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ON CONSTRUCTION OF BALANCED TERNARY DESIGNS

By G. M. SAHA
Indian Statistical Institute

SUMMARY. Two methods of construction of balanced ternary block designs for V treatments in blocks of size K, where $K < V$, are described in this paper. Unlike many of the existing balanced ternary designs (BTD) which usually have either $K > V$, or $K < V$, but B very large, where B denotes the number of blocks in the design, the BTD constructed in this paper are likely to be useful as incomplete block designs in many practical situations. The first method is simple, and it yields a number of efficient designs in a useful range of varieties and replications. A table of such designs with their efficiency factors is presented herein. The second method of construction of BTD uses 'initial blocks'. An application of this method is made to obtain two new series of balanced ternary designs. Some useful BTD belonging to these series and their efficiency factors are reported in another table.

1. Introduction

A balanced n-ary block design has been defined by Tocher (1952) as one which has : (i) the incidence matrix $N = (n_{ij})$, where n_{ij} is the number of times the i-th treatment occurs in the j-th block, n_{ij} being one of the n distinct numbers $0, 1, 2, \ldots, n-1$; $i = 1, 2, \ldots, V$; $j = 1, 2, \ldots, B$; and (ii) the property that the variances of comparisons between any two treatments are the same. Tocher (1952) gave some examples of balanced ternary designs and discussed their uses as incomplete block designs. He did not, however, consider any method of construction of these designs.

The above definition of n-ary designs with frequencies $(0, 1, 2, \ldots, n-1)$ replaced by a set of any n distinct positive integers $(f_0, f_1, \ldots f_{n-1})$ is a more general version of the definition of a balanced n-ary block design. Now it is known that if for an n-ary block design :

(i) $\sum_{i=1}^{V} n_{ij} = K$, a constant for all j;

(ii) $\sum_{j=1}^{B} n_{ij} = R$, a constant for all i; and

(iii) $\sum_{j=1}^{B} n_{ij}\, n_{i'j} = \Lambda$, a constant for all i and i'; $i \neq i' = 1, 2, \ldots V$;

then the design is balanced in the sense explained earlier. Since

$$\sum_{j=1}^{B} n_{ij}^2+\Lambda(V-1) = \sum_{j=1}^{B}\left[n_{ij}\left(n_{ij}+\sum_{\substack{i'=1\\ i'\neq i}}^{V} n_{i'j}\right)\right] = \left(\sum_{j=1}^{B} n_{ij}\right)\left(\sum_{i=1}^{V} n_{ij}\right) = RK$$

we see that $\sum_{j=1}^{B} n_{ij}^2$ is also a constant for all i and is equal to $= RK-\Lambda(V-1)$. One can also easily establish that the efficiency factor of a balanced n-ary design having the *parameters* V, B, R, K, Λ, defined in the same fashion as in the case of BIB designs, is given by $E = \Lambda V/RK$.

Murty and Das (1967), Das and Rao (1969) have given methods of construction of balanced n-ary, in particular ternary, designs. But these designs are all useless as incomplete block designs, for, they require $K \geqslant V$. These designs were found useful for construction of weighing designs only. Recently, Dey (1970) has used the incidence matrices of GD and affine α-resolvable BIB designs to construct BTD. His designs have $K < V$, but they require in general a large number of blocks. The existing literature, thus, does not appear to have any method of construction of balanced ternary designs which may really be of any use as incomplete block designs. This brings us to the problem considered in this paper.

Two methods of construction of balanced ternary block designs in small block sizes are presented in this paper, of which the first method is simple to apply, and leads to a number of useful designs, and the second method utilizes 'initial blocks'. A table of some efficient BTD, obtained through the first method, is presented. The second method is based on a result of Saha and Dey (1973) on construction of BTD using differences. An application of the second method of construction of BTD yields us two new series of BTD which are reported in the last section of this paper. Some designs in useful range are reported.

2. A method of construction of BTD

The ternary designs to be constructed in this section have the entries 0, 1 and 2 in the incidence matrix N. Let v, b, r, k, λ be the parameters of a BIB design whose $v\times b$ incidence matrix is denoted by M, and such that (i) $2k < v$, and (ii) v is even. Let $m_1, m_2, \ldots, m_v$ denote the row vectors of M. Setting $2V = v$, we define the vectors

$$n_q = m_i+m_j; \quad q = 1, 2, \ldots, V; \; i \neq j = 1, 2, \ldots, v, \qquad \ldots \quad (2.1)$$

so that the pairs of symbols (i, j) form any V mutually disjoint subsets of the set $(1, 2, \ldots, v)$. One can then prove

Theorem 1 : *The incidence matrix $N = (n_1', n_2', \ldots, n_V')'$, where n_q's are as defined above and n_q' is the transpose of n_q, is that of a balanced ternary design having the parameters,*

$$V = v/2, B = b, R = 2r, K = k, \Lambda = 4\lambda,$$

v, b, r, k, λ being as explained above. We note that Theorem 1 is valid even without the condition $2k < v$ which only ensures $K < V$. We also note that $E = \Lambda V/RK$ for these BTD reduces to $E = \lambda v/rk$.

Proof : Let $n_{q^*} = m_{i^*} + m_{j^*}$, and n_q be as in (2.1) where $i \neq i^*$, $j \neq j^*$, $q \neq q^*$. Then the inner product of the vectors n_q and n_{q^*} is

$$n_q n_{q^*} = (m_i + m_j)(m_{i^*} + m_{j^*}) = 4\lambda.$$

Hence the theorem.

The above method of addition of row vectors of the incidence matrices of BIB designs in a disjoint fashion can be extended to the incidence matrices of certain group divisible (GD) designs. In a GD design $v = mn$ and the treatments can be divided into m groups of n each such that any two treatments of the same group are first associates and occur together in λ_1 blocks, while two treatments from different groups are second associates and occur together in λ_2 blocks of the design. One can immediately establish the following.

Theorem 2 : (a) *The addition of row vectors corresponding to treatments of the same group of the incidence matrix of a GD design having $n = 2$, $\lambda_1 \neq r$, $\lambda_1 > 0$, $\lambda_2 > 0$, gives us a BTD with the parameters :*

$$V = m, B = b, R = 2r, K = k, \Lambda = 4\lambda_2,$$

$v = 2m$, $b, r, k, m, n = 2$, λ_1, λ_2 being the parameters of GD design taken.

(b) *Similarly, for GD design having $m = 2$, $\lambda_2 \neq r$, $\lambda_2 > 0$, the addition of row vectors, in a disjoint fashion, corresponding to two treatments belonging to different groups, gives us a BTD with*

$$V = n,\ B = b,\ R = 2r,\ K = k;\ \Lambda = 2(\lambda_1 + \lambda_2),$$

$v = 2n$, $b, r, k, m = 2, n, \lambda_1, \lambda_2$ being the parameters of the GDD taken. We note that we have to take only those GD designs which satisfy $2k < v$, to obtain BTD having $K < V$.

Proof: We note that the inner product of any two row vectors of the incidence matrix of a GD design is λ_1 and λ_2 according as they correspond to first associate or second associate pair of treatments. Λ's of the resulting BTD's can now be evaluated on the same lines as in the proof of the Theorem 1.

The condition (i) $0 < \lambda_1 \neq r$ in (a) and (ii) $0 < \lambda_2 \neq r$ in (b) are imposed to make the resulting design a ternary design with frequencies 0, 1 and 2. It is easily seen that if these conditions are violated, the resulting designs become binary with frequencies 0 and 2.

Example 1 : Let us construct a BTD having $V = 8$, $B = 20$, $R = 10$, $K = 4$, $\Lambda = 4$. The initial block (1, 4, 5, 10, 12), mod 21, provides a solution for the BIBD ($v = b = 21$, $r = k = 5$, $\lambda = 1$). (See Fisher and Yates, 1967). Delete from this solution the last block and treatments of the last block from the remaining $b-1$ blocks. This obviously is a solution of the BIBD ($v = 16$, $b = 20$, $r = 5$, $k = 4$, $\lambda = 1$) with the incidence matrix,

$$M = \begin{pmatrix} m_1 \\ \vdots \\ m_{16} \end{pmatrix},$$

where the treatment and block numbers are taken in ascending order of magnitude for writing down M. Define $n_i = m_{2i-1} + m_{2i}$, $i = 1, 2, \ldots, 8$. Then,

$$N = \begin{pmatrix} n_1 \\ \vdots \\ n_8 \end{pmatrix},$$

is the incidence matrix of the BTD ($V = 8$, $B = 20$, $R = 10$, $K = 4$, $\Lambda = 4$), the solution of which is given by (the numbers 1 through 8 denote the eight treatments and parentheses are used to denote a block) :

(1, 2, 4, 4)	(1, 2, 2, 5)	(2, 3, 4, 5)	(3, 3, 5, 6)	(2, 3, 5, 6)
(2, 4, 6, 7)	(3, 4, 6, 7)	(3, 4, 7, 8)	(4, 5, 7, 8)	(4, 5, 5, 8)
(1, 5, 6, 8)	(1, 4, 6, 6)	(1, 5, 6, 7)	(1, 5, 7, 7)	(2, 6, 7, 8)
(2, 6, 8, 8)	(2, 3, 7, 8)	(1, 2, 3, 7)	(1, 1, 3, 8)	(1, 3, 4, 8)

The efficiency factor of this design is evidently 0.8.

3. Some useful BTD

Some efficient balanced ternary designs having $K < V$, and $R \leqslant 20$, constructible from BIB and GD designs are presented in Table 1.

TABLE 1. SOME USEFUL BALANCED TERNARY DESIGNS

sl. no.	V	B	R	K	Λ	E	type of the original design	reference of the original design*
1	3	15	10	2	4	0.60	BIB	u
2	4	28	14	2	4	0.57	BIB	u
3	4	24	18	3	10	0.74	GD	*R*7
4	5	45	18	2	4	0.50	BIB	u
5	5	30	18	3	8	0.74	BIB	15
6	5	15	12	4	8	0.83	BIB	3
7	6	24	12	3	4	0.67	GD	*R*17
8	6	32	16	3	6	0.75	GD	*R*19
9	6	12	8	4	4	0.75	GD	*R*15
10	8	20	10	4	4	0.80	BIB	o.s.
11	8	16	12	6	8	0.89	BIB	5
12	8	24	18	6	12	0.89	BIB	17
13	9	54	18	3	4	0.67	GD	*R*41
14	14	63	18	4	4	0.78	BIB	21
15	14	36	18	7	8	0.89	BIB	22
16	15	75	20	4	4	0.75	GD	*R*57
17	32	72	18	8	4	0.89	BIB	o.s.

* For BIBD, refer to the Statistical Tables by Fisher and Yates (Reprinted, 1967), and for GDD, refer to the tables of partially balanced designs by Bose, Clatworthy and Shrikhande (1954).

4. A method of construction of BTD using initial blocks

Saha and Dey (1973) have shown that, like the balanced incomplete block designs, the balanced ternary block designs can also be constructed through suitably chosen initial blocks. In fact, if these initial blocks have symmetrically repeated *non-zero* differences, they yield a BTD. In this section, a method of obtaining the initial blocks for BTD in moderate block sizes is put forward. An application of this method yields us two new series of BTD.

Let $(0, 1, 2, \ldots, v-1)$ be the elements of a finite Abelian group $G(v)$ of order v, addition mod v being the operation of the group. Let the set of t initial blocks

$$I_i = (a_{i1}, a_{i2}, \ldots, a_{ik}), \quad i = 1, 2, \ldots, t;\ a_{ij} \in \quad (0, 1, 2, \ldots, v-1),$$

when developed modulo v, yield a BIBD with the parameters $v, b = tv$, $r = tk$, k, λ. Let A denote the set $(I_i, -I_i \mid i = 1, 2, \ldots, t)$, where the set $-I_i$ is that of the additive inverses of the elements of I_i. Let, further, every non-zero element of $G(v)$ occur c times in A. For such a set of initial blocks $\{I_i\}$ of a BIBD, we define another set of initial blocks $\{I_i^*\}$ by

$$I_i^* = (\overbrace{0, 0, \ldots, 0}^{p \text{ times}}; a_{i1}, a_{i2}, \ldots, a_{ik}), \quad i = 1, 2, \ldots, t.$$

We then have the following.

Theorem 3 : *The initial blocks $\{I_i^*\}$ as defined above yield a balanced n-ary design with the parameters*

$$V = v, B = tv, R = t(k+p), K = k+p, \Lambda = \lambda+cp.$$

When none of the I_i's contains the zero (0) of $G(v)$, we have $n = 3$ and 0, 1, p as the frequencies; and when each of the I_i's contains 0, we still have $n = 3$ with frequencies 0, 1 and $p+1$. If, however, some but not all of the I_i's contain 0, n is then equal to four and the frequencies are 0, 1, p and $p+1$.

Proof: Expressions for V, B, R and K are obvious. Only Λ need be proved. Consider the set S of $t(k+p)(k+p-1)$ differences arising from the t initial blocks $\{I_i^*\}$. Obviously, every non-zero element of $G(v)$ occurs exactly $\lambda+cp$ times in S. Now, one can argue out, as in Saha and Dey (1973), that the number of times a non-zero element of $G(v)$ occurs in the set of differences arising from the initial blocks of a BTD is equal to the value of Λ. Hence, for the BTD given by $\{I_i^*\}$, we have $\Lambda = \lambda+cp$.

Example 2 : It is known that (0, 1, 3, 9) is an initial block for the BIBD with $v = b = 13$, $r = k = 4$, $\lambda = 1$. But, unfortunately, $A = (0, 1, 3, 9, 0, 12, 10, 4)$ does not contain every non-zero element of $G(13)$ equally often. We however, observe that

$I_1 = (0, 1, 3, 9)$ and $I_2 = (0, 2, 6, 5)$ together also yield a BIBD having $v = 13$, $b = 26$, $r = 8$, $k = 4$, $\lambda = 2$. Further in this case, $A = (0, 1, 3, 9, 0, 12, 10, 4, 0, 2, 6, 5, 0, 11, 7, 8)$ which satisfies the required condition with $c = 1$, We take $p = 1$ to obtain a BTD with frequencies 0, 1 and 2. Thus, the initial blocks for a BTD having $V = 13$, $B = 26$, $R = 10$, $K = 5$, $= 3$, are

$$I_1^* = (0, 0, 1, 3, 9) \text{ and } I_2^* = (0, 0, 2, 6, 5).$$

Since it is wellknown (refer book by Raghavarao, 1971) that when $v = 4n+3$ $(n > 0)$ is a prime power, the initial block $(x^0, x^2, \ldots, x^{4n})$ developed modulo v always provides a solution for the balanced incomplete block design with the parameters $v = b = 4n+3$, $r = k = 2n+1$, $\lambda = n$, provided x is a primitive element of $GF(v)$, we have as an immediate consequence of Theorem 3 the following corollary :

Corollary 3.1 : *When $v = 4n+3$ $(n > 0)$ is a prime power and x is a primitive element of the Galois Field, $GF(v)$, the initial block*

$$I = (\overbrace{0, 0, \ldots, 0}^{p \text{ times}}; x^0, x^2, x^4, \ldots, x^{4n}), \quad 4 > 2$$

yields a BTD with frequencies 0, 1 *and p and the parameters*

$$V = B = 4n+3, \; R = K = 2n+p+1, \; \Lambda = n+p.$$

It is also wellknown that (see Raghavarao, 1971) *when $v = 4n+1$ $(n > 0)$ is a prime power, the two initial blocks*

$$I_1 = (x^0, x^2, x^4, \ldots, x^{4n-2}), \; \textit{and} \; I_2 = (x, x^3, x^5, \ldots, x^{4n-1}),$$

developed modulo v, where x is a primitive element of $GF(v)$, provide solution for the balanced incomplete block design with the parameters $v = 4n+1$, $b = 2(4n+1)$, $r = 4n$ $k = 2n$ $\lambda = 2n-1$. Hence we have

Corollary 3.2 : *Let x be a primitive element of $GF(v)$ where $v = 4n+1$ $(n > 0)$ is a prime power. The two initial blocks*

$$I_1 = (\overbrace{0 \;\; 0, \ldots, 0}^{p \text{ times}}), x^0, x^2, x^4, \ldots, x^{4n-2}),$$

and

$$I_2 = (\overbrace{0, 0, \ldots, 0}^{p \text{ times}}; x, x^3, x^5, \ldots, x^{4n-1}), \quad p > 2$$

yield a BTD with frequencies $0, 1, p$ *and the parameters*

$$V = 4n+1, \; B = 2(4n+1), \; R = 2(2n+p), \; K = 2n+p, \; \Lambda = 2(n+p)-1.$$

5. Some more useful BTD

As in Section 3, we can apply the method of construction of Section 4 and obtain a few more efficient balanced ternary block designs having $K < V$

and $R \leqslant 20$. The designs obtained have parameters and E-values as given in Table 2.

TABLE 2. SOME USEFUL BALANCED TERNARY DESIGNS WITH FREQUENCIES 0, 1, 2, CONSTRUCTIBLE THROUGH THE METHOD OF SECTION 4

serial no.	V	B	R	K	Λ	E
1	5	10	8	4	5	0.78
2	7	7	5	5	3	0.84
3	9	18	12	6	7	0.88
4	11	11	7	7	4	0.90
5	13	26	16	8	9	0.91
6	17	34	20	10	11	0.94
7	19	19	11	11	6	0.94
8	23	23	13	13	7	0.95
9	27	27	15	15	8	0.96
10	31	31	17	17	9	0.96

Note : It is noted that E tends to unity as n increases indefinitely.

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