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ON STATIONARY RECTILINEAR VORTICES IN THE CORNER OF PLANE WALLS MEETING AT RIGHT ANGLE

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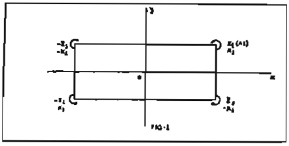
Abstract : The possibility of existence of one or more stationary rectilinear vortices in a frictionless fluid at the corner of two perpendicular plane walls has been studied The case of two vortices has been studied in detail. It has been found that two vortices of opposite signs and of different strengths can remain stationary in a stagnation flow in a corner. Stream lines for a particular case have been traced The question of stability has not been discussed.

1. **Introduction** : Recirculating flows occur in many practical problems. Flow in rectangular cavities, flow beneath a hovercraft, flow between the tubes of a superheated boiler, flow behind a bluff body, all have recirculating flows. In many problems the heat transfer is very much dependent on recirculating flows. The Heat transfer will be much affected if instead of one there are two or more recirculating flows. Decelerated stagnation flow with separation is a particular case of recirculating flow. Foettinger's (see Schlichting) well-known photographs show the existence of a big vortex in the corner with a few small ones rotating in the same or in the opposite directions though they are not necessarily stationary. In the present study the possibility of existence of one or more stationary vortices at the corner of two perpendicular plane walls for a frictionless fluid has been investigated. One or more vortices of different strengths have been placed in a stagnation flow near the corner and conditions have been found out so that it they may remain standing. The case for one vortex is quite simple and one finds that the line joining the centre of the vortex to the corner makes angle of 45° with each wall. The distance of the vortex from the corner increases as the magnitude of the oncoming flow is decreased. The conditions that two vortices may remain stationary give rise to four complicated equations with four unknowns. They have been solved taking particular values for the strength of the vortices and the magnitude of the stagnation flow. Solutions could be obtained only when the vortices are of opposite signs. The position of the vortices with their strengths for a few particular cases are given in Table 1. The stream lines for a particular case when $\frac{k_1}{k_2} = -1 \cdot 0.35$ has been traced in fig. 8. Since the flow is frictionless the vortices with their images on the wall parallel to the oncoming stream will also be stationary for a stagnation flow perpendicular to a plane. The case for three stationary vortices involves six equations with six unknowns and hence it will be much more difficult to obtain the solution. Here no attempt has been made to find them. The question of stablity of the vortices has not been studied here.

2. One stationary vortex at the corner of two perpendicular walls.



Let Ox and Oy be the two plane walls meeting at right angles at O. The velocity distribution in frictionless two dimensional potential flow in the neighbourhood of the stagnation point $x=0$, $y=0$ is given by

$$\begin{aligned} u &= -ax \\ v &= +ay \end{aligned} \qquad \dots \quad (2.1)$$

where a is a constant.

The complex potential of the stagnation flow is $\frac{az^2}{2}$. Let there be a vortex of stength k_1 at a point A_1 (z_1). Then the image system is $-k_1$ at $\bar{z}_1$, $-k_1$ at $-\bar{z}_1$ and k_1' at $-z_1$.

To find the velocity of the vortex at A_1 one takes the complex potential

$$W_1 = \frac{az^2}{2} + ik_1 \; log \frac{z+z_1}{z^2-\bar{z}_1^2} \qquad \dots \quad (2.2)$$

The vortex at A_1 will be at rest if

$$\frac{dw_1}{dz} = 0 \text{ when } z = z_1 \qquad \dots \quad (2.3)$$

This leads to the relation

$$az_1 = ik_1 \frac{3z_1^2 + \bar{z}_1^2}{2z_1(z_1^2 - \bar{z}_1^2)} \qquad \dots \quad (2.4)$$

Putting $z = r_1 e^{i\theta_1}$, one gets

$$\left. \begin{aligned} r_1^2 &= \frac{k_1}{2a} \\ \text{and} \qquad \theta &= \pi/4 \end{aligned} \right\} \qquad \dots \quad (2.5)$$

When the above conditions (2.5) are satisfied the vortex can remain in a stationary state. The vortex always lies on the line $\theta = \pi/4$. As the magnitude of the flow velocity is inereased by increasing the value of the constant "a", the distance of the vortex from the origin decreases but as the strength of the vortex is increased it gradually moves away from the origin.

The complex potential of the fluid motion is given by

$$w=\frac{az^2}{2}+ik_1\ log\frac{(z-z_1)(z+z_1)}{(z-\bar{z}_1)(z+\bar{z}_1)} \qquad \dots \quad (2.6)$$

The stream function Ψ is given by

$$\frac{\Psi}{a}=xy+\frac{k_1}{2a}log\ \frac{\{(x-x_1)^2+(y-y_1)^2\}\{(x+x_1)^2+(y+y_1)^2\}}{\{(x-x_1)^2+(y+y_1)^2\}\{(x+x_1)^2+(y-y_1)^2\}} \qquad \dots \quad (2.7)$$

The stream-lines for the particular case

$r_1=\sqrt{\frac{k_1}{2a}}=1$ are given by

$$\frac{\Psi}{a}=xy+log\ \frac{x^4+y^4+1+2x^2y^2-4xy}{x^4+y^4+1+2x^2y^2+4xy} \qquad \dots \quad (2.8)$$

and have been traced in fig. 2.

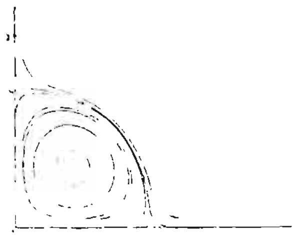
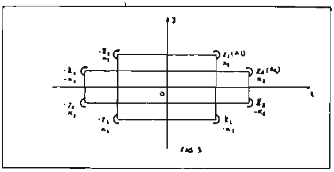


FIG. 2

There are two stagnation points S_1 and S_2 on the walls as shown in the figure. The fluid inside the stream line S_1 S_2 recirculates near the corner.

3. Two stationary vortices at the corner of two perpendicular walls

Let there be two vortices, one of strength k_1 at A_1 (z_1) and another of strength k_2 at $A_2(z_2)$ in the stagnation flow (2.1). The vortex at A_1 (z_1) has images $-k_1$ at $\bar{z}_1, -k_1$ at $-\bar{z}_1$ and k_1 at $-z_1$. The vortex at $A_2(z_2)$ has images $-k_2$ at $\bar{z}_2$ and $-\bar{z}_2$ and $+k_2$ at $-z_2$. (Fig. 3)



The velocity of the vortex at A_1 can be obtained from the complex potential function

$$w(A_1) = \frac{az^2}{2} + ik_1 \log \frac{z+z_1}{z^2 - \bar{z}_1^2} + ik_2 \log \frac{z^2 - z_2^2}{z^2 - \bar{z}_2^2} \qquad \dots \quad (3.1)$$

The velocity of the vortex at A_2 is given by the complex potential function

$$w(A_2) = \frac{az^2}{2} + ik_1 \log \frac{z^2 - z_1^2}{z^2 - \bar{z}_1^2} + ik_2 \log \frac{z + z_2}{z^2 - \bar{z}_2^2} \qquad \dots \quad (3.2)$$

The vortex at A_1 will be at rest if

$$\frac{dw(A_1)}{dz} = 0 \text{ when } z = z_1$$

which gives

$$ik_1 \frac{3z_1^2 + \bar{z}_1^2}{2z_1(z_1^2 - \bar{z}_1^4)} = az_1 + ik_2 \left[\frac{2z_1}{z_1^2 - z_2^2} - \frac{2z_1}{z_1^2 - \bar{z}_2^2} \right] \qquad \dots \quad (3.3)$$

Similarly the vortex at A_2 will be at rest if

$$\frac{dw(A_2)}{dz} = 0 \text{ when } z = z_2$$

which gives

$$ik_2 \frac{3z_2^2 + \bar{z}_2^2}{2z_2(z_2^2 - \bar{z}_2^2)} = az_2 + ik \left[\frac{2z_2}{z_2^2 - z_1^2} - \frac{2z_2}{z_2^2 - \bar{z}_1^2} \right] \qquad \dots \quad (3.4)$$

Substituting $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ in equations (3.3) and (3.4) and equating real and imaginary parts one gets the following relations

$$k_1 \begin{matrix} \cos 2\theta_1 \\ \sin 2\theta_1 \end{matrix} = ar_1^2 \cos 2\theta_1 + 2k_2 r_1^2 \left\{ \frac{r_2^2 \sin 2(\theta_1 - \theta_2)}{r_1^4 + r_2^4 - 2r_1^2 r_2^2 \cos 2(\theta_1 - \theta_2)} \right.$$

$$\left. \frac{r_2^2 \sin 2(\theta_1 + \theta_2)}{r_1^4 + r_2^4 - 2r_1^2 r_2^2 \cos 2(\theta_1 + \theta_2)} \right\} \qquad (3.5)$$

$$\frac{k_1}{2}=ar_1^2\sin 2\theta_1+2k_2r_1^2\left\{\frac{r_1^2-r_2^2\cos 2(\theta_1-\theta_2)}{r_1^4+r_2^4-2r_1^2r_2^2\cos 2(\theta_1-\theta_2)}-\frac{r_1^2-r_2^2\cos 2(\theta_1+\theta_2)}{r_1^4+r_2^4-2r_1^2r_2^2\cos 2(\theta_1+\theta_2)}\right\} \quad \ldots (3.6)$$

$$\frac{k_2\cos 2\theta_2}{\sin 2\theta_2}=ar_2^2\cos 2\theta_2+2k_1r_2^2\left\{\frac{r_1^2\sin 2(\theta_2-\theta_1)}{r_1^4+r_2^4-2r_1^2r_2^2\cos 2(\theta_2-\theta_1)}-\frac{r_1^2\sin 2(\theta_2+\theta_1)}{r_1^4+r_2^4-2r_1^2r_2^2\cos 2(\theta_1+\theta_2)}\right\} \quad \ldots (3.7)$$

$$\frac{k_2}{2}=ar_2^2\sin 2\theta_2+2k_1r_2^2\left\{\frac{r_2^2-r_1^2\cos 2(\theta_2-\theta_1)}{r_1^4+r_2^4-2r_1^2r_2^2\cos 2(\theta_2-\theta_1)}-\frac{r_2^2-r_1^2\cos 2(\theta_2+\theta_1)}{r_1^4+r_2^4-2r_1^2r_2^2\cos 2(\theta_2+\theta_1)}\right\} \quad (3.8)$$

These are the four equations involving the four unknowns r_1, r_2, θ_1 and θ_2 when k_1, k_2 and a are assumed to be known constants.

4. Solutions of Equations for two stationary vortices

Eliminating a between (3.5) and (3.6) one gets

$$\frac{k_1\cos 2\theta_1}{4k_2r_1^2(r_2^2\cos 2\theta_2-r_1^2\cos 2\theta_1)}=\frac{1}{r_1^4+r_2^4-2r_1^2r_2^2\cos 2(\theta_1-\theta_2)}-\frac{1}{r_1^4+r_2^4-2r_1^2r_2^2\cos 2(\theta_1+\theta_2)} \quad (4.1)$$

Similarly, eliminating a between (3.7) and (3.8), one gets

$$\frac{k_2\cos 2\theta_2}{4k_1r_2^2(r_1^2\cos 2\theta_1-r_2^2\cos 2\theta_2)}=\frac{1}{r_1^4+r_2^4-2r_1^2r_2^2\cos 2(\theta_1-\theta_2)}-\frac{1}{r_1^4+r_2^4-2r_1^2r_2^2\cos 2(\theta_1+\theta_2)} \quad (4.2)$$

Since the right hand sides in equations (4.1) and (4.2) are the same, equating the left hand side expressions one can get the following simple relation

$$\frac{k_1^2}{r_1^2}\cos 2\theta_1+\frac{k_2^2}{r_2^2}\cos 2\theta_2=0 \quad \ldots (4.3)$$

From this relation it is evident that if $\cos 2\theta_1$ is negative then $\cos 2\theta_2$ is positive. So if θ_1 lies between $\frac{\pi}{4}$ and $\frac{\pi}{2}$ then θ_2 lies between o and $\frac{\pi}{4}$. Again multiplying (3.6) by k_1 and (3.8) by k_2 and adding one obtains the following relation

$$\frac{k_1^2+k_2^2}{2a}=r_1^2k_1\sin 2\theta_1+r_2^2k_2\sin 2\theta_2 \quad \ldots (4.4)$$

Now, solving for r_1^2 and r_2^2 from the equations (4.3) and (4.4) and substituting those values in equations (4.1) and (3.7) one obtains the following two equations involving the two unknowns θ_1 and θ_2

$$(k_1^4\cos^2 2\theta_1+k_2^4\cos^2 2\theta_2)^2+4k_1^2k_2^2(k_1^4\cos^2 2\theta_1+k_2^4\cos^2 2\theta_2)\cos^2 2\theta_1\cos^2 2\theta_2$$
$$+4k_1^4k_2^4\cos^2 2\theta_1\cos^2 2\theta_2\cos 2(\theta_1-\theta_2)\cos 2(\theta_1+\theta_2)=16k_2^3k_1^3\cos 2\theta_1\cos 2\theta_2\sin 2\theta_1$$
$$\sin 2\theta_2(k_1^2\cos^2 2\theta_1+k_2^2\cos^2 2\theta_2) \quad \ldots (4.5)$$

$$(k_2^7 - 2k_2^5 k_1^2)\cos^3 2\theta_2 \sin 2\theta_2 - 3k_2^4 k_1^3 \cos 2\theta_1 \sin 2\theta_1 \cos^2 2\theta_2$$

$$-3k_1^4 k_2^3 \cos 2\theta_2 \sin 2\theta_2 \cos^2 2\theta_1 + (k_1^7 - 2k_2^2 k_1^5)\cos^3 2\theta_1 \sin 2\theta_1 = 0 \quad \ldots \quad (4.6)$$

These equations involve k_1 and k_2 but not "a". So θ_1 and θ_2 are independent of "a" i.e. of the magnitude of the stagnation flow velocity. Once θ_1 and θ_2 are determined, the distance of the centres of the vortices from the origin i.e. r_1 and r_2 will be determined from equations (4.3) and (4.4) and hence will involve 'a'. To solve the above two equations (4.5) and (4.6) one makes the following substitutions

$$\cos 2\theta_1 = X \,;\; \cos 2\theta_2 = Y \,;\; \frac{k_1}{k_2} = A$$

and transforms the equations to the following form

$$A^8 X^4 + Y^4 - 2A^4 X^2 Y^2 + 4(A^6 + A^4) X^4 Y^2 + 4(A^4 + A^2) X^2 Y^4$$

$$= 16A^3(A^2 X^3 Y + XY^3)\sqrt{(1 - X^2)(1 - Y^2)} \quad \ldots \quad (4.7)$$

$$(1 - 2A^2) Y^3 \sqrt{1 - Y^2} - 3A^3 X \sqrt{1 - X^2}\, Y^2 - 3A^4 Y X^2 \sqrt{1 - Y^2}$$

$$+ (A^7 - 2A^5)\sqrt{(1 - X^2)} = 0 \quad \ldots \quad (4.8)$$

The method adopted to find the solutions X and Y was to draw the graphs of equation (4.7) and (4.8), find the point of intersection which gives an approximate value of the solution and then to improve the solution. While attempting to find solutions so that θ_1 and θ_2 lie between 0 and $\frac{\pi}{2}$ it was found that solution was only possible when A was negative which means that the two vortices have opposite signs. For tracing the graphs a constant negative value of A was assumed and then for a particular value of X the value of Y was determined by a method of iteration. Then the value af X was gradually varied and the corresponding values of Y were determined. In some cases two solutions for Y were found for a particular of X, but in the figures only the curves which intersected with the curve obtained for the other equation have been traced. Fig. 4 gives the curves for $A = -0.5$. Curve I has been obtained from equation (4.7) and Curve II from equation (4.8). Fig. 5, Fig. 6 and Fig. 7 give the same curves when $A = -0.6$, $A = -0.4$ and $A = -\dfrac{1}{0.35}$ respectively. Positions of the rectilinear vortices can now be calculated from equations (4.3) and (4.4). When one interchanges the strengths of the vortices, the absolute magnitudes of $\cos 2\theta_1$ and $\cos 2\theta_2$ are interchaned and the value of new θ_2 is complementary to old θ_1 and vice-versa. The values for $A = -0.35$ and $A = -1/0.35$ are shown in Table I, which gives the values of $r_1, r_2, \theta_1, \theta_2$ and (x_1, y_1) and (x_2, y_2) for different values of A and $\dfrac{2a}{\kappa_2}$.

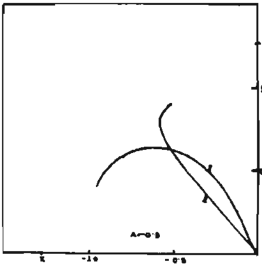


Fig. 4

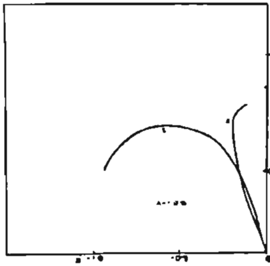
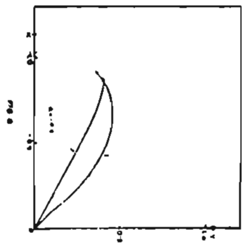


Fig. 5



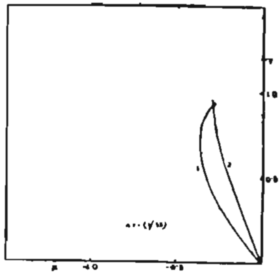


Fig. 7

To trace the stream-lines one has to consider the complex potential of the fluid motion which is given by

$$w=\frac{az^2}{2}+ik_1 \log \frac{z^2-z_1^2}{z^2-\bar{z}_1^2}+ik_2 \log \frac{z^2-z_2^2}{z^2-\bar{z}_2^2} \tag{4.9}$$

The stream function ψ is given by

$$\frac{\psi}{a}=xy+\frac{k_1}{2a} \log \frac{\{(x-x_1)^2+(y-y_1)^2\}\{(x+x_1)^2+(y+y_1)^2\}}{\{(x-x_1)^2+(y+y_1)^2\}\{(x+x_1)^2+(y-y_1)^2\}}$$

$$+\frac{k_2}{2a} \log \frac{\{(x-x_2)^2+(y-y_2)^2\}\{(x+x_2)^2+(y+y_2)^2\}}{\{(x-x_2)^2+(y+y_2)^2\}\{(x+x_2)^2+(y-y_2)^2\}} \quad \ldots \tag{4.10}$$

The stream lines for the particular case when $A=-\dfrac{1}{0.35}$ and $\dfrac{2a}{k_2}=-1$ are given by

$$\frac{2\Psi}{k_2} = -xy - 2.857142\ log\frac{\{(x-x_1)^2+(y-y_1)^2\}\{(x+x_1)^2+(y+y_1)^2\}}{\{(x-x_1)^2+(y+y_1)^2\}\{(x+x_1)^2+(y-y_1)^2\}}$$

$$+\ log\frac{\{(x-x_2)^2+(y-y_2)^2\}\{(x+x_2)^2+(y+y_2)^2\}}{\{(x-x_2)^2+(y+y_2)^2\}\{(x+x_2)^2+(y-y_2)^2\}} \qquad \dots \quad (4.11)$$

This has been traced in fig. 8 which gives a general idea of the position of the stream lines for all such cases.

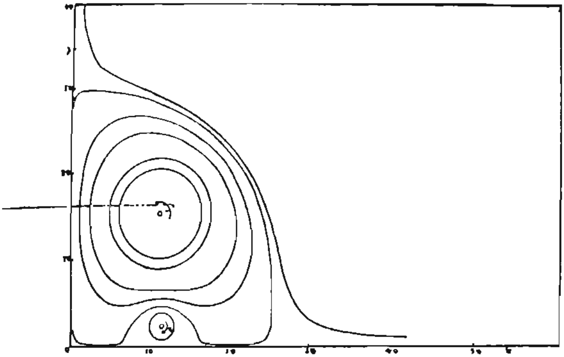


FIG. 8

5. Conclusion : Here solutions have been obtained only for particular values of $A(=\frac{k_1}{k_2})$and $\frac{2a}{k_2}$. It has not been possible to find a general expression for the solution. The equation being very much involved one had to take recourse to numerical methods. While trying to find out solutions for particular values of A it was found that one could get solutions in the approximate ranges $-0.6<A<-0.35$ and $-0.6\leqslant\frac{1}{A}\leqslant-0.35$. In fact one could not obtain a solution for $A=-0.3$ and $A=-0.7$ by the method adopted in the present study.

For more than two vortices the conditions for them to be stationary are much more complicated. No attempt was made to find the solutions. But it is interesting to note that relations similar to (4.3) and (4.4) are still valid. For n stationary vortices they are of the form

$$\sum_{m=1}^{n}\frac{k_m^2}{r_m^2}cos2\theta_m=0 \qquad \dots \quad (5.1)$$

and

$$\frac{\sum_{m=1}^{n}k_m^2}{2a}=\sum_{m=1}^{n}r_m^2k_m sin2\theta_m \qquad \dots \quad (5.2)$$

Table I

$A=\frac{k_1}{k_2}$	$X=\cos 2\theta_1$	$Y=\cos 2\theta_2$	r_1^2	r_2^2	$\frac{2a}{k_2}$	r_1	r_2	θ_1	θ_2	(x_1, y_1)	(x_2, y_2)
−0.35	−0.928	0.2882	.488584	1.23865	1	.6989	1.1129	79°4′	36°37′.5	(.1329, .6861)	(.8932, .6637)
−1/0.35	−0.2882	0.928	3.539	1.395956	−1	1.88122	1.181148	53°22′.5	10°56′	(1.12233, 1.50967)	(1.16009, 0.224126)
−0.4	−0.8655	0.4035	0.433721	1.26377	1	0.65857	1.12417	74°58′	33°6′	(0.17083, 0.63604)	(0.941717, 0.613909)
−0.5	−0.5116	0.6205	0.370439	1.797167	1	0.608637	1.34058	60°23′	25°49′	(0 30078, 0.52914)	(1.20679, 0.5838)
−0.6	−0.150964	0.465	0.194593	1.664964	1	0.441127	1.29033	49°20′.5	31°8′.5	(0.287394, 0.334638)	(1.10439, 0.667229)

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Schlichting, H. (1962) Boundary Layer Theory, Fourth Edition, p 33

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