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A note on the countable chain condition and sigma-finiteness of measures

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The objectives of this paper are the following:

(1) to show that a theorem of Ficker is incorrect;

(2) to show that a stronger version of Ficker's Theorem is valid for a certain class of measures;

(3) characterize all σ-algebras on which every measure is a countable sum of finite measures.

1. Introduction, notation and definitions

A measure μ is an extended real valued, nonnegative, countably additive function defined either on a σ-algebra A of subsets of a set X or on a boolean σ-algebra B vanishing at the empty set $\emptyset$ or the zero element of B. Ficker [1, p. 242] proved the following theorem.

THEOREM (*). *Let* μ *be a measure on a* σ*-algebra* A *of* X *and* N *denote the collection of all sets in* A *of* μ*-measure zero. Then* $A - N$ *satisfies countable chain condition* (CCC) *if and only if* μ *can be written as a countable sum of finite measures.*

We give an example to show that this Theorem (*) is incorrect.

2. Example

Let B be a boolean σ-algebra satisfying CCC such that there is no strictly positive, finite measure on B. For example, one can take the

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boolean σ-algebra of all Borel subsets of the real line modulo first category Borel sets. Let X be the Stone space of B, $\mathcal{A}$ the Baire σ-algebra on X and I the collection of all first category Baire subsets of X. By Loomis' Theorem (see, for example, [2, p. 102]), the quotient boolean σ-algebra $\mathcal{A}/I$ and B are σ-isomorphic. Since I is a σ-ideal, the function μ defined by the formula, $\mu(A) = 0$ if $A \in I$, $\mu(A) = \infty$ if $A \in \mathcal{A} - I$, is a measure on $\mathcal{A}$. Note that $\mathcal{A}/I$ satisfies CCC and so $\mathcal{A} - I$ satisfies CCC. If Ficker's Theorem (*) were to be true, we can write μ as a countable sum of finite measures on $\mathcal{A}$ which implies that μ is equivalent to a finite measure λ on $\mathcal{A}$. Since I is precisely the collection of all λ-null sets, we have a strictly positive finite measure on $\mathcal{A}/I$. But this is a contradiction.

3. Semi-finite measures

A measure μ on a σ-algebra $\mathcal{A}$ of X is said to be semi-finite if $F \in \mathcal{A}$, $\mu(F) = \infty$ implies there exists $E \in \mathcal{A}$ such that E is contained in F and $0 < \mu(E) < \infty$. For a measure μ on $\mathcal{A}$, there are two definitions of μ-atoms.

(I) A set A in $\mathcal{A}$ is said to be a μ-atom if

(i) $\mu(A) > 0$ and

(ii) $B \in \mathcal{A}$, B contained in A implies $\mu(B) = 0$ or $= \mu(A)$.

(II) A set A in $\mathcal{A}$ is said to be a μ-atom if

(i) $\mu(A) > 0$ and

(ii) $B \in \mathcal{A}$, B contained in A implies $\mu(B) = 0$ or $\mu(A-B) = 0$.

These definitions are not equivalent. It is easy to construct an example. However, when μ is semi-finite these two definitions are equivalent. Ficker [1] adopted definition (II) in the course of his proof of the Theorem (*). Under this definition, his Lemma 3 [1, p. 239] is not correct. However, if μ is semi-finite all his proofs are valid and hence for such a class of measures his Theorem (*) is true.

Here we prove a stronger version of his Theorem (*) directly for semi-finite measures.

THEOREM 1. *Let* μ *be a semi-finite measure on a* σ*-algebra* $\mathcal{A}$ *of*

X . *Let* N *denote the collection of sets of* μ*-measure zero. Then* $A - N$ *satisfies* CCC *if and only if* μ *is* σ*-finite.*

Proof. If μ is σ-finite, it is obvious that $A - N$ satisfies CCC . Conversely, if $\mu(X) < \infty$, there is nothing to prove. If $\mu(X) = \infty$, choose A_1 in A such that $0 < \mu(A_1) < \infty$. Choose A_2 in A such that A_2 is contained in $X - A_1$ and $0 < \mu(A_2) < \infty$. Thus we can find a sequence of disjoint sets $A_1, A_2, \ldots$ in A such that each $A_i \in A - N$ and $\mu(A_i) < \infty$. If $\mu\left(X - \bigcup_{i\geq 1} A_i\right) < \infty$, then we have a decomposition of X which implies that μ is σ-finite. If $\mu\left(X - \bigcup_{i\geq 1} A_i\right) = \infty$, choose A_ω in A such that A_ω is contained in $X - \bigcup_{i\geq 1} A_i$ and $0 < \mu(A_\omega) < \infty$, where ω is the first countable ordinal. Continue this process. Since $A - N$ satisfies CCC , there exists a countable ordinal α such that $\mu\left(X - \bigcup_{\beta<\alpha} A_\beta\right) < \infty$. This implies that μ is σ-finite.

4. Some characterizations

Let A be a σ-algebra on a set X . A set A in A is said to be an atom of A if

(i) $A \neq \emptyset$ and

(ii) B in A , B contained in A implies $B = \emptyset$ or $= A$.

A σ-algebra A on X is said to be atomless if there are no atoms of A .

The following result is known. See Remark 11 of [3, p. 203]. For completeness sake, we give a proof of this result.

PROPOSITION. *Let* A *be a* σ*-algebra on a set* X *.* A *is atomless if and only if every nonempty set in* A *contains* $\aleph_1$ *disjoint nonempty sets in* A *.*

Proof. Let A in A be nonempty. Fix $x \in A$. Find A_1 in A such that $x \notin A_1$, $A_1 \neq \emptyset$ and A_1 is contained in A . Choose A_2 in A such that $x \notin A_2$, $A_2 \neq \emptyset$ and A_2 is contained in $A - A_1$.

Continuing this process, we obtain a family $A_\alpha : \alpha < \Omega$ of nonempty disjoint sets contained in A, where Ω is the first uncountable ordinal. The converse part is trivial.

THEOREM 2. *Let* $\mathcal{A}$ *be a* σ*-algebra on a set* X. *The following statements are all true:*

(i) $\mathcal{A}$ *satisfies* CCC *if and only if* $\mathcal{A}$ *is isomorphic to the power set, that is, the class of all subsets, of some countable (finite or infinite) set;*

(ii) there exists a strictly positive finite measure on $\mathcal{A}$ *if and only if* $\mathcal{A}$ *is isomorphic to the power set of some countable set;*

(iii) every measure on $\mathcal{A}$ *can be written as a countable sum of finite measures if and only if* $\mathcal{A}$ *is isomorphic to the power set of some countable set;*

(iv) every measure on $\mathcal{A}$ *is equivalent to a finite measure if and only if* $\mathcal{A}$ *is isomorphic to the power set of some countable set.*

Proof. A proof of *(i)* can be obtained using the Proposition proved earlier. Since $\mathcal{A}$ satisfies CCC, the number of atoms of $\mathcal{A}$ is countable. From X remove all atoms of $\mathcal{A}$. In view of the Proposition the remaining part is empty. The proofs of *(ii)*, *(iii)* and *(iv)* are easy.

Professor Ashok Maitra suggested an alternative proof of *(i)*. Since $\mathcal{A}$ satisfies CCC, it is complete as a boolean algebra. For x in X, the infimum of all sets in $\mathcal{A}$ containing x is an atom of $\mathcal{A}$. This implies that $\mathcal{A}$ is atomic. Again by CCC, the number of atoms of $\mathcal{A}$ is at most countable.

References

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