{j,n}(1-\xi_{j,n}))^{1/2+\epsilon}]$$
$$(j = 1, 2, \ldots, c_n).$$

Use the inequality

$$n^{1/2}(\xi_{j,n}(1-\xi_{j,n}))^{1/2-\epsilon} \geqslant n^{1/2}n^{-(2+\delta)(1/2-\epsilon)}(1-n^{-1})^{1-2\epsilon} \geqslant n^{-1/2(1+\delta)+\epsilon(2+\delta)}\left(\frac{1}{2}\right)^{1-2\epsilon}$$
for $n > 2$.

Also,

$$n^{1/2}(\log n)^{-1}(\xi_{j,n}(1-\xi_{j,n}))^{1/2+\epsilon} < n^{1/2}(\log n)^{-1}(n^{-1})^{1/2+\epsilon} = n^{-\epsilon}(\log n)^{-1};$$

choose $\epsilon = \frac{1}{2}(1+\delta)/(2+\delta)\left(<\frac{1}{2}\right)$. It follows now that

$$g_{j,n} > \frac{\frac{1}{\sqrt{2}}K_2'\left(\frac{1}{2}\right)^{1-2\epsilon}\log n}{2[1+K_2'^{-1}n^{-\epsilon}(\log n)^{-1}]} > C\log n \text{ for } n > n_2',$$

C and n_2' both depending on K' and ϵ i.e. K_2' and δ. Hence, $2\sum_{j=1}^{c_n}\exp(-g_{j,n}) <$ $2c_nn^{-C} < 2n^{1+\delta-C} < 2n^{-1-\delta}$ if $C > 2+\delta$. Thus (2.10) is proved. Hence, the lemma.

The lemma has independent interest apart from proving the theorem. It gives a useful estimate of the fluctuation of the empirical process, and is expected to be useful in other contexts as well. For proving our theorem, the following corollary to the above lemma is used.

Corollary : *For every $\delta \geqslant 0$, there is a $K > 0$ such that with probability 1,*

$$\sup_{0<u<1} (u(1-u))^{\frac{1}{2(2+\delta)}} |U_n(u)-u| \leqslant Kn^{-\frac{1}{2}} \log n \text{ as } n \to \infty.$$

Proof: The proof is immediate from the above lemma and the Borel-Cantelli lemma.

3. Proof of the main theorem

J'' bounded $\to J$ and J' are bounded. Using the mean value theorem, one can write $I_{2n} = \int_0^1 G(u)(U_n(u)-u)^2 J''(\theta U_n(u)+(1-\theta)u)dU_n(u)$, $0 < u < 1$. Using the corollary to the lemma and conditions (i) and (ii) of the theorem, it follows that $I_{2n} = O(n^{-1}(\log n)^2)$ a.s. as $n \to \infty$. Also, with probability 1, $I_{3n} = \frac{1}{2} \int_0^1 G(u)J'(u)d(U_n(u)-u)^2 + \frac{1}{2} \int_0^1 G(u)J'(u)(d(U_n(u))^2$. Now, $\int_0^1 G(u)J'(u)(d(U_n(u))^2 = n^{-2} \sum_{i=1}^{n} G(F(X_i))J'(F(X_i))$. $G(F(X_i))J'(F(X_i))$'s are i.i.d.r.v. with expectation $\int_0^1 G(u)J'(u)du = E$(say). Hence, $|E| \leqslant$ const. $\int_0^1 |G(u)|du < \infty$ from (ii). Using the strong law of large numbers, $n^{-1} \sum_{i=1}^{n} G(F(X_i))J'(F(X_i)) \to \int_0^1 G(u)J'(u)du < \infty$ a.s. as $n \to \infty$. So, $\int_0^1 G(u)J'(u)(dU_n(u))^2 = O(n^{-1})$ a.s. as $n \to \infty$. Again, integrating by parts, one gets with probability 1, $\int_0^1 G(u)J'(u)d(U_n(u)-u)^2 = -\int_0^1 (U_n(u)-u)^2 G(u)J''(u)du - \int_0^1 (U_n(u)-u)^2 J'(u)\, dG(u)$. Using the corollary and the conditions (i) and (ii) of the theorem, it follows again that each of the above two terms is $O(n^{-1}(\log n)^2)$ a.s. as $n \to \infty$. Hence, the theorem.

Remarks: An interesting question would be to replace the boundedness condition of J'' by milder conditions on J, J' and J'' under which a similar theorem can be proved. We do not know, however, whether the same order of the remainder term still holds true. It would also be worthwhile to carry out the investigation under the milder and more natural condition $E(|X_1|) = \int_0^1 |G(u)|du < \infty$ than our condition (ii).

Acknowledgement

Thanks are due to Professor J. K. Ghosh for suggesting investigation in this area and for many helpful discussions during the preparation of this manuscript. Thanks are also due to the referee for several useful comments.

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Paper received : April, 1972.
Revised : June, 1972.