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LINEAR INVARIANCE AND ADMISSIBILITY IN SAMPLING FINITE POPULATIONS

By BIKAS K. SINHA[1] and SASTRY G. PANTULA

North Carolina State University, Raleigh, U.S.A.

SUMMARY. Using a technique of Patel and Dharmadhikari (1977), we introduce certain classes of linear invariant and sub-invariant admissible unbiased estimators of the mean in finite population sampling. The motivation rests on a study of the variance functions of such estimators. We also deduce explicit forms of linear invariant estimators for two familiar sampling schemes, viz., (a) ppswor sampling of size two, (b) Midzuno scheme of size n.

1. Introduction

For a finite population having identifiable units, the nonexistence of a uniformly minimum variance unbiased estimator (based on an arbitrary sampling design) for the population mean of a study-character was initially pointed out by Godambe (1955) and, subsequently, rigorously established by Basu (1966)... Roy and Chakravarti (1960) pleaded for the property of *linear invariance* as desirable of any estimator of the population mean whenever the study-character possesses certain units of measurement. Patel and Dharmadhikari (1977) used the concept of linear invariance in developing a nice tool for constructing *admissible* (linear invariant) unbiased estimators. However, they were unable to deduce explicit forms of such estimators. Subsequently, they modified the technique (Patel and Dharmadhikari (1978)) and introduced certain other classes of admissible estimators.

In Section 2, we review the results of Patel and Dharmadhikari (1977). We reemphasize the use of the linear invariance as a tool for constructing admissible estimators. We also obtain an alternate set of equations for obtaining linear invariant admissible estimators. In Section 3, we introduce a related concept of linear sub-invariance and use it as another tool to construct certain other admissible estimators in the class of linear unbiased estimators. The motivation lies in a study of the nature of the variance functions of such estimators. In the Appendix, we derive explicit forms of admissible linear invariant unbiased estimators of the population mean for two familiar sampling schemes, viz., (i) ppswor of sample size two and (ii) Midzuno (1950) scheme of sample size n.

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[1]On leave from the Indian Statistical Institute

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2. Linear invariance and admissibility

Consider a finite population $u_N = \{1, 2, \ldots, N\}$ of N identifiable units. Let (S, P) denote a sampling design for u_N so that $p(s) > 0$ for every $s \in S$ and $\sum_{s \in S} p(s) = 1$. A linear estimator $l(s, \bar{Y})$, for the population mean $\bar{Y}$ of a character Y under study, based on a sample s, has the form

$$l(s, \bar{Y}) = \sum_{i \in s} \beta(s, i) Y_i. \qquad \ldots \quad (1)$$

See Raj (1968) for the general notation and definitions. The conditions of unbiasedness and linear invariance are respectively,

$$\sum_{s \supset \{i\}} \beta(s, i) p(s) = 1/N, \quad i = 1, 2, \ldots, N \qquad \ldots \quad (2)$$

and

$$\sum_{i \in s} \beta(s, i) = 1, \; s \in S. \qquad \ldots \quad (3)$$

We assume that the sampling design is connected. That is, any two units (samples) are connected through one or more chains of samples (units) and units (samples). See Patel and Dharmadhikari (1977) for details. Let $\pi_i = \sum_{s \supset \{i\}} p(s)$ and $\pi_{ij} = \sum_{s \supset \{i, j\}} p(s)$, denote the inclusion probabilities of the unit i and the units i and j, respectively. Following the notation of Patel and Dharmadhikari (1977), let

$$c_{ii} = \pi_i - \sum_{s \supset \{i\}} \{p(s)/n(s)\}, \quad i = 1, 2, \ldots, N$$

$$c_{ij} = -\sum_{s \supset \{i, j\}} \{p(s)/n(s)\}, \quad i \neq j = 1, 2, \ldots, N \qquad \ldots \quad (4)$$

where $n(s)$ denotes the size of the sample s. The $N \times N$ matrix $\boldsymbol{C} = ((c_{ij}))$ is called the $\boldsymbol{C}$-matrix of the sampling design. Note that the matrix $\boldsymbol{C}$ is symmetric and that $\boldsymbol{C}\boldsymbol{1} = \boldsymbol{0}$ where $\boldsymbol{1} = (1, 1, \ldots, 1)'$ and $\boldsymbol{0} = (0, 0, \ldots, 0)'$. Since the design is assumed to be connected, the rank of the matrix $\boldsymbol{C}$ is $N-1$. For the sake of completeness we state the results of Patel and Dharmadhikari (1977) in the following theorem.

Theorem 2.1: (Patel and Dharmadhikari, 1977): *Assume that the design is connected. The estimator $l(s, \bar{Y})$ with $\beta(s, i) = [1/n(s)] + \lambda_i - \bar{\lambda}_s$, where the λ_i's satisfy*

$$C\boldsymbol{\lambda} = \boldsymbol{d}, \qquad \ldots \quad (5)$$

$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)'$, $d_i = N^{-1} - \sum_{s \supset \{i\}} [p(s)/n(s)]$, $i = 1, 2, \ldots, N$ and $\bar{\lambda}_s = \sum_{i \in s} \lambda_i/n(s)$, is linear invariant, unbiased for $\bar{Y}$, and minimizes the sum of the variances at the points $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_N$. The system of equations (5) is consistent. Also, the estimator l is unique and, hence, l is admissible within the class of all linear unbiased estimators of $\bar{Y}$. Here $\mathbf{Y}_i$ stands for an $N \times 1$ vector with 1 in the i-th position and 0 elsewhere.

The system (5) seems intractable in general terms. However, for at least two familiar schemes of sampling, viz., (i) ppswor of sample size two and (ii) Midzuno scheme, it is possible to obtain an algebraic expression for λ satisfying (5). The calculations as well as the resulting expressions for linear invariant estimators for these two designs are given in the Appendix. It may be noted that for the scheme (i), Sengupta (1980) has established admissibility of the symmetrized Des Raj estimator which is not, however, linear invariant. For the scheme (ii), the Horvitz-Thompson estimator (HTE) is also not linear invariant.

Patel and Dharmadhikari (1977) deduced that a choice of $\beta(s, i)$ as $\beta(s, i) = \lambda_i + a_s$ together with the conditions (2) and (3) results in $\beta(s, i) = [1/n(s)] + \lambda_i - \bar{\lambda}_s$ where λ_i's satisfy the system (5). Likewise, it may be seen that such a $\beta(s, i)$ also has an alternative representation given by $\beta(s, i) = \{1/N\pi_i\} + \omega_s - \bar{\omega}_i$ where $\bar{\omega}_i = \sum_{s \supset \{i\}} \omega_s p(s)/\pi_i$, the ω_s's satisfy an analogous system

$$\mathbf{D}\,\boldsymbol{\omega} = \mathbf{q}, \qquad \ldots \quad (6)$$

with

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_M)',$$

$$\mathbf{D} = (\,(d_{ss'})\,),\ M \times M,$$

$$\mathbf{q} = (q_1, q_2, \ldots, q_M)',$$

$$d_{ss} = n(s)p(s) - p^2(s) \sum_{i \in s} (1/\pi_i),$$

$$d_{ss'} = -p(s)p(s') \sum_{i \in s \cap s'} (1/\pi_i),\ s \neq s',$$

$$q_s = p(s) - p(s) \sum_{i \in s} (N\pi_i)^{-1},$$

and M is the cardinality of S. Note that the study of admissibility essentially rests on the reduced forms of sampling designs wherein no two samples are equivalent. Therefore, M is at most $2^N - 1$ and the matrix equation (6) represents a finite system.

It is readily verified that $D\,1 = 0$ and $q'1 = 0$ so that the system (6) is consistent and the solution is unique (up to $\omega_s - \bar{\omega}_i$). This reminds one of similar studies in the block-design set-up. It can be shown that the rank of D is $M-1$, if and only if, the rank of C is $N-1$. In practice, therefore, given a connected sampling design, one might first check the aspect of simplicity in solving the systems (5) and (6). If M is smaller than N, one might prefer the system (6) to (5).

The two alternative representations of the admissible linear invariant unbiased estimator $l(s, \bar{Y})$ e.g.

$$l(s, \bar{Y}) = \bar{y}(s) + \sum_{i \in s} (\lambda_i - \bar{\lambda}_s) Y_i$$

and

$$l(s, \bar{Y}) = \frac{1}{N} \sum_{i \in s} (Y_i/\pi_i) + \sum_{i \in s} (\omega_s - \bar{\omega}_i) Y_i$$

where $\bar{y}_s = \sum_{i \in s} Y_i/n(s)$, have interesting interpretations. In the first case, it is the sample mean adjusted for unbiasedness by an error function and in the second case, it is the HTE adjusted for linear invariance by another error function. This latter representation allows us to explore further the behavior of variance functions of such estimators and investigate the question of availability of similar other admissible estimators.

3. Linear sub-invariance and admissibility

In this section we generalize the concept of linear invariance and obtain other types of admissible estimators. Let us first examine the variance curves of the two admissible estimators: HTE and $l(s, \bar{Y})$ defined in Theorem 2.1. The HTE has the least variance at the points Y_i, $i = 1, 2, \ldots, N$, among all linear unbiased estimators. Also, the curve of the variance of HTE *usually* stays above the X-axis, except at the origin. On the other hand, the curve of the variance of $l(s, \bar{Y})$ defined in Theorem 2.1, hits the X-axis at the point 1 and stays above the curve of the variance of HTE at the points Y_i, $i = 1, 2, \ldots, N$. Naturally, no estimator is better than the other and both estimators are admissible. (See also Godambe, 1960).

The following two questions naturally arise:

Q1: Do there exist other admissible variance curves that intersect the curve of V(HTE) at one or more points $Y_1, Y_2, \ldots, Y_N$ and yet hit the X-axis at a point ?

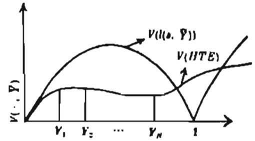

Fig. 1

Q2 : Do there exist other admissible variance curves that hit the X-axis at more than one point ? (It must be understood that we are referring to points that are not proportional.)

We will now establish that such admissible curves do exist. We define systems analogous to (6) and construct admissible estimators for the population mean based on such admissible curves. We first introduce the concept of sub-invariance.

Definition 3.1 : An unbiased estimator is said to be type I—linear sub-invariant of order m if for some choice of $(i_1, i_2, \ldots, i_m) \subset \{1, 2, \ldots, N\}$, the variance curve of the estimator attains the minimum value at each of the points $Y_{i_1}, Y_{i_2}, \ldots, Y_{i_m}$ and zero value at the point $1 - \sum_{j=1}^{m} Y_{i_j}$, simultaneously. For $m = 0$, it is a linear invariant estimator.

Note that, for $m = 0$, we know that a linear invariant estimator exists. Also, for $m = N$, the HTE is linear sub-invariant of order N. For $m = N-1$, a linear sub-invariant estimator of order $N-1$ has to assume minimum value at all points $Y_1, Y_2, \ldots, Y_N$ except at one such point where it has to assume zero value. Clearly, this will be possible if and only if the sampling design would provide $\pi_i = 1$ for some i and the HTE would then serve the purpose. So, for $m = 0$, $N-1$ and N, we have linear sub-invariant estimators and they are clearly admissible. For other values of m, $1 \leq m \leq N-2$, we establish the existence of such estimators in the following lemma.

Lemma 3.1 : *Assume that the sampling design is connected. In addition, suppose for some m, $1 \leq m \leq N-2$ and for some $(i_1, i_2, \ldots, i_m)$, the design is such that (i) $p(s) = 0$ whenever $s \subseteq \{i_1, i_2, \ldots, i_m\}$ and (ii) the reduced sampling* *design obtained by deleting $(i_1, i_2, \ldots, i_m)$ from the original samples is also connected. Then, there exists at least one type-I linear invariant estimator of order m.*

Proof: For the simplicity of notation let $i_j = j$, $j = 1, 2, \ldots, m$. Consider a linear estimator $l(s, \overline{Y}) = \sum_{i \epsilon s} \beta(s, i) Y_i$. We now minimize the sum of the variance of $l(s, Y)$ at the points $Y_{m+1}, \ldots, Y_N$ subject to the conditions of unbiasedness, zero variance at the point $1 - \sum_{i=1}^{m} Y_i$ and the minimum variance at the points $Y_1, Y_2, \ldots, Y_m$, simultaneously. The conditions are respectively,

$$\text{(i)} \quad \sum_{s \supset \{i\}} \beta(s, i) p(s) = N^{-1}, \; i = 1, 2, \ldots, N,$$

$$\text{(ii)} \quad \sum_{i \epsilon s - u_m} \beta(s, i) = N^{-1}(N-m), \text{ for all } s, \text{ and} \quad \ldots (7)$$

$$\text{(iii)} \quad \beta(s, i) = (N\pi_i)^{-1}, \; i \epsilon S \cap u_m$$

where $u_m = \{1, 2, \ldots, m\}$. Following the minimization technique of Patel and Dharmadhikari (1977) and the arguments of Section 2, we obtain the estimator $l^{(m)}(s, \overline{Y}) = \sum_{i \epsilon s} \beta^{(m)}(s, i) Y_i$, with

$$\beta^{(m)}(s, i) = \begin{cases} (N\pi_i)^{-1}, & i \epsilon s, \; i = 1, 2, \ldots, m \\ (N\pi_i)^{-1} + \omega_s - \overline{\omega}_i, & i \epsilon s, \; i = m+1, \ldots, N \end{cases} \quad \ldots (8)$$

$$\overline{\omega}_i = \pi_i^{-1} \sum_{s \supset \{i\}} \omega_s p(s), \; i = m+1, \ldots, N,$$

and ω_s's satisfy,

$$D^{(m)} \omega = q^{(m)}, \quad \ldots (9)$$

where $\quad D^{(m)} = ((d_{ss'})),$

$$d_{ss}^{(m)} = n^*(s) p(s) - p^2(s) \sum_{i \epsilon s - u_m} \pi_i^{-1},$$

$$d_{ss'}^{(m)} = -p(s) p(s') \sum_{i \epsilon s \cap s' - u_m} \pi_i^{-1}, \; s \neq s',$$

$$q_s^{(m)} = N^{-1}(N-m) p(s) - p(s) \sum_{i \epsilon s - u_m} (N\pi_i)^{-1}$$

and $\quad n^*(s) = n(s) - n(s \cap u_m).$

Note that, with $u_m' = \{m+1, m+2, \ldots, N\}$,

$$1'q^{(m)} = N^{-1}(N-m) \sum_{s \not\subseteq u_m} p(s) - \sum_{i \in u_m'} (N\pi_i)^{-1} \sum_{s \supset \{i\}} p(s)$$

which is zero since we assumed that $p(s) = 0$ for every $s \subseteq u_m$. Similarly, one can show that $D^{(m)}1 = 0$. Now, as the resulting sampling scheme obtained by deleting u_m from all the samples is still connected, the rank of $D^{(m)}$ is equal to the number of rows of $D^{(m)}$ minus one. Therefore, the system of equations in (9) is consistent and the estimator $l^{(m)}(s, \underline{Y})$ given in (8) is a type I-linear sub-invariant estimator of order m.

Remark 3.1 : Note that one cannot exclude the condition that $p(s) = 0$ for $s \not\subseteq u_m$ in Lemma 3.1, as otherwise $1'q^{(m)} \neq 0$. Also, the value of the estimator $l(s, \underline{Y})$ at the point $1 - \sum_{j=1}^{m} Y_j$ is zero for $s \subseteq u_m$ and is $N^{-1}(N-m)$ for $s \not\subseteq u_m$. Therefore, $l(s, \underline{Y})$ would not have achieved zero variance at the point $1 - \sum_{j=1}^{m} Y_j$ unless $p(s) = 0$ for all $s \subset u_m$. On the other hand, connectedness on the reduced sampling design can be replaced by appropriate conditions on the different disconnected parts of the reduced design. (See Patel and Dharmadhikari, 1977, for such conditions).

We will now answer Q1 by establishing the admissibility of the estimator $l^{(m)}(s, \underline{Y})$ defined in (8).

Theorem 3.1 : *Assume that the conditions of Lemma 3.1 hold. Then for every m, $1 \leqslant m \leqslant N-2$, the estimator $l^{(m)}(s, \underline{Y})$ is admissible in the class of linear unbiased estimators.*

Proof : Note that the estimator $l^{(m)}(s, \underline{Y})$ determined through (8) and (9) is the *unique* linear unbiased estimator that minimizes the sum of the variances at the points $Y_{m+1}, \ldots, Y_N$ subject to attaining the minimum variance at each of the points $Y_1, Y_2, \ldots, Y_m$ and also zero variance at the point $1 - \sum_{j=1}^{m} Y_j$. Therefore, the estimator is admissible.

We now introduce another type of sub-invariance that is related to Q2

Definition 3.2 : A linear unbiased estimator is said to be a type II linear sub-invariant estimator of order m if for some $(i_1, i_2, \ldots, i_m) \subset \{1, 2, \ldots, N\}$, the variance of the estimator is zero at each of the points $1 - Y_{i_1}, 1 - Y_{i_2}, \ldots, 1 - Y_{i_m}$, simultaneously. For $m = 0$, it is a linear invariant estimator.

We now establish the existence of a type II linear sub-invariant estimator.

Lemma 3.2 : *Assume that the conditions of Lemma 3.1 hold. In addition, assume that* $\pi_{i_1, i_2, \ldots, i_m}$ (the m-th order inclusion probability of $\{i_1, i_2, \ldots, i_m\}$ $= u_m$ (say)), is strictly positive. Then there exists a type II linear sub-invariant estimator of order m.

Proof: For simplicity of the notation, set $i_j = j$, $j = 1, 2, \ldots, m$, so that $\pi_{12\ldots m} > 0$. Following the arguments similar to that of Lemma 3.1, we define

$$l_{(m)}(s, \bar{Y}) = \sum_{i \in s} \beta_{(m)}(s, i) Y_i \qquad \ldots \quad (10)$$

with

$$\beta_{(m)}(s, i) = \begin{cases} (N \pi_{12\ldots m})^{-1} & \text{for } i = 1, 2, \ldots, m, s \supseteq u_m \\ 0 & \text{for } i = 1, 2, \ldots, m, s \not\supseteq u_m \\ (N \pi_i)^{-1} + \omega_s - \bar{\omega}_i & \text{for } i \in s \cap u_m^{*} \end{cases}$$

where
$$\bar{\omega} = \pi_i^{-1} \sum_{s \supset \{i\}} \omega_s p(s)$$

and $u_m^{*} = (m+1, m+2, \ldots, N)$.

The coefficients $\beta_{(m)}(s, i)$ are well defined because $\pi_{12\ldots m}$ is assumed to be positive. In order that $l_{(m)}(s, \bar{Y})$ is type II linear sub-invariant, it is necessary and sufficient that ω_s satisfy,

$$D^{(m)} \omega = q_{(m)} \qquad \ldots \quad (11)$$

where $D^{(m)}$ is as defined in (9),

$$q_{(m)s} = \left(1 - \frac{1}{N}\right) p(s) - \frac{(m-1)\, p(s)}{N \pi_{12\ldots m}} \delta_{s:u_m} - \sum_{i \in s - u_m} (N\pi_i)^{-1} p(s),$$

and
$$\delta_{s:u_m} = \begin{cases} 1 & \text{if } s \supseteq u_m \\ 0 & \text{otherwise.} \end{cases}$$

In view of connectedness of the reduced design leading to $D^{(m)}$ and the fact $1' q_{(m)} = 0$, we get the consistency and uniqueness (up to $\omega_s - \bar{\omega}_i$) of ω in (11).

We now state the result on admissibility of $l_{(m)}(s,)\bar{Y}$.

Theorem 3.2 : *Assume that the conditions of Lemma 3.2 hold. The estimator $l_{(m)}(s, \bar{Y})$ defined in (10) with ω_s satisfying (11) is admissible within the class of linear unbiased estimators of the population mean.*

The proof is similar to that of Theorem 3.1 and hence is omitted.

Remark 3.2 : The system (9) can be translated into the corresponding system $C^{(m)} \lambda = d^{(m)}$ where

$$c_{ii}^{(m)} = \pi_i - \sum_{s \supset (i)} \{p(s)/n^*(s)\}.$$

$$c_{ij}^{(m)} = - \sum_{s \supset (i, j)} \{p(s)/n^*(s)\},$$

and

$$d_i^{(m)} = N^{-1} - N^{-1}(N-m) \sum_{s \supset (i)} \{p(s)/n^*(s)\},$$

for $m+1 \leqslant i \neq j \leqslant N$.

Similarly, the system (11) has the counterpart $C^{(m)} \lambda = d_{(m)}$ where

$$d_{(m), i} = N^{-1} - N^{-1}(N-1) \sum_{s \supset (i)} \{p(s)/n^*(s)\}$$

$$+ (N\pi_{12\ldots m})^{-1}(m-1) \sum_{s \supset (i)} \delta_{s:u_m} \{p(s)/n^*(s)\}$$

with $\delta_{s:u_m}$ and $n^*(s)$ as defined in Lemma 3.2.

We conclude this section with the following examples based on SRSWOR (N, n) sampling procedure. For $m < n$, the linear sub-invariant estimators given in Lemma 3.1 and Lemma 3.2 are

$$\text{(i)} \quad l^{(m)}(s, \bar{Y}) = \left\{1 - \frac{n^*(s)}{n}\right\} \bar{y}(s \cap u_m) + \left\{1 - \frac{m}{N}\right\} \bar{y}(s \cap u_m')$$

and

$$\text{(ii)} \quad l_{(m)}(s, \bar{Y}) = \frac{(N-1)^{(m-1)}}{n^{(m)}} \bar{y}(u_m)\delta_{s:u_m}$$

$$+ \left\{1 - \frac{1}{N} - \frac{(N-1)^{(m-1)}}{n^{(m)}} \delta_{s:u_m}\right\} \bar{y}(s \cap u_m')$$

where $\bar{y}(\,.\,)$ = mean of the elements in the set $(\,.\,)$,

$$n^{(m)} = n(n-1)\ldots(n-m+1)$$

and $u_m' = u_N - u_m$. If $m = 0$ we get $l^{(0)}(s, \bar{Y}) = l_{(0)}(s, \bar{Y}) = \bar{y}(s)$.

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Appendix

Here we derive explicit expressions for linear invariant estimators for two familiar sampling schemes by solving the system $C\lambda = d$ where C, λ and d are defined in (5). Note that the system $C\lambda = d$ with $C\mathbf{1} = 0$ and $d'\mathbf{1} = 0$ where C is symmetric, is equivalent to $(C+\alpha\alpha')\lambda = d$ for any α satisfying $\alpha'\mathbf{1} \neq 0$. Now a suitable choice of α can be made such that (i) the rank of $C+\alpha\alpha' = N$ and (ii) explicit algebraic expressions for the elements of $(C+\alpha\alpha')^{-1}$ are available. Moreover, in our applications, we are concerned with contrasts involving the λ_i's so that in any solution to λ, the terms proportional to $\mathbf{1}$ can be ignored.

(i) *PPSWOR* (N, $n = 2$, $\mathbf{p} = (p_1, p_2, \ldots, p_N)'$) *sampling scheme* :

Note that for this scheme, $\pi_i = p_i\{1+\sum_{j\neq i} p_j(1-p_j)^{-1}\}$ and $\pi_{ij} = p_ip_j \{(1-p_i)^{-1}+(1-p_j)^{-1}\}$. Then, the system (5) reads as

$$H\lambda = \mathbf{h}, \qquad \ldots \quad (A.1)$$

where

$$H = M-(\mathbf{p}\boldsymbol{\delta}'+\boldsymbol{\delta}\mathbf{p}'),$$

$$M = \text{diagonal}\ (\theta_1, \theta_2, \ldots, \theta_N),$$

$$\mathbf{h} = 2\,N^{-1}\mathbf{1}-\boldsymbol{\theta}+2(\boldsymbol{\delta}-\mathbf{p}),$$

$$\mathbf{1} = (1, 1, \ldots, 1)',$$

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_N)',$$

$$\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_N)',$$

$$\theta_i = a\,p_i+\delta_i$$

$$a = \boldsymbol{\delta}'\mathbf{1}$$

and

$$\delta_i = p_i(1-p_i)^{-1}.$$

We take $\alpha = (\mathbf{p}+\boldsymbol{\delta})$ in the above formulation. Then,

$$\lambda = (H+\alpha\alpha')^{-1}\mathbf{h},$$

and after some routine algebra, it can be shown that

$$\lambda = 2N^{-1}\,M^{-1}\mathbf{1}-[(a+1)\,(2-z)+y]\,M^{-1}\,\mathbf{p}+(l+2-z)\,\mathbf{1}$$

and hence

$$\lambda_i-\lambda_j = \frac{(p_i-p_j)}{N(a+1-ap_i)(a+1-ap_j)}$$

$$\left[\frac{2(N+1-ax)}{(a+1)x}+\frac{N}{x}-\frac{2}{p_ip_j}\{a(1-p_i)(1-p_j)+1\}\right] \quad \ldots \quad (A.2)$$

where $z = (\delta-p)'M^{-1}h[x(x+1)(a+1)^2]^{-1}$,

$l = -(zx+1)$

and $x = \sum_{i=1}^{N}(p_i^2/\theta_i)$.

See Sinha and Pantula (1982) for details.

The linear invariant unbiased estimator, for the population mean, of Patel and Dharmadhikari (1977) is then given by

$$l(s, \overline{Y}) = \tfrac{1}{2}(Y_i+Y_j)+\tfrac{1}{2}(Y_i-Y_j)(\lambda_i-\lambda_j) \quad \ldots \quad (A.3)$$

where $s = \{i, j\}$ and $(\lambda_i-\lambda_j)$ is as defined in (A.2).

(ii) *Midzuno scheme for samples of size* n ;

For this scheme, $\pi_i = (N-1)^{-1}\{(n-1)+(N-n)p_i\}$ and $\pi_{ij} = \{(N-1)(N-2)\}^{-1}\{(n-1)(n-2)+(n-1)(N-n)(p_i+p_j)\}$. The system (5), then, reads as

$$R\lambda = r \quad \ldots \quad (A.4)$$

where

$R = P+b\,1\,1'-c(p1'+1p')$,

$P = \text{diagonal}\,(a_1, a_2, \ldots, a_N)$,

$b = -[n(N-1)(N-2)]^{-1}(n-1)(n-2)$,

$c = [n(N-1)(N-2)]^{-1}(n-1)(N-n)$,

$a_i = g+fp_i,\ i = 1, 2, \ldots, N$

$g = \{n(N-1)(N-2)\}^{-1}(n-1)(Nn-N-n)$,

$f = \{n(N-1)(N-2)\}^{-1}(n-1)N(N-n)$,

$r = \{n(N-1)\}^{-1}(N-n)\{N^{-1}1-p\}$,

and $p = (p_1, p_2, \ldots, p_N)'$.

We take $\alpha = (1+p)$ in the above formulation and obtain,

$$\lambda = (R+\alpha\alpha')^{-1}r.$$

After some routine algebra (see Sinha and Puntula, 1982), it can be shown that

$$\lambda_i = x\, q_i^{-1} + l,\ i = 1, 2, \ldots, N \qquad \ldots \quad (A.5)$$

where

$$x = \{nN(N-1)(1+dy)g\}^{-1}(N-n)(N+1)f(f+c)$$

$$q = f^2 + cg^2y - cgN + cf - dfc(N-gy)$$

$$y = (n-1)^{-1}n(N-1)(N-2)z$$

$$z = \sum_{i=1}^{N}[(Nn-N-n)+N(N-n)p_i]^{-1}$$

$$d = b+c$$

and l is some constant.

The linear invariant unbiased estimator of Patel and Dharmadhikari (1977) is, then, given by

$$l(s, \overline{Y}) = \bar{y}(s) + \sum_{i \in s}(\lambda_i - \bar{\lambda}_s)Y_i \qquad \ldots \quad (A.6)$$

where λ_i's are given in (A.5) and $\bar{\lambda}_s = [n(s)]^{-1}\sum_{i \in s}\lambda_i$.

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