ZERO-SETS OF QUATERNIONIC AND OCTONIONIC ANALYTIC FUNCTIONS WITH CENTRAL COEFFICIENTS

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ABSTRACT

We prove that the zero set of any quaternionic (or octonionic) analytic function f with central (that is, real) coefficients is the disjoint union of codimension two spheres in $\mathbb{R}^4$ or $\mathbb{R}^8$ (respectively) and certain purely real points. In particular, for polynomials with real coefficients, the complete root-set is geometrically characterisable from the lay-out of the roots in the complex plane. The root-set becomes the union of a finite number of codimension 2 Euclidean spheres together with a finite number of real points. We also find the preimages $f^{-1}(A)$ for any quaternion (or octonion) A.

We demonstrate that this surprising phenomenon of complete spheres being part of the solution set is very markedly a special 'real' phenomenon. For example, the quaternionic or octonionic Nth roots of any non-real quaternion (respectively octonion) turn out to be precisely N distinct points. All this allows us to do some interesting topology for self-maps of spheres.

Introduction

Let $\mathbb{R}^4 \equiv \mathbb{H}$ be identified with the space of quaternions, correspondingly the octonions are $\mathbb{O} \equiv \mathbb{R}^8$.

Our aim in this note is to describe geometrically the solution sets of analytic equations (in particular polynomials) of the form $f(V) = A$ where A is any quaternion (or octonion) and $f(V)$ is any analytic function (Laurent series with real coefficients about real centres) over the quaternions (or octonions). Our chief method of attack is to utilise certain geometrical interpretations of such quaternionic and octonionic analytic functions which we had discovered and discussed in previous papers [1, 3]. We had called that the theory of the Fueter transform. It is important to recall that the very nature of this transform imposed the restriction that the coefficients of our analytic functions or polynomials be from the centre of the algebra $\mathbb{H}$ (or $\mathbb{O}$). Namely, these coefficients must be real numbers.

In §2 we are able to provide direct algebraic proofs of some of our results for the case of polynomials. Certain algebraic aspects of these proofs appear to be of independent interest.

An amusing topological application of our results is to exhibit natural self-maps of the Euclidean unit spheres of dimensions 3 and 7 (namely, the quaternionic and octonionic unit spheres) which are of topological degree N (N any integer) such that almost every fibre has precisely $|N|$ distinct points, while all the exceptional fibres actually contain codimension 1 subspheres. The number of exceptional fibres is one for $N = 2$ and two otherwise. Using the Fueter transform we are also able to study a natural generalisation of these self-mappings on spheres of arbitrary dimension.

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REMARK. Because of the non-commutativity of H and O the only polynomial equations which are unambiguously defined, independent of the positions of the coefficients in each term, are precisely of the general form $f(V) = A$ that we have treated in this article. Happily, non-associativity in O never causes any problems since any two octonions generate an associative subalgebra.

§1. *Zero-sets via the Fueter theory*

In [1, 3] we discussed some generalisations of complex analytic mappings obtained from holomorphic mappings by a geometrical process of 'rotation around the real axis'. This goes as follows.

Let D be a domain in the upper half plane U. Suppose

$$\varphi = \zeta + i\eta : D \longrightarrow \mathbf{C}$$

is a holomorphic mapping. The n-dimensional Fueter transform of φ denoted by $F_n(\varphi)$ is a C^∞ (in fact C^ω) mapping from an open domain of $\mathbf{R}^n$, denoted by $F_n(D)$, into $\mathbf{R}^n$. Here,

$$F_n(D) = \{x_0 + e_1 x_1 + \dots + e_{n-1} x_{n-1} : (x_0, (x_1^2 + \dots + x_{n-1}^2)^{\frac{1}{2}}) \in D\} \subseteq \mathbf{R}^n \tag{1}$$

and $F_n(\varphi) : F_n(D) \to \mathbf{R}^n$ is given by

$$x_0 + e_1 x_1 + \dots + e_{n-1} x_{n-1} \longmapsto \zeta(x_0, x) + \left(\frac{e_1 x_1 + \dots + e_{n-1} x_{n-1}}{x}\right) \eta(x_0, x) \tag{2}$$

where $x = (x_1^2 + \dots + x_{n-1}^2)^{\frac{1}{2}}$ (positive square root).

Note that if φ has real boundary values where the real axis abuts D then a direct application of the reflection principle guarantees that we can define $F_n(\varphi)$ real analytically on the revolved domain $F_n(D)$ together with corresponding portions of x_0-axis.

A fundamental observation for our purposes is that if φ has a Laurent expansion with real coefficients about real centres (that is,

$$\varphi(z) = \sum_{n=0}^{\infty} a_n (z-c)^n + \sum_{m=1}^{\infty} b_m/(z-c)^m, \tag{3}$$

where a_n, b_m, c are reals; the annulus of convergence is $r < |z-c| < R$), then

$$F_4(\varphi)(V) = \sum_{n=0}^{\infty} a_n (V-c)^n + \sum_{m=1}^{\infty} b_m/(V-c)^m, \tag{4}$$

where $V = x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3$ is a quaternionic variable. Similarly, $F_8(\varphi)$ will be represented by the 'same' Laurent series with V an octonionic variable. The corresponding domains of convergence are the ring-domains $r < \|V-c\| < R$ in Euclidean spaces $\mathbf{R}^4$ and $\mathbf{R}^8$ respectively.

It is important to interpret the Fueter transforms of functions and domains via rigid rotations around the real axis.

We can identify $'\mathbf{R}^n = \mathbf{R}^n - \{x_0\text{-axis}\}$ with $U \times S^{n-2}$ using the mapping

$$(x_0 + e_1 x_1 + \dots + e_{n-1} x_{n-1}) \longmapsto \left(x_0 + ix, \left(\frac{x_1}{x}, \dots, \frac{x_{n-1}}{x}\right)\right) \in U \times S^{n-2}$$

where $x = (x_1^2 + \dots + x_{n-1}^2)^{\frac{1}{2}}$. Here S^{n-2} denotes the standard Euclidean unit sphere in $\mathbf{R}^{n-1}$.

One can then think of $U_\sigma = U \times \{\sigma\}$, for any $\sigma \in S^{n-2}$, as the rotated position of the standard half plane $U \times \{(1,0,\ldots,0)\}$ in $\mathbf{R}^n$. Then $\mathbf{C}_\sigma = U_\sigma \cup \{x_0\text{-axis}\} \cup U_{-\sigma}$ becomes the new position of the standard complex plane $\mathbf{C} \equiv \mathbf{C} \times \{(1,0,\ldots,0)\}$ sitting in $\mathbf{R}^n$. The axis of rotation is of course the x_0-axis.

The Fueter mapping on $F_n(D)$ is then the 'function of revolution' obtained by revolving the function φ and its domain D around x_0-axis. We state this as the 'Revolution Principle': a Fueter map $F_n(\varphi)$ preserves each plane $\mathbf{C}_\sigma$; that is, $F_n(\varphi)$ maps $\mathbf{C}_\sigma$ into itself. In fact, $F_n(\varphi)$ restricted to $\mathbf{C}_\sigma \cap F_n(D)$ (for any $\sigma \in S^{n-2}$) is identifiable with the original map φ on D when $\mathbf{C}$ is identified with $\mathbf{C}_\sigma$ by rotation.

From this geometrical interpretation it is evident that, whenever φ has Laurent expansion (3) we will have:

$$\{V \in \mathbf{R}^n : F_n(\varphi)(V) = A\} = F_n(\{z : \varphi(z) = A\}) \tag{5}$$

for any *real* number A.

We see immediately the following:

THEOREM 1.1. *Let θ be any Laurent series with central coefficients (in $\mathbf{H}$ or $\mathbf{O}$ variable V), as in (4), convergent in $r < \|V-c\| < R$; namely*

$$\theta(V) = \sum_{n=0}^{\infty} a_n (V-c)^n + \sum_{m=1}^{\infty} b_m/(V-c)^m$$

(a_n, b_m, c are reals). The zero-set of this function θ, namely $\{V : \theta(V) = 0\}$, is simply the above rotation-transform F_4 or F_8 applied to the zero-set of the complex analytic function $\varphi(z)$ (defined by (3)) in $r < |z-c| < R$.

The set F_4 of a point is a 2-sphere orthogonal to each of the planes $\mathbf{C}_\sigma$ provided the point is not on the x_0-axis. On the x_0-axis of course rotation changes nothing. In fact,

$$F_4(\{\alpha + i\beta\}) = \{V = \alpha + e_1 x_1 + e_2 x_2 + e_3 x_3 : x_1^2 + x_2^2 + x_3^2 = \beta^2\}.$$

Similarly for F_8.

In particular, for polynomials we state what we now know separately.

COROLLARY 1.2. *Let $\{\alpha_1 \pm i\beta_1, \ldots, \alpha_m \pm i\beta_m, \gamma_1, \ldots, \gamma_k : \alpha_j, \beta_j, \gamma_p$ reals, $\beta_j > 0$; $j = 1, \ldots, m, p = 1, \ldots, k\}$ be the set of complex roots of the polynomial equation*

$$a_N V^N + \ldots + a_1 V + a_0 = 0, \quad a_j \in \mathbf{R}; j = 1, \ldots, N. \tag{6}$$

Then the quaternionic (octonionic) roots of (6) form the set

$$\bigcup_{j=1}^{m} S_{\alpha_j, \beta_j} \cup \{\gamma_1, \ldots, \gamma_k\}$$

where

$$S_{\alpha_j, \beta_j} = F_n(\{\alpha_j \pm i\beta_j\}), \quad n = 4 \text{ (or 8)}.$$

REMARK. In the above situation it appears reasonable to think of the sphere S_{α_j, β_j} as occurring with multiplicity $2m_j$, where m_j is the multiplicity of the root $(\alpha_j + i\beta_j)$ of the complex polynomial (6). The total multiplicity over all components of the quaternionic or octonionic solution set then adds up to the degree N.

When we wish to solve $\theta(V) = A$, $A \notin \mathbf{R}$, we can still apply our rotation process. Since $A \notin \mathbf{R}$, all the roots must lie in precisely the same $\mathbf{C}_\sigma$ which contains A itself. Therefore it only remains to rotate A into the standard position of the complex plane (namely, $A = a_0 + \sum_{j=1}^{n-1} e_j a_j \mapsto a_0 + ia$, where $a = (\sum_{j=1}^{n-1} a_j^2)^{\frac{1}{2}} > 0$, $n = 4$ (or 8)) and consequently the roots of $\theta(V) = A$ are nothing but the roots of $\varphi(z) = a_0 + ia$ rotated back into the $\mathbb{C}_\sigma$ position. Note, σ here is $(a_1/a, \ldots, a_{n-1}/a) \in S^{n-2}$. We state therefore the following theorem.

Theorem 1.3. *The root-set of $\theta(V) = A$, $A \in \mathbb{R}$, is*

$$S = \left\{\alpha + \beta \sum_{j=1}^{n-1} e_j \frac{a_j}{a} : \alpha + i\beta \text{ is a root of } \varphi(z) = a_0 + ia \text{ in } \mathbb{C}\right\}$$

where $A = a_0 + \sum_{j=1}^{n-1} e_j a_j$, $a = (\sum_{j=1}^{n-1} a_j^2)^{\frac{1}{2}} > 0$. The multiplicity of $\alpha + \beta \sum_{j=1}^{n-1} e_j a_j/a$ is the same as that of $\alpha + i\beta$ as a root of $\varphi(z) = a_0 + ia$.

[Here $n = 4$ or 8 according as V is a quaternionic or octonionic variable.]

Remark. The set identity (5) for *any* quaternion $A = a_0 + e_1 a_1 + e_2 a_2 + e_3 a_3$, is

$$\{V \in \mathbb{R}^n : F_n(\varphi)(V) \in F_n(\{A\})\} \equiv F_n(\{z : \varphi(z) = a_0 + ia\}) \tag{7}$$

where $a = (a_1^2 + a_2^2 + a_3^2)^{\frac{1}{2}} > 0$.

§2. *An algebraic proof of Corollary 1.2*

The Corollary 1.2 can be proved by straightforward algebra. We deal only with the quaternionic case since no new ideas come in for octonions.

Firstly note that any quaternionic polynomial with real coefficients can be factored into the product of quadratic and linear polynomials with real coefficients. Namely,

$$V^N + a_{N-1} V^{N-1} + \ldots + a_1 V + a_0 \equiv (V^2 + b_1 V + c_1) \ldots (V^2 + b_m V + c_m)(V + d_{2m+1}) \ldots (V + d_N) \tag{8}$$

with $b_j^2 - 4c_j < 0$; $j = 1, \ldots, m$. Consider therefore such a quadratic polynomial

$$V^2 + bV + c = 0 \tag{9}$$

where $b, c \in \mathbb{R}$, $b^2 - 4c < 0$ (and therefore $c > 0$). If $V = x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3$ is a root of (9) then one notes that $\|V\|^2 = c$ and $V^2 = 2x_0 V - \|V\|^2 = 2x_0 V - c$. Consequently, $2x_0 + b = 0$. This implies that V lies on the sphere

$$\{V = x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 : x_0 = -b/2, x_1^2 + x_2^2 + x_3^2 = (4c - b^2)/4\}.$$

The Corollary 1.2 now follows since there are no zero divisors in $\mathbb{H}$ or $\mathbb{O}$.

It is to be noted that when there exist infinitely many roots of a polynomial like (8) then the polynomial actually allows infinitely many distinct factorisations.

§3. *A recursive representation of powers of a hypercomplex variable, with applications*

As any non-zero quaternion is the product of a non-negative real number and a quaternion of norm 1, we will consider only quaternions or octonions with norm 1. In fact,

$$\{V : V^N = A\} \equiv \{\|A\|^{1/N} . W : W^N = A/\|A\|\}, (\|A\|^{1/N} > 0).$$

Now $\|V\| = 1$ is equivalent to

$$V^2 = 2x_0 V - 1, \quad \text{where } V = x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3. \tag{10}$$

(Again V could be an octonion without any extra trouble.) This gives, inductively,

$$V^k = P_k(x_0)\, V - P_{k-1}(x_0), \tag{11}$$

where P_k is a real polynomial of degree $(k-1)$ in the single (real) variable x_0. The P_k satisfy the recursive relations

$$P_{k+2} - 2x_0 . P_{k+1} + P_k = 0. \tag{12}$$

Note that $P_1 = 1$ and $P_2 = 2x_0$ from equation (10). Equations (11) and (12) provide a convenient representation of powers of a $\mathbb{H}$ or $\mathbb{O}$ variable.

As an application note the analysis of

$$V^N = A = a_0 + e_1 a_1 + e_2 a_2 + e_3 a_3. \tag{13}$$

Then V is a root of (13) precisely when

$$x_0 P_N(x_0) - P_{N-1}(x_0) = a_0 \tag{14}$$

and

$$P_N(x_0)\, x_j = a_j, \quad j = 1, 2, 3. \tag{15}$$

If A is real then the solutions of (13) are described in Corollary 1.2 (and may be obtained from (14) and (15) also). If A is non-real, then note that $P_N(x_0) \neq 0$, because otherwise $V^N = -P_{N-1}(x_0)$ would be real. In this case (14) has exactly N solutions, all real. (Indeed, the real parts of the complex roots of $z^N = a_0 + i(a_1^2 + a_2^2 + a_3^2)^{\frac{1}{2}}$ are precisely the roots of (14).) Therefore, by (15), $V^N = A$ has exactly N distinct solutions in $\mathbb{H}$. This confirms the conclusion of Theorem 1.3.

NOTE. The roots of $1/V^N = A$ also behave similarly, since

$$\frac{1}{V^N} = \frac{\bar{V}^N}{\|V\|^{2N}}.$$

REMARK. The polynomials $P_n(x)$ above are universal for all the algebras $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. They are essentially related to Tchebysheff's polynomials T_n, defined by $T_n(\cos\theta) = \cos n\theta$. In fact, both systems are solutions of the same difference equation (12), with the respective initial conditions:

$$\begin{cases} P_1(x) = 1 \\ P_2(x) = 2x \end{cases} \qquad \begin{cases} T_0(x) = 1 \\ T_1(x) = x. \end{cases}$$

They are related by the following formulae:

$$2T_n = P_{n+1} - P_{n-1}$$

$$P_{n+1} = 2(T_n + T_{n-2} + \dots).$$

In the case of $\mathbb{C}$ these formulae can be thought of as consequences of de Moivre's formula. We are indebted to Dr B. Bagchi for suggestions generating this remark.

§4. *The topology of the maps $V \mapsto V^N$*

Consider the natural maps $f_N(V) = V^N$, $(N = \pm 1, \pm 2, \dots)$, as self-maps of the unit spheres S^3 or S^7 – V being a quaternionic (respectively octonionic) variable of unit norm. From Theorem 1.3 and by §3 we know that the preimage via f_N of every non-real point is precisely N distinct points. The preimages of the two real points ± 1 are described in Corollary 1.2. Let us denote by $\lambda(N, \pm 1)$ the number of codimension one subspheres which are contained in $f_N^{-1}(\pm 1)$. Then, Corollary 1.2 implies:

$$\lambda(N, 1) = \begin{cases} \dfrac{|N|-1}{2} & \text{if } n \text{ is odd,} \\ \dfrac{|N|}{2} - 1 & \text{if } N \text{ is even,} \end{cases}$$
$$\lambda(N, -1) = \begin{cases} \dfrac{|N|-1}{2} & \text{if } N \text{ is odd,} \\ \dfrac{|N|}{2} & \text{if } N \text{ is even.} \end{cases} \tag{16}$$

It is convenient to note that we need not restrict to spheres of dimensions 3 or 7 only because we have the Fueter transform of $\varphi_N(z \mapsto z^N)$ at our disposal in any dimension. Thus $f_N = F_{d+1}(\varphi_N)$ is again a real-analytic self-map of S^d. The fibres of these general f_N mappings are also describable just as above because the 'Revolution Principle' of §1 applies.

It is natural to ask for the topological degree of the maps f_N and whether their restriction above $S^d - \{\pm(1, 0, \dots, 0)\}$ is a $|N|$-sheeted covering space or not. The answers are interesting and provided below.

First of all we notice from Corollary 1.2 that the $\lambda(N, 1) + \lambda(N, -1)(=|N|-1)$ codimension one subspheres in S^d separate $S^d - \{\pm(1, 0, \dots, 0)\}$ into $|N|$ cylinders (that is, $S^{d-1} \times (0, 1)$) each of which maps homeomorphically onto $S^d - \{\pm(1, 0, \dots, 0)\}$. The two ideal boundary components in each cylinder are getting collapsed to the points $(1, 0, \dots, 0)$ and $(-1, 0, \dots, 0)$. Thus:

Proposition 4.1. *The restriction of f_N to $S^d - \{f_N^{-1}(\pm(1, 0, \dots, 0))\}$ is a trivial $|N|$-fold covering of $S^d - \{\pm(1, 0, \dots, 0)\}$.*

As for the degree, the answers are given in:

Proposition 4.2. *If d is odd, $f_N: S^d \to S^d$ has degree* $\deg(f_N) = N$. *If d is even,*

$$\deg(f_N) = \begin{cases} +1 & \text{if } N \text{ is odd,} \\ 0 & \text{if } N \text{ is even.} \end{cases}$$

Remark. Algebraic topologists have already been interested in special cases of the above. See Dold [2, p. 65] for the complex and quaternionic case.

It is convenient to prove a lemma first.

Lemma 4.3. *For $N \geq 1$, f_N is homotopic to $I \bullet (-I) \bullet I \bullet (-I) \bullet \dots \bullet ((-1)^{N-1} I)$. Here I denotes the identity map on S^d, $-I$ denotes the antipodal map, and $\bullet$ denotes the usual operation by which maps are composed in the homotopy group $\pi_d(S^d)$.* (We will give a formal definition of $\bullet$ in the proof.)

Proof. Let us parametrise S^d by polar coordinates $(\theta_0, \ldots, \theta_{d-1})$:

$$\left.\begin{aligned} x_0 &= \cos\theta_0, \\ x_1 &= \sin\theta_0 \cos\theta_1, \\ x_2 &= \sin\theta_0 \sin\theta_1 \cos\theta_2, \\ &\vdots \\ x_{d-1} &= \sin\theta_0 \sin\theta_1 \sin\theta_2 \ldots \cos\theta_{d-1}, \\ x_d &= \sin\theta_0 \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-1}, \end{aligned}\right\} \quad (17)$$

$\theta_1, \ldots, \theta_{d-1} \in [0, \pi)$, $\theta_0 \in [-\pi, \pi)$. In that case one realises (say by the 'Revolution principle') that $F_{d+1}(\varphi_N)(\theta_0, \ldots, \theta_{d-1}) = (N\theta_0, \theta_1, \ldots, \theta_{d-1})$.

Now define $\bullet$ as follows: we will say two self-maps $f, g\colon S^d \to S^d$ are $\bullet$-composable if

$$g(0, \theta_1, \ldots, \theta_{d-1}) = f(\pm\pi, \theta_1, \ldots, \theta_{d-1}) \text{ for all } (\theta_1, \ldots, \theta_{d-1}) \in [0, \pi)^{d-1}$$

Then define

$$(f \bullet g)(\theta_0, \ldots, \theta_{d-1}) = \begin{cases} f(2\theta_0, \theta_1, \ldots, \theta_{d-1}) & \text{if } \theta_0 \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right], \\ g(2\theta_0 - \pi, \theta_1, \ldots, \theta_{d-1}) & \text{if } \theta_0 \in \left[-\pi, -\dfrac{\pi}{2}\right] \cup \left[\dfrac{\pi}{2}, \pi\right]. \end{cases} \quad (18)$$

This is clearly the usual $\bullet$ product in the definition of composition in $\pi_d(S^d)$. It is now trivial to check that

$$I \bullet (-I) \bullet I \bullet \ldots \bullet ((-1)^{N-1} I)(\theta_0, \ldots, \theta_{d-1}) = (N\theta_0, \theta_1, \ldots, \theta_{d-1})$$

for all $(\theta_0, \ldots, \theta_{d-1}) \in [-\pi, \pi) \times [0, \pi)^{d-1}$.

Proof of Proposition 4.2. Recall the following standard facts about the degree:

$$\deg(f \circ g) = \deg(f) \cdot \deg(g), \quad (19)$$

$$\deg(f \bullet g) = \deg(f) + \deg(g). \quad (20)$$

Using these relations we immediately obtain the claimed values of $\deg(f_N)$ for N positive. For N negative, notice that

$$V^N = V^{-|N|} = \bar{V}^{|N|} = j \circ V^{|N|},$$

where j is the 'conjugation map', that is,

$$j(x_0, x_1, \ldots, x_d) = (x_0, -x_1, \ldots, -x_d).$$

But $\deg(j) = (-1)^d$ (see for example Vick [4]), consequently the proposition is proved completely.

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