ON GENERALIZED INVERSES OF DOUBLY STOCHASTIC MATRICES

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SUMMARY. It is well known that a nonsingular doubly stochastic matrix has doubly stochastic inverse if and only if it is a permutation matrix. Permutation matrices are the only doubly stochastic isometries. This problem is studied with respect to g-inverse of doubly stochastic matrices. In this paper it is established that a doubly stochastic matrix has a doubly stochastic g-inverse if and only if it is partial isometry.

0. Introduction

It is well known that a nonsingular doubly stochastic matrix has doubly stochastic inverse if and only if it is a permutation matrix. Permutation matrices are the only doubly stochastic isometrics. This problem is studied with respect to g-inverse of doubly stochastic matrices. In this paper it is established that a doubly stochastic matrix has a doubly stochastic g-inverse if and only if it is partial isometry.

1. Notation and definitions

Matrices are denoted by capital letters, A, G, U etc. and vectors by lower case letters x, y etc. A' and A^* denote the transpose and complex conjugate transpose of A respectively and $\mathcal{M}(A)$ and $R(A)$ denote column space of A and rank of A respectively. $\|\cdot\|$ denotes the Euclidean norm. I denotes an identity matrix. R^n denotes the n-dimensional Euclidean space.

An $n\times n$ matrix $A = (a_{ij})$ is said to be doubly stochastic if

$$a_{ij} \geqslant 0 \quad \text{and} \quad \sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{kj} = 1 \quad \text{for} \quad i, j = 1, \ldots, n.$$

It can be easily verified that product of doubly stochastic matrices is also doubly stochastic.

Let A be an $m\times n$ matrix and G be an $n\times m$ matrix. Consider

$$AGA = A \qquad \ldots (1)$$

$$GAG = G \qquad \ldots (2)$$

$$(AG)^* = AG \qquad \ldots (3)$$

$$(GA)^* = GA \qquad \ldots (4)$$

If G satisfies (1) it is called a generalized inverse of A (g-inverse) and is denoted by A^-. If G satisfies (1) and (2) it is called a reflexive g-inverse of A and is denoted by A_r^-. If G satisfies (1) and (3) it is called a least squares g-inverse of A, denoted by A_l^-. If G satisfies (1) and (4) it is called minimum norm g-inverse of A and is denoted by A_m^-. If G satisfies (1), (3) and (4) it is called a minimum norm least squares g-inverse of A denoted by A_{lm}^- and if G satisfies (1), (2), (3) and (4) it is called the Moore-Penrose inverse of A and is denoted by A^+.

For the properties and uses of g-inverses the reader is referred to Rao and Mitra (1971).

An $n \times n$ matrix A is said to be an isometry if

$$\|Ax\| = \|x\| \quad \forall x \in R^n.$$

An $m \times n$ matrix A is said to be a partial isometry if

$$\|Ax\| = \|x\| \quad \forall x \in \mathcal{M}(A').$$

2. Partial isometry

From the definitions it immediately follows that A is an isometry if and only if $A' = A^{-1}$ and A is a partial isometry if and only if $A' = A^+$.

Theorem 1 : *A is partial isometry if and only if one of the following equivalent conditions holds.*

(i) *$(AA')^pA$ is partial isometry for some non-negative integer p and* (ii) *$(AA')^p$ is a partial isometry for some positive integer p.*

Proof :

(1) $(AA')^p A$ is partial isometry

$\Longleftrightarrow (AA')^pAA'(AA')^{2p}A = (AA')^pA$

$\Longleftrightarrow (AA')^{3p+1}A = (AA')^pA$

$\Longleftrightarrow (AA')^{2p+1}A = A$

$\Longleftrightarrow C(AA'A-A) = 0$ where $C = [(AA')^{2p}+(AA')2^{p-1}+\ldots+(AA')+I]$

$\Longleftrightarrow AA'A = A \because C$ is positive definite

$\Longleftrightarrow A$ is partial isometry.

(2) $(AA')^p$ is partial isometry

$\Longleftrightarrow (AA')^{3p} = (AA')^p$

$\Longleftrightarrow (AA')^{2p+1} = AA'$

$\Longleftrightarrow C(AA'AA'-AA') = 0$ where $C = [(AA')^{2p-1}+(AA')^{2p-2}+\ldots$
$+(AA')+I]$

$\Longleftrightarrow AA'AA' = AA' \because C$ is positive definite

$\Longleftrightarrow A$ is partial isometry.

Hence the theorem.

3. g-inverse of doubly stochastic matrices

In the sequel we need the following result of Sinkhorn (1968) which we state below for completeness.

Lemma 1 : *If A is a doubly stochastic idempotent matrix then it is symmetric..*

Theorem 2 : *Let A be a doubly stochastic matrix possessing a doubly stochastic g-inverse. Then A^+ is doubly stochastic.*

Proof : Let G_1 be a doubly stochastic g-inverse of A. Let $G = G_1AG_1$. Observe that G is doubly stochastic and is a reflexive g-inverse of A. Further GA and AG are idempotent and doubly stochastic and hence by Lemma 1 are symmetric. Hence $G = A^+$.

Theorem 3 : *Let A be a normal doubly stochastic matrix. Then the following statements are equivalent.*

(a) *A has a doubly stochastic g-inverse,*

(b) *each non-zero eigen value of A is of modulus unity and*

(c) *A is a partial isometry.*

Proof: Let $R(A) = r$. Since A is normal there exists a unitary matrix U such that $A = U \Lambda U^*$ where $\Lambda = \text{diag}\ (\lambda_1, \ldots, \lambda_n)$ where $\lambda_1, \ldots, \lambda_n$ are eigen values of A. Without loss of generality let $\lambda_1, \ldots, \lambda_r$ be non-zero and $\lambda_{r+1}, \ldots, \lambda_n$ be zero.

(a) $\Longrightarrow$ (b)

A has a doubly stochastic g-inverse implies by Theorem 1 that A^+ is doubly stochastic. Since A is doubly stochastic $|\lambda_i| \leqslant 1$ for $i = 1, \ldots, r$. Clearly $A^+ = U \Lambda^+ U^*$ where $\Lambda^+ = \text{diag}\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_r}, 0, \ldots, 0\right)$. Again since A^+ is doubly stochastic $\left|\frac{1}{\lambda_i}\right| \leqslant 1$ for $i = 1, \ldots, r$. Hence $|\lambda_i| = 1$ for $i = 1, \ldots, r$.

(b) $\Longrightarrow$ (c)

First observe that $A^* = A'$ since A is real. $|\lambda_i| = 1$ for $i = 1, \ldots, r$ implies that $\Lambda\Lambda^*\Lambda = \Lambda$ and hence $AA^*A = A$. Hence $A' = A^* = A^+$.

(c) $\Longrightarrow$ (a) trivial.

Theorem 4 : *A doubly stochastic matrix A possesses a doubly stochastic g-inverse if and only if A is a partial isometry.*

Proof: 'if' part is trivial.

'only if' part : Let $G = A^+$.

A has a doubly stochastic g-inverse

$\Longrightarrow G$ is doubly stochastic by Theorem 2

$\Longrightarrow G'G = (AA')^+$ is also doubly stochastic

$\Longrightarrow AA'$ is a partial isometry by Theorem 3

$\Longrightarrow A$ is a partial isometry by Theorem 1.

References

Rao, C. R. and Mitra, S. K. (1971) : *Generalized Inverse of Matrices and Its Applications*, John Wiley and Sons, New York.

Sinkhorn, R. (1968) : Two results concerning doubly stochastic matrices. *Amer. Math. Monthly*, 632-634.

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