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ON THE ROBUSTNESS OF LRT IN SINGULAR LINEAR MODELS

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SUMMARY. This is a equel to Mathew and Bhimasankaram (1983). We consider singular covariance structure and study the robustness of LRT in a linear model with respect to specification errors in the dispersion martix.

1. Introduction

In an earlier paper (Mathew and Bhimasankaram, 1983) we considered the robustness of LRT under specification errors in a linear model with positive definite covariance structure. In this paper, we consider the singular covariance structure and study the following problem. Consider the model $(Y, X\beta, \sigma^2V_1)$ where V_1 is possibly singular, the hypothesis $H_0 : A\beta = 0$ where $A\beta$ is estimable and the corresponding LRT statistic. We obtain the class of all models $(Y, X\beta, \sigma^2V)$ for which the LRT statistic remains the same for testing H_0. Unlike in the case of positive definite covariance structure, here it turns out that eventhough the LRT statistics may be the same with probability one under alternative models, the corresponding F distributions (null) need not have the same degrees of freedom. These problems are discussed in detail in Section 3 where we assume multivariate normality for Y. Khatri's (1981) main result comes out as a corollary to one of our results.

Consider $(Y, X\beta, \sigma^2V_1)$ where V_1 may be singular. Let $A\beta$ be estimable. In Section 2 we obtain the class of all models $(Y, X\beta, \sigma^2V)$ such that a specific linear representation/some linear representation/every linear representation of BLUE of $A\beta$ under $(Y, X\beta, \sigma^2V_1)$ remains its BLUE under $(Y, X\beta, \sigma^2V)$. The corresponding problem for $X\beta$ was solved by Mitra and Moore (1973).

For a matrix B, $M(B)$, $N(B)$ and $r(B)$ denote the column space, null space and rank of B respectively. B^- denotes any matrix satisfying $BB^-B = B$. $B^\perp$ denotes a matrix of maximum rank satisfying $B'B^\perp = 0$. For any n.n.d matrix N, $P_{B,N}$ denotes $B(B'NB)^-B'N$ and P_B stands for $P_{B;I}$.

For matrices X and A we denote $X_0 = X(I-A^-A)$. Z and $Z_0 = (Z : Z_1)$ are semiorthogonal matrices such that $Z = X^\perp$ and $Z_0 = X_0^\perp$.

2. Robustness of BLUES

Mitra and Moore (1973) have established that under the linear model $(Y, X\beta, \sigma^2 V_1)$ every linear representation of the BLUE of an estimable parametric function $A\beta$ is of the form $A(X'GX)^-X'GY$ where G is a n.n.d g-inverse of V_1+XX'. In fact, it can be shown that if we vary over nnd g-inverses of $V+XX'$ with any specified rank, we get all possible representations of the BLUE. We now proceed to characterise n.n.d matrices V such that a given linear representation/some linear representation/every linear representation of the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 V_1)$ continues to be its BLUE under $(Y, X\beta, \sigma^2 V)$ also. The characterisations are given in Theorems 2.1, 2.2, 2.3 and 2.4. We state a lemma given in Mathew and Bhimasankaram (1983)

Lemma 2.1 : *Let X_0, Z_0 and Z_1 be as defined in Section 1. Then*

$$M(A') = M(X'Z_0) = M(X'Z_1).$$

Theorem 2.1 : *Let G be a given g-inverse of V_1+XX' and let W be such that $M(W) = N(X'G)$. Then $A(X'GX)^-X'GY$ is BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 V)$ if and only if*

$$V = XD_1X' + WD_2W' + X_0D_3W' + WD'_3X'_0,$$

where D_1, D_2 and D_3 are arbitrary matrices subject to the condition that V is n.n.d.

Now consider the spectral representation of V relative to V_1+XX' as defined in Mitra and Moore (1973) given by

$$V = \lambda_1 E_1 + \lambda_2 E_2 + \ldots + \lambda_p E_p$$

$$V_1 + XX' = E_1 + E_2 + \ldots + E_u, \quad u \leqslant p$$

where λ_i are scalars and E_i are n.n.d matrices such that

$$M(E_1 : E_2 : \ldots \; E_p) = R^n.$$

Let G be a p.d. g-inverse of V_1+XX' satisfying $E_iGE_j = \delta_{ij}E_i$ We now state

Theorem 2.2 : *Let $\mathcal{G}$ be the class of all p.d. g-inverses of V_1+XX'. For $G \in \mathcal{G}$, let W satisfy $M(W) = N(X'G)$. Then $A\beta$ has a common BLUE under $(Y, X\beta, \sigma^2 V_1)$ and $(Y, X\beta; \sigma^2 V)$ if and only if $V = V_G$ for some $G \in \mathcal{G}$ where*

$$V_G = XD_1X' + WD_2W' + X_0D_3W' + WD'_3X'_0,$$

D_1, D_2 and D_3 being arbitrary matrices subject to the condition that V_G is n.n.d

Theorem 2.3 : *The condition on V given in Theorem 2.2 is equivalent to each of the following conditions*

(i) $M(0:0:A)' \subset M(VZ:V_1Z:A)'$

(ii) $M(VZ:V_1Z) \cap M(A) = \{0\}$

(iii) $M\begin{pmatrix} Z'V \\ Z'V_1 \end{pmatrix} = M\begin{pmatrix} Z'V \\ Z'V_1 \end{pmatrix} A^{\perp}.$

Theorem 2.4 : *Every linear representation of the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 V_1)$ continues to be its BLUE under $(Y, X\beta, \sigma^2 V)$ if and only if $M(VZ) \subset M(X_0 : V_1Z)$ or equivalently*

$$V = XD_1X' + V_1ZD_2Z'V_1 + X_0D_3Z'V_1 + V_1ZD'_3X'_0,$$

where D_1, D_2 and D_3 are arbitrary matrices subject to the condition that V is n.n.d.

We shall prove Theorem 2.4 only. Applying Lemma 2.1 the proofs of Theorems 2.1, 2.2 and 2.3 are seen to be similar to the proofs of Theorem 2.1(b), Theorem 3.1 and Theorem 3.2 in Mitra and Moore (1973).

Proof of Theorem 2.4 : Using Lemma 2.1, we see that we want to characterise V such that $L'(X : V_1Z) = (Z_0'X : 0) \Longrightarrow L'VZ = 0$. Using Theorem 2.3.1(c) in Rao and Mitra (1971b, p. 24), we get

$$VZ = (X : V_1Z)K = XK_1 + V_1ZK_2$$

for some $K = (K_1' : K_2')'$. Hence

$$L'VZ = 0 \Longleftrightarrow L'(X : V_1Z)K = 0 \Longleftrightarrow (Z_0'X : 0)K = 0$$

$$\Longleftrightarrow Z_0'XK_1 = 0 \Longleftrightarrow XK_1 = X_0K_3$$

for some K_3. Hence

$$VZ = X_0K_3 + V_1ZK_2 \Longleftrightarrow M(VZ) \subset M(X_0 : V_1Z).$$

Since $M(V) \subset M(X : VZ) \subset M(X : V_1Z)$, we can write

$$V = XD_1X' + V_1ZD_2Z'V_1 + XDZ'V_1 + V_1ZD'X'$$

for some D_1, D_2 and D. Using $M(VZ) \subset M(X_0 : V_1Z)$, we get

$$V_1ZD_2Z'V_1Z + XDZ'V_1Z = X_0U_1 + V_1ZU_2,$$

for some U_1 and U_2

$$\Longrightarrow XDZ'V_1Z = X_0U_1, \text{ using } M(X) \cap M(V_1Z) = \{0\}$$

$$\Longrightarrow M(XDZ'V_1) \subset M(X_0)$$

and hence
$$XDZ'V_1 = X_0D_3Z'V_1 \text{ for some } D_3.$$

The proof of Theorem 2.4 is thus complete.

Corollary 2.1 : *The BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ is its BLUE under $(Y, X\beta, \sigma^2 V)$ if and only if $Z'VZ_1 = 0$ or equivalently*

$$V = XD_1X' + ZD_2Z' + X_0D_3Z' + ZD'_3X'_0,$$

where D_1, D_2 and D_3 are arbitrary matrices subject to the condition that V is n.n.d.

Remark 2.1 : If $r(A) = r(X)$, then $X_0 = 0$ and Theorems 2.1, 2.2, 2.3 and 2.4 reduce to the results of Mitra and Moore (1973).

3. Robustness of the LRT-statistic

In this section we shall derive the necessary and sufficient conditions under which the LRT statistic for testing $H_0 : A\beta = 0$ under $(Y, X\beta, \sigma^2 V)$ coincides with probability one with the LRT statistic under $(Y, X\beta, \sigma^2 V_1)$. Since $M(A') = M(X'Z_0)$, the above hypothesis is equivalent to $H_0 : Z_0'X\beta = 0$. Under $(Y, X\beta, \sigma^2 V)$, the BLUE of $Z_0'X\beta$ is $u = Z_0'X(X'GX)^-X'GY$ with dispersion matrix $\sigma^2 D$, where $D = Z_0'X(X'GX)^-X'GVGX(X'GX)^-X'Z_0$ where $G = (V+XX')^-$. The hypothesis is consistent with the model if and only if $u \in M(D)$ [See Rao and Mitra (1971a) p 300, Rao (1972), p 371 or Mitra (1973) p 680]. They observe that if this condition is violated then the null hypothesis stands rejected. It can be shown that $u \in M(D)$ if and only if $Y \in M(X_0 : V)$ Hence, when we consider the LRT statistic for testing H_0, we will consider only those Y's which satisfy $Y \in M(X_0 : V)$. For testing H_0, the LRT statistic is given by

$$L_V = \delta \left(\frac{Y'Z_0(Z_0'VZ_0)^-Z_0'Y}{Y'Z(Z'VZ)^-Z'Y} - 1 \right)$$

$$= \delta \left[\frac{Y'(I-P_{X_0})((I-P_{X_0})V(I-P_{X_0}))^-(I-P_{X_0})Y}{Y'(I-P_X)((I-P_X)V(I-P_X))^-(I-P_X)Y} - 1 \right]$$

for $Y \in M(X_0 : V)$, where

$$\delta = \frac{r(VZ)}{r(VZ_0) - r(VZ)}$$

It can be shown that L_V as defined above coincides with the F-statistic given by Rao (1972, equation (4.6)), whenever $Y \in M(X_0 : V)$

Under the model $(Y, X\beta, \sigma^2 V_1)$ the LRT statistic for testing H_0 is defined only for $Y \epsilon M(X_0 : V_1)$. Hence, if we want the LRT statistic under $(Y, X\beta, \sigma^2 V)$ to coincide with probability one with the LRT statistic under $(Y, X\beta, \sigma^2 V_1)$, we should necessarily have $M(X_0 : V) \subset M(X_0 : V_1)$ or equivalently $M(V) \subset M(X_0 : V_1)$.

Khatri (1981) gives necessary and sufficient conditions under which $L_V = L_I$ for all Y. It turns out that when $L_V = L_I$ for all Y, then the degrees of freedom associated with the F-distributions (null) of L_V under $(Y, X\beta, \sigma^2 V)$ and L_I under $(Y, X\beta, \sigma^2 I)$ are the same. In other words the F-tests for testing H_0 under $(Y, X\beta, \sigma^2 V)$ and $(Y, X\beta, \sigma^2 I)$ are the same. However, if we want $L_V = L_I$ for $Y \epsilon M(X_0 : V)$, then eventhough the F-statistics under $(Y, X\beta, \sigma^2 V)$ and $(Y, X\beta, \sigma^2 I)$ coincide with probability one, the associated F-distributions (null) will have different degrees of freedom under the models and hence the F-tests are no longer the same. But $L_V = L_I$ for $Y \epsilon M(X_0 : V)$ together with the condition $r(VZ) = r(Z)$ and $r(VZ_0) = r(Z_0)$ will imply the F-statistics coincide with probability one and the associated F-distributions (null) have the same degrees of freedom under $(Y, X\beta, \sigma^2 V)$ and $(Y, X\beta, \sigma^2 I)$. These facts are stated in Theorem 3.1 and Corollary 3.1. We also consider the equality $L_V = L_{V_1}$ for $Y \epsilon M(X_0 : V)$ in Theorem 3.2 and Theorem 3.3.

Lemma 3.1 : *Let $H_0 : A\beta = 0$ be a hypothesis consistent with the model $(Y, X\beta, \sigma^2 V)$. Then for testing H_0, if the LRT-statistic under $(Y, X\beta, \sigma^2 V)$ coincides with probability one with the LRT-statistic under $(Y, X\beta, \sigma^2 V_1)$ where $M(V) \subset M(X_0 : V_1)$, then the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 V_1)$, irrespective of its linear representation, continues to be its BLUE under $(Y, X\beta, \sigma^2 V)$ also.*

Proof : Let

$$\delta = \frac{r(VZ)}{r(VZ_0) - r(VZ)} \quad \text{and} \quad \delta_1 = \frac{r(V_1 Z)}{r(V_1 Z_0) - r(V_1 Z)} .$$

Then for testing H_0, the F-statistics under $(Y, X\beta, \sigma^2 V)$ and $(Y, X\beta, \sigma^2 V_1)$ coincide with probability one if and only if

$$\delta \left(\frac{Y' Z_0 (Z_0' V Z_0)^- Z_0' Y}{Y' Z (Z' V Z)^- Z' Y} - 1 \right)$$

$$= \delta_1 \left(\frac{Y' Z_0 (Z_0' V_1 Z_0)^- Z_0' Y}{Y' Z (Z' V_1 Z)^- Z' Y} - 1 \right), \forall\ Y \epsilon M(X_0 : V). \quad \dots \quad (3.1)$$

In view of Theorem 2.4, the lemma will be proved if we show that (3.1) implies $M(VZ) \subset M(X_0 : V_1Z)$. For arbitrary θ putting $Y = VZ\theta$ in (3.1) we get

$$Z'VZ_0(Z_0'V_1Z_0)^-Z_0'VZ = Z'VZ(Z'V_1Z)^-Z'VZ. \quad \dots (3.2)$$

Since V satisfies the condition $M(V) \subset M(X_0 : V_1) = M(X_0 : V_1Z_o)$, we can write

$$VZ = X_0K_0 + V_1ZK + V_1Z_1K_1. \quad \dots (3.3)$$

Substituting in (3.2) and simplifying, we get

$$K_1'Z_1'(V_1Z_0(Z_0'V_1Z_0)^-Z_0'V_1 - V_1Z(Z'V_1Z)^-Z'V_1)Z_1K_1 = 0$$

$$\Longleftrightarrow V_1Z_0(Z_0'V_1Z_0)^-Z_0'V_1Z_1K_1 = V_1Z(Z'V_1Z)^-Z'V_1Z_1K_1$$

$$\Longleftrightarrow Z_0'V_1Z_1K_1 = Z_0'V_1Z(Z'V_1Z)^-Z'V_1Z_1K_1$$

$$\Longleftrightarrow Z_1'V_1Z_1K_1 = Z_1'V_1Z(Z'V_1Z)^-Z'V_1Z_1K_1$$

$$\Longleftrightarrow Z_1'(V_1 - V_1Z(Z'V_1Z)^-Z'V_1)Z_1K_1 = 0$$

$$\Longrightarrow V_1Z_1K_1 = V_1Z(Z'V_1Z)^-Z'V_1Z_1K_1$$

$$\Longleftrightarrow M(V_1Z_1K_1) \subset M(V_1Z).$$

Hence from (3.3) we get $M(VZ) \subset M(X_0 : V_1Z)$ and this concludes the proof of Lemma 3.1.

Remark 3.1 : For $V_1 = I$, Lemma 3.1 is proved in Mathew and Bhimasankaram (1983).

Theorem 3.1 : *Let $r(VZ) = r$ and $r(VZ_1) = s$ and let H_0 be consistent with $(Y, X\beta, \sigma^2V)$. Then under $(Y, X\beta, \sigma^2V)$, $L_V = L_I$ with probability one if and only if V satisfies $Z'VZ = kU_1U_1'$, $Z_1'VZ_1 = lU_2U_2'$ where U_1 and U_2 are semiorthogonal matrics of ranks r and s respectively and $Z'VZ_1 = 0$ or equivalently $V = XD_1X' + kZU_1U_1'Z' + X_0D_2Z' + ZD_2'X_0'$, where D_1 and D_2 are arbitrary subject to the conditions V is n.n.d. and $Z_1'XD_1X'Z_1 = lU_2U_2'$, k and l being positive real numbers satisfying $\frac{k}{l} \cdot \frac{r}{s} = \frac{r(Z)}{r(Z_1)}$.*

Proof : We want conditions under which

$$\delta\left(\frac{Y'Z_0(Z_0'VZ_0)^-Z_0'Y}{Y'Z(Z'VZ)^-Z'Y} - 1\right) = \delta_1\frac{Y'Z_1Z_1'Y}{Y'ZZ'Y} \quad \forall\ Y \in M(X_0 : V) \quad \dots (3.4)$$

where

$$\delta = \frac{r(VZ)}{r(VZ_0) - r(VZ)} \text{ and } \quad \delta_1 = \frac{r(Z)}{r(Z_1)}$$

From Lemma 3.1 and Theorem 2.4, we see that (3.4) holds if and only if

$$\mathcal{M}(VZ) \subset \mathcal{M}(X_0 : Z) \iff Z_1'VZ = 0.$$

Hence (3.4) simplifies to

$$\frac{r(VZ)}{r(VZ_1)} \cdot \frac{Y'Z_1(Z_1'VZ_1)^{-}Z_1'Y}{Y'Z(Z'VZ)^{-}Z'Y} = \frac{r(Z)}{r(Z_1)} \cdot \frac{Y'Z_1Z_1'Y}{Y'ZZ'Y} \;\forall\; Y \in \mathcal{M}(X_0 : V) \quad \ldots \quad (3.5)$$

Putting $Y = VZ\theta + VZ_1\theta_1$ in (3.5), we get

$$\frac{r(VZ)}{r(VZ_1)} \cdot \frac{\theta_1'Z_1'VZ_1\theta_1}{\theta'Z'VZ\theta} = \frac{r(Z)}{r(Z_1)} \cdot \frac{\theta_1'Z_1'VZ_1Z_1'VZ_1\theta}{\theta'Z'VZZ'VZ\theta}$$

$$\iff \frac{\theta'Z'VZZ'VZ\theta}{\theta'Z'VZ\theta} \cdot \frac{\theta_1'Z_1'VZ_1\theta_1}{\theta_1'Z_1'VZ_1Z_1'VZ_1\theta_1} = \frac{r(VZ_1)}{r(Z_1)} \; \frac{r(Z)}{r(VZ)} . \quad \ldots \quad (3.6)$$

A necessary and sufficient condition for the above to hold is

$$Z'VZ = kU_1U_1'$$

and

$$Z_1'VZ_1 = lU_2U_2' \quad \ldots \quad (3.7)$$

where U_1 and U_2 are semiorthogonal matrices of ranks r and s respectively, for some positive scalars k and l. From (3.6) it is clear that k and l should satisfy $\frac{k}{l} = \frac{s}{r} \cdot \frac{r(Z)}{r(Z_1)}$. Using Theorem (2.3), we get

$$V = XD_1X' + ZD_2Z' + X_0D_3Z' + ZD_3'X_0'$$

for some D_1, D_2 and D_3. The conditions on D_1 and D_2 given in the theorem are necessary and sufficient for (3.7) to hold.

Corollary 3.1 : *Let $r(VZ) = r(Z)$, $r(VZ_0) = r(Z_0)$ and let $H_0 : A\beta = 0$ be consistent with $(Y, X\beta, \sigma^2V)$. Then under $(Y, X\beta, \sigma^2V)$, $L_V = L_I$ with probability one if and only if any one of the following equivalent conditions holds :*

(i) $(I - P_{X_0})(V - kI)(I - P_{X_0}) = 0$ *for some* $k > 0$

(ii) $\begin{pmatrix} I - P_X \\ LP_X \end{pmatrix} (V - kI)(I - P_X : P_XL') = 0$ *for some* $k > 0$

where L is a matrix satisfying $LX = A$.

(iii) $V = XD_1X' + k(I - P_X) + X_0D_3Z' + ZD_3'X_0'$, *where D_1 and D_3 are arbitrary matrices and k is an arbitrary positive real number subject to the conditions*

(a) *V is n.n.d. and*

(b) $(P_X - P_{X_0})XD_1X'(P_X - P_{X_0}) = k(P_X - P_{X_0})$.

Proof: Since $r(VZ) = r(Z)$ and $r(VZ_0) = r(Z_0)$, from the proof of Theorem 3.1 it is clear that $L_V = L_I$ with probability one under $(Y, X\beta, \sigma^2 V)$ if and only if $Z'VZ_1 = 0$, $Z'VZ = kI_r$ and $Z_1'VZ_1 = kI_s$ which are equivalent to the condition

$$Z_0'VZ_0 = kI_{r+s},$$

which proves part (i) of the corollary, since $Z_0Z_0' = I - P_{X_0}$. The equivalence of (i) and (ii) is easily established. Since $P_X - P_{X_0} = Z_1Z_1'$, the equivalence of (i) and (iii) is also clear.

Remark 3.2 : The main result proved by Khatri (1981) states that $L_V = L_I$ for all Y if and only if $\begin{pmatrix} I-P_X \\ LP_X \end{pmatrix} (V-kI)(I-P_X : P_XL') = 0$ for some $k > 0$, or equivalently the covariance matrix of $\begin{pmatrix} I-P_X \\ LP_X \end{pmatrix}$ Y under $(Y, X\beta, \sigma^2 V)$ is a scalar multiple of its covariance matrix under $(Y, X\beta, \sigma^2 I)$. This equivalent form of the condition is a conjecture of J. K. Ghosh stated in Khatri's paper.

Remark 3.3 : For a positive definite V Corollary 3.1 (i) and (iii) have been obtained by Mathew and Bhimasankaram (1983).

Theorem 3.2 : *Let $r(VZ) = r(V_1Z)$, $r(VZ_0) = r(V_1Z_0)$ and let $H_0 : A\beta = 0$ be consistent with $(Y, X\beta, \sigma^2 V)$ where $M(V) \subseteq M(X_0 : V_1)$. Then for testing H_0 under $(Y, X\beta, \sigma^2 V)$, $L_V = L_{V_1}$ with probability one if and only if anyone of the following equivalent conditions holds* :

(i) $(I - P_{X_0})(V - kV_1)(I - P_{X_0}) = 0$ *for some* $k > 0$.

(ii) $\begin{pmatrix} I-P_{X,G} \\ LP_{X,G} \end{pmatrix}(V-kV_1)(I-P'_{X,G} : P'_{X,G}L') = 0$ *for some* $k > 0$, *where L is any matrix satisfying $LX = A$ and G is any g-inverse of $V_1 + XX'$.*

(iii) $V = XD_1X' + kV_1Z(Z'V_1Z)^-Z'V_1 + X_0D_3Z'V_1 + V_1ZD_3'X_0'$ *where D_1 and D_3 are arbitrary matrices and k is an arbitrary positive real number subject to the conditions*

(a) *V is n.n.d.*

(b) $Z_1'XD_1X'Z_1 = kZ_1'(V_1 - V_1Z(Z'V_1Z)^-Z'V_1)Z_1$ *and*

(c) $M(XD_1X') \subseteq M(X_0 : V_1)$.

Proof: Since $r(VZ) = r(V_1Z)$, $r(VZ_0) = r(V_1Z_0)$ and H_0 is consistent with $(Y, X\beta, \sigma^2 V)$, we see that $L_V = L_{V_1}$ with probability one under $(Y, X\beta, \sigma^2 V)$ if and only if

$$\frac{Y'Z_0(Z_0'VZ_0)^-Z_0'Y}{Y'Z(Z'VZ)^-Z'Y} = \frac{Y'Z_0(Z_0'V_1Z_0)^-Z_0'Y}{Y'Z(Z'V_1Z)^-Z'Y} \quad \forall\, Y \in M(X_0 : V). \quad \ldots \quad (3.8)$$

Let $V = CC'$. Writing $Z_0'Y = Z_0'C\theta$ in (3.8) we get

$$\frac{\theta'C'Z_0(Z_0'VZ_0)^-Z_0'C\theta}{\theta'C'Z(Z'VZ)^-Z'C\theta} = \frac{\theta'C'Z_0(Z_0'V_1Z_0)^-Z_0'C\theta}{\theta'C'Z(Z'V_1Z)^-Z'C\theta} \;\forall\, \theta. \quad \ldots \quad (3.9)$$

Using Lemma 3.1 we see that a necessary condition for (3.9) to hold is V admits the representation given in Theorem 2.4. Using this observation it can be shown that the matrices $C'Z_0(Z_0'VZ_0)^-Z_0'C$, $C'Z(Z'VZ)^-Z'C$, $C'Z_0(Z_0'V_1Z_0)^-Z_0'C$ and $C'Z(Z'V_1Z)^-Z'C$ commute pairwise and hence can be reduced to diagonal forms using the same orthogonal matrix P. Let the corresponding diagonal matrices be $\begin{pmatrix} I_{r+s} & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, Λ_0 and Λ where $r' = r(V_1Z)$ and $r+s = r(V_1Z_0)$. Writing $P'\theta = t = (t_1, t_2, \ldots, t_n)$, $\Lambda_0 = \text{diag}(\lambda_{01}, \lambda_{02}, \ldots, \lambda_{0r+s}, 0, \ldots, 0)$ and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0)$ we get, (3.9) holds $\forall\, \theta$ if and only if

$$\frac{\sum_{i=1}^{r+s} t_i^2}{\sum_{i=1}^{r} t_i^2} = \frac{\sum_{i=1}^{r+s} \lambda_{0i} t_i^2}{\sum_{i=1}^{r} \lambda_i t_i^2} \;\forall\, t_i$$

$$\Longleftrightarrow \lambda_{0i} = \lambda_j = \lambda, \quad i = 1, 2, \ldots, r+s$$

$$j = 1, 2, \ldots, r$$

$$\Longleftrightarrow VZ(Z'VZ)^-Z'V = \lambda VZ(Z'V_1Z)^-Z'V \quad \ldots \quad (3.10)$$

and

$$VZ_0(Z_0'VZ_0)^-Z_0'V = \lambda VZ_0(Z_0'V_1Z_0)^-Z_0'V \quad \ldots \quad (3.11)$$

$$(3.11) \Longleftrightarrow Z_0'VZ_0 = \lambda Z_0'VZ_0(Z_0'V_1Z_0)^-Z_0VZ_0$$

$\Longleftrightarrow$ every g-inverse of $\frac{1}{\lambda} Z_0'V_1Z_0$ is a g-inverse of $Z_0'\; VZ_0$. Since $r(VZ_0) = r(V_1Z_0)$, applying Theorem 1 in Rao, Mitra and Bhimasankaram (1972), we get

$$Z_0'VZ_0 = kZ_0'V_1Z_0 \text{ for some } k > 0 \quad \ldots \quad (3.12)$$

$$\Longleftrightarrow (I-P_{X_0})(V-kV_1)(I-P_{X_0}) = 0, \text{ since } Z_0Z_0' = I-P_{X_0}$$

This proves (i). Observe that (i) implies the condition $M(V) \subset M(X_0 : V_1)$, as required. The equivalence of (i) and (ii) can be established by showing that

$$M\{(V-kV_1)(I-P_{X_0})\} = M\{(V-kV_1)(I-P'_{X,G} : P'_{X,G}L')\}.$$

To prove (iii), observe that V should necessarily be of the form

$$V = XD_1X'+V_1ZD_2Z'V_1+X_0D_3Z'V_1+V_1ZD_3'X_0' \text{ for some } D_1, D_2 \text{ and } D_3$$

(3.12) then gives

$$Z_0'(XD_1X'+V_1ZD_2Z'V_1)Z_0 = kZ_0'V_1Z_0 \text{ for some } k > 0.$$

$$\Longleftrightarrow (Z'V_1Z)D_2(Z'V_1Z) = kZ'V_1Z \quad \ldots \quad (3.13)$$

and

$$Z_1'XD_1X'Z_1+Z_1'V_1ZD_2Z'V_1Z_1 = kZ_1V_1Z_1 \quad \ldots \quad (3.14)$$

(3.13) gives $D_2 = k(Z'V_1Z)^-$. Then from (3.14) we get

$$Z_1'XD_1X'Z_1 = kZ_1'(V_1-V_1Z(Z'V_1Z)^-Z'V_1)Z_1.$$

The condition $M(XD_1X') \subset M(X_0 : V_1)$ guarantees that $M(V) \subset M(X_0 : V_1)$.

Remark 3.4 : Matrices D_1 satisfying

$$M(XD_1X') \subset M(X_0 : V_1)$$

and

$$Z_1'XD_1X'Z_1 = kZ_1'(V_1-V_1Z(Z'V_1Z)^-Z'V_1)Z_1$$

could be characterised as follows. Let Q denote the parallel sum (see Rao and Mitra, 1971b, p. 189) of XX' and $V_1+X_0X_0'$. Then

$$M(XD_1X') \subset M(X_0 : V_1) \Longleftrightarrow XD_1X' = QDQ'.$$

D is obtained from

$$Z_1'QDQ'Z_1 = kZ_1'(V_1-V_1Z(Z'V_1Z)^-Z'V_1)Z_{\cdot 1}$$

Theorem 3.3 : *Let* $V_1 = C_1C_1'$, $TT' = C_1'Z(Z'V_1Z)^-Z'C_1$, *and*

$$QQ' = C_1'Z_0(Z_0'V_1Z_0)^-Z_0'C_1-C_1'Z(Z'V_1Z)^-Z'C_1.$$

Then under $(Y, X\beta, \sigma^2V)$ *where* $M(V) \subset M(X_0 : V_1)$, *for testing a hypothesis* $H_0 : A\beta = 0$ *consistent with the model,* $L_V = L_{V_1}$ *with probability one if and only if* $V = CC'$ *with* $C = X_0B_1+C_1B_2$, *where* B_1 *is arbitrary and* B_2 *is obtained from the equation*

$$\begin{pmatrix} T' \\ Q' \end{pmatrix} B_2B_2'(T : Q) = \begin{pmatrix} kU_1U_1' & 0 \\ 0 & lU_2U_2' \end{pmatrix}$$

U_1 and U_2 being semiorthogonal matrices and k and l are positive scalars satisfying

$$k\frac{r(V_1Z)}{r(V_1Z_0-r(V_1Z)} = l\frac{r(VZ)}{r(VZ_0)-r(VZ)}.$$

Proof : Using arguments similar to those given in the proof of Theorem 3.2, we get $L_V = L_{V_1}$ with probability one if and only if

$$C'Z(Z'VZ)^-Z'C = \lambda C'Z(Z'V_1Z)^-Z'C \quad \ldots \quad (3.15)$$

and

$$C'Z_0(Z_0'VZ_0)^-Z_0'C - C'Z(Z'VZ)^-Z'C$$

$$= \lambda.\frac{\delta_1}{\delta}(C'Z_0(Z_0'V_1Z_0)^-Z_0'C - C'Z(Z'V_1Z)^-Z'C) \quad \ldots \quad (3.16)$$

where

$$\delta_1 = \frac{r(V_1Z)}{r(V_1Z_0)-r(V_1Z)} \text{ and } \delta = \frac{r(VZ)}{r(VZ_0)-r(VZ)}.$$

Since the matrices on the left hand sides of (3.15) and (3.16) are symmetric and idempotent, they can be represented as P_1P_1' and P_2P_2' respectively, where P_1 and P_2 are semiorthogonal matrices satisfying $P_2'P_1 = 0$. Since we want $M(V) \subseteq M(X_0 : V_1)$, we can write $C = X_0B_1+C_1B_2$. Then (3.15) and (3.16) become

$$\lambda B_2'TT'B_2 = P_1P_1'$$

$$\lambda.\frac{\delta}{\delta_1}B_2'QQ'B_2 = P_2P_2'.$$

Using Lemma 2.2 of Bhimasankaram and Majumdar (1980), we see that the above equations are equivalent respectively to the equations

$$T'B_2 = k^{1/2}U_1P_1' \quad \ldots \quad (3.17)$$

and

$$Q'B_2 = l^{1/2}U_2P_2' \quad \ldots \quad (3.18)$$

where U_1 and U_2 are semiorthogonal matrices with

$$M(P_1') \subseteq M(U_1'),\ M(P_2') \subseteq M(U_2'),\ k^{-1} = \lambda \text{ and } l^{-1} = \lambda\frac{\delta}{\delta_1}.$$

(3.17) and (3.18) are together equivalent to

$$\begin{pmatrix} T' \\ Q' \end{pmatrix} B_2B_2'(T : Q) = \begin{pmatrix} kU_1U_1' & 0 \\ 0 & lU_2U_2' \end{pmatrix}$$

where k and l satisfy $k\delta_1 = l\delta$.

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