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ON THE MINIMAL THICK SETS OF A MEASURE SPACE

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Summary

Let (X, S, μ) be a finite measure space. A subset A of X is called a thick set of (X, S, μ) if the μ-inner measure of its complement is zero. A thick set A is called a minimal thick set if no proper subset of A is a thick set.

If (X, S, μ) admits a minimal thick set A, then A is countable and μ is atomic. Finite Cartesian product of minimal thick sets is a minimal thick set. Topological aspects of sets of measures, admitting a given subset as a minimal thick set, are studied. If X is a complete, separable metric space without isolated points, S the σ-field generated by open subsets of X, then the following conditions are equivalent :

(i) (X, S, μ) is non-atomic.

(ii) (X, S, μ) admits a thick set whose complement is also a thick set.

(iii) (X, S, μ) admits a decreasing sequence of thick sets tending to the empty set.

1. *Introduction*

Let X be any set, and S any σ-field of subsets of X; Let μ be a finite measure on (X, S). The system (X, S, μ) is called a measure space. A σ-field S is strictly separable if it is generated by a countably many measurable sets. Two sub σ-fields S_1, S_2 of the σ-field S in a measure space (X, S, μ) are equivalent if to every set E, in either one of them there corresponds a set F in the other such that the symmetric difference $(E-F) \cup (F-E)$ has μ measure zero. A measure space is separable if there exists a strictly separable σ-field S_1 contained in and equivalent to S. A measure space (X, S, μ) is properly separable if there exists a strictly separable σ-field $S_1 \subset S$ such that to every $E \in S$ there corresponds an $F \in S_1$ with $E \subset F$ and $\mu(F-E)=0$. A measure space (X, S, μ) is non-atomic if every measurable set of μ-positive measure contains measurable subsets of smaller μ-positive measure. A countable sequence $A_1, A_2, \ldots, A_n, \ldots$ of subsets of X is a separating sequence if to every pair of points $x \neq y$, we can find an integer n with $x \in A_n$ and $y \in X-A_n$.

If (X_1, S_1, μ_1) and (X_2, S_2, μ_2) are measure spaces, a point isomorphism T between X_1 and X_2 is a one-to-one mapping from almost all of X_1 to almost all of X_2 such that $E_1 \in S_1$ if and only if $E_2 = TE_1 \in S_2$, and $\mu_1(E_1)=\mu_2(E_2)$. If such a mapping T exists, X_1 and X_2 are called point isomorphic.

A measure space is proper if it is complete, properly separable, and non-atomic, and if it contains a separating sequence of measurable sets. A proper measure space is normal if to each real-valued univalent function $f(X)$ there corresponds a set X_0 of measure zero such that the range $f(X-X_0)$ is a Borel set.

Let (X, S, μ) be a measure space. A set $A \subset X$ is called a thick set if the inner measure

$$\mu_*(X-A)=\sup\{\mu(D) / D \in S, D \subset X-A\}$$

is zero. A thick set A_0 is called a minimal thick if no proper subset of A_0 is a thick set.

We say that A is a null set if $\mu(A)=0$. We say that two measurable sets A, B are μ-equivalent if they coincide upto μ-null sets. We say that a non-empty set $A \in S$ is a μ-atom if every measurable subset of A is equivalent either to empty set or to A. A measure μ is said to be atomic if X is a countable disjoint union of μ-atoms of positive measure. A is an atom of a σ-field S if $A \in S$ and B is a proper subset of A, $B \in S \Rightarrow B$ is empty.

2.0. We observe that a measure space admits a minimal thick set if, and only if, it admits a maximal set of inner measure zero. The following theorem is a generalisation of this statement in one direction.

2.1. THEOREM. *Let (X, S, μ) be a measure space which admits a maximal set of μ-measure zero, then the measure space admits a minimal thick set which is countable.*

Proof. Let N_0 be the maximal set of measure zero. This implies that no subset of $X-N_0$ has measure zero. For every $x \in X-N_0$, define $S_x=\{E \in S/x \in E$ and $E \subset X-N_0\}$. Let $a_x=\inf\{\mu(E) : E \in S_x\}$. Then there exists a sequence $A_n \in S_x$ such that $\mu(A_n) \to a_x$. Let $E_x=\bigcap_{n=1}^{\infty} A_n$. Then $x \in E_x \subset X-N_0$ and $\mu(E_x)=a_x$. The maximality of N_0 implies that E_x is unique and $a_x>0$. Define $x \sim y$ if $E_x=E_y$. This is an equivalence relation and $X-N_0$ is divided into a family of disjoint equivalence classes of measurable sets of positive measure. Since the measure space is finite, this family is countable. Let A_0 be the set which contains one point from each of E_x from the equivalence class of sets (E_x). Then A_0 is a minimal thick set.

2.2 Example. *The converse of the above theorem is not true.*

Let X_α : closed unit interval for every $\alpha \in [0, 1]$,

S_α : discrete σ-field.

Define $\mu_\alpha(A)=1$ if A contains the point $1/2$,
$=0$ otherwise.

Let $X=\Pi X_\alpha$, $S=\Pi S_\alpha$, $\mu=\Pi \mu_\alpha$. Then the point x, whose α-th coordinate is equal to $1/2$ for all α, is the minimal thick set. But the measure space does not admit a maximal set of measure zero.

2.3. Theorem. *Let A_0 be a minimal thick set of a measure space (X, S, μ), then A_0 is countable and μ is atomic.*

Proof. Let $x \in A_0$. Define $S_x=\{E \in S/x \in E\}$ and $a_x=\inf\{\mu(E), E \in S_x\}$. Then there exists a set $E_x \in S_x$ such that $\mu(E_x)=a_x$. This set need not be unique. Further,

(i) $\mu(E_x)>0$ for every $x \in A_0$,

(ii) $x \neq y$ implies $E_x \neq E_y$,

(iii) $x \neq y$ implies $\mu(E_x \cap E_y)=0$.

(i) If $\mu(E_x)=0$ for some x, the following inequality

$$\text{(1)} \qquad \mu_*[X-(A_0-\{x\})] \leqslant \mu_*(X-A_0)+\mu^*(\{x\}) \leqslant \mu_*(X-A_0)+\mu(E_x)=0,$$

implies that $A_0-\{x\}$ is a thick set, contradicting the minimality of A_0.

(ii) Suppose for some $x \neq y$, $E_x=E_y$. Let $D \subset (X-A_0) \cup \{y\}$, $D \in S$. If $y \in D$, since $x \notin D$, if $\mu(E_x \cap D)=0$, then $A_0-\{y\}$ is a thick set, by (1). If

$\mu(E_x \cap D)>0$, then $\mu(E_x \cap D^c)=0$. In this case, $A_0-\{x\}$ is a thick set, by (1). Therefore, $y \notin D$. This means that $D \subset X-A_0$ implies $\mu(D)=0$, since A_0 is a thick set. Therefore, $\mu_*[X-(A_0-\{y\})]=0$ implies $A_0-\{y\}$ is a thick set.

(iii) Suppose for some $x \neq y$, $\mu(E_x \cap E_y)>0$, then x and y belong to $E_x \cap E_y$. If D is any measurable set then x and y both belong to D or both do not belong to D. Therefore, every measurable set contained in $[X-(A_0-\{y\})]$ is contained in $X-A_0$, which implies that $A_0-\{y\}$ is a thick set, which is a contradiction.

For each $x \in A_0$, associate a fixed E_x. Since μ is finite, this family $\{E_x\}$ is countable, which implies A_0 is countable. Let $A_0=(x_1, x_2, x_3, \ldots)$. Let E_{x_i}'s be the corresponding fixed sets. Then each E_{x_i} is a μ-atom. Let $B_i=E_{x_i}-\bigcup_{i\neq j}(E_{x_i} \cap E_{x_j})$. Then $X=\bigcup B_i \cup D$, where B_i's are disjoint atoms of positive measure and $\mu(D)=0$.

Corollary. *A minimal thick set A_0 is measurable if, and only if, for every $x \in A_0$, $\{x\}$ is measurable.*

Proof. Since A_0 is countable, *if part* is obvious. Conversely, if A_0 is measurable, then $A_0 \cap E_x=\{x\}$ for every $x \in A_0$, implies $\{x\} \in S$. Further, in this case, $\mu(\{x\})>0$, for every $x \in A_0$.

2.4. Example. *The converse of the above theorem is not true.*

X=any uncountable set,

S=countable, co-countable σ-field on X.

For every $E \in S$, define $\mu(E)=0$ if E is countable,

$=1$ otherwise.

Then (X, S, μ) is an atomic measure space. But the measure space does not admit any minimal thick set.

2.5. Theorem. *The Cartesian product of two thick sets is a thick set.*

Proof. Let A and B be two thick sets of the measure spaces (X_1, S_1, μ_1) and (X_2, S_2, μ_2) respectively. Let $D \subset X_1 \times X_2-A \times B$, $D \in S_1 \times S_2$. Since the finite disjoint union of measurable rectangles F_0, is a field, given any positive number ϵ, we can always find a set $E \in F_0$, $E=\bigcup_{i=1}^{n} A_i \times B_i$, $E \subset D$ and $A_i \in S_1, B_i \in S_2$ such that $\mu_1 \times \mu_2(D-E)<\epsilon$.

Now each $A_i \times B_i$ is entirely contained in one of the sets $X_1 \times (X_2 - B)$, $(X_1 - A) \times X_2$ and

$$\mu_1 \times \mu_2(A_i \times B_i) \leqslant (\mu_1 \times \mu_2)_*(X_1 \times \{X_2 - B\})$$

or

$$\leqslant (\mu_1 \times \mu_2)_*\{[X_1 - A] \times X_2\}$$

$$\leqslant \mu_{2*}(X_2 - B) = 0, \text{ since } B \text{ is thick,}$$

or

$$\leqslant \mu_{1*}(X_1 - A) = 0, \text{ since } A \text{ is thick.}$$

Therefore, $\mu_1 \times \mu_2(D) < \epsilon$. This being true for every ϵ, $\mu_1 \times \mu_2(D) = 0$, which in turn implies that $A \times B$ is a thick set.

Corollary 1. *Finite Cartesian product of thick sets is a thick set.*

Corollary 2. *Finite Cartesian product of minimal thick sets is a minimal thick set.*

Proof. Let A_0, B_0 be two minimal thick sets of (X_1, S_1, μ_1) and (X_2, S_2, μ_2) respectively. Then by the above theorem, $A_0 \times B_0$ is a thick set and the measure spaces are atomic. Let a proper subset A of $A_0 \times B_0$ be a thick set. Let $(x, y) \epsilon A_0 \times B_0$ and $(x, y) \not\in A$. Let E_x and F_y be the atoms containing x and y respectively. Then $E_x \times F_y \subset (X_1 \times X_2) - A$ and $\mu_1 \times \mu_2(E_x \times F_y) > C$, which is a contradiction. Therefore, $A_0 \times B_0$ is a minimal thick set.

2.6. Example : *The countable Cartesian product of minimal thick sets need not be a minimal thick set.*

Let $X_i = \{0, 1\}$; S_i=discrete σ-field on X_i

$$\mu_i\{0\} = \mu_i\{1\} = 1/2, \text{ for every } i = 1, 2, 3, \ldots.$$

Let X be the product space of X_i's, S, the product σ-field, and μ, the product measure. Then (X, S, μ) is a non-atomic measure space. Hence it does not admit a minimal thick set.

The following lemma and example are used in the next section.

2.7. Lemma. *If A_0 is a minimal thick set of (X, S, μ), then A_0 is a minimal thick set of (X, S, μ_1) iff $\mu \equiv \mu_1$.*

Proof. If A_0 is a minimal thick set of (X, S, μ_1) and if $\mu(A) = 0$, then $A \cap A_0 = \phi$, otherwise $\mu^*(A \cap A_0) = 0$, contradicting the minimality of A_0. Therefore, $A \subset X - A_0$. But $\mu_{1*}(X - A_0) = 0$, which implies that $\mu_1(A) = 0$. Therefore, $\mu_1 << \mu$, and similarly, $\mu << \mu_1$. The second part is trivial.

2.8. Example. *Let X be the real line, S, the σ-field of Borel sets, μ, the Lebesgue measure. Then there exists a decreasing sequence of thick sets tending to the empty set.*[1]

Let A, E_0 be the sets as in the theorems C, D of Halmos[2] then the μ-inner measure of E_0 is zero. Since A is countable, $(r_1, r_2, r_3, \ldots)$, say, define $E_n = \cup(E_0 + r_i)$, the union being taken over $1 \leqslant i \leqslant n$.

Then $\bigcup_{n=0}^{\infty} E_n = X$, and by the same argument as given by Halmos[3], it can be shown that each E_n has μ-inner measure zero. Therefore, for each n, $X - E_n$ is a thick set and $\bigcap_{n=1}^{\infty} X - E_n = \phi$.

Section 3

3.0. Let X : Complete separable metric space,

S : σ-field generated by open subsets of X,

M : The set of all probability measures on S,

J : Weak topology on M.

Let $\mu \epsilon M$, which admits a minimal thick set X_0. Let

$$\mathcal{J}_\mu = \{\lambda \epsilon M : X_0 \text{ is a minimal thick set for } \lambda\}.$$

3.1. Lemma : *$\mathcal{J}_\mu$ is compact in (M, J) iff μ is a degenarate measure.*

Proof. Suppose μ is a degenarate measure, then X_0 is a set consisting of only one point. Therefore, $\mathcal{J}_\mu = \{\mu\}$, hence compact. Conversely, suppose $\mathcal{J}_\mu$ is compact. Since X_0 is a minimal thick set, X_0 is countable. If X_0 is denumerable, $(x_1, x_2, \ldots)$, say, define a sequence of probability measures μ_n as follows :

$$\mu_n(x_1) = 1/(n+2),\ \mu_n(x_2) = [1/2] - [1/(n+2)],\ \mu_n(x_i) = 1/2^{i-1} \text{ for } i > 2;$$

$$\mu_0(x_1) = 0,\ \mu_0(x_2) = 1/2,\ \mu_0(x_i) = 1/2^{i-1} \text{ for } i > 2.$$

Then $\mu_n \to \mu_0$ in the weak topology and $\mu_n \epsilon \mathcal{J}_\mu$, but $\mu_0 \notin \mathcal{J}_\mu$. Since (M, J) is a metric space[4], $\mathcal{J}_\mu$ is not closed and so not compact. Therefore, X_0 is finite. Let $X_0 = (y_1, y_2, \ldots, y_n)$, and if $m \geqslant 2$, define

1. Halmos (2), see example 3, section 4.2.
2. Halmos (2), see section 16.8.
3. Halmos (2).
4. See Varadarajan (6).

$$\mu_n(y_1)=1/[n+2]$$

$$\mu_n(y_2)=(1/2)-[1/(n+2)]+[1/2(m-1)]$$

$$\mu_n(y_i)=1/2(m-1) \text{ for } i>2;$$

$$\mu_0(y_1)=0;\ \mu_0(y_2)=[1/2]+[1/2(m-1)]$$

$$\mu_0(y_i)=1/2(m-1) \text{ for } i>2.$$

Then $\mu_n \rightarrow \mu_0$ weakly and $\mu_n \epsilon \mathcal{J}_\mu$, but $\mu_0 \notin \mathcal{J}_\mu$. Therefore $\mathcal{J}_\mu$ is not compact. Hence, X_0 is a set consisting of a single point. Therefore, μ is a degenerate measure.

3.2. Lemma : *$\mathcal{J}_\mu$ is open iff X_0 is finite and $X_0=X$.*

Proof : If $X_0=X=(x_1, x_2, \dots x_n)$, say, then any function is continuous. Define

$$f_i(x_i)=0,\ f_i(x_j)=1, \text{ if } i\neq j \text{ for } i, j=1, 2, \dots, n.$$

Let $\mu_1 \epsilon \mathcal{J}_\mu,\ \mu_1(x_i)=q_i>0,\ i=1, 2, \dots, n.$

Define, $\epsilon_i=\min\,[q_i/2,\ (1-q_i)/2]$, and

$$\mathcal{N}=\mathcal{N}(f_1, f_2, \dots, f_n; \epsilon_1, \epsilon_2, \dots, \epsilon_n; \mu_1)$$

$$=\{\xi : |\textstyle\int f_i d\xi-\sum_{j=1}^{n} f_i(x_j)q_j|<\epsilon_i,\ i=1, 2, \dots, n\}.$$

Let $\xi \epsilon \mathcal{N}$, $\xi(x_i)=\xi_i$ implies

$$|\sum_{j=1}^{n} f_i(x_j)(\xi_j-q_j)|<\epsilon_i, \text{ for } i=1, 2, \dots, n,$$

which in turn implies $|\xi_i-q_i|<\epsilon_i$ for every i. This inequality holds iff ξ_i is different from 0 and 1. Therefore, $\xi\equiv\mu$, which implies $\xi \epsilon \mathcal{J}_\mu$, and $\mathcal{N}$ is contained in $\mathcal{J}_\mu$. Hence, $\mathcal{J}_\mu$ is open. Next, we shall prove that if X_0 is denumerable, then $\mathcal{J}_\mu$ will not be open. Let $X_0=(x_1, x_2, \dots)$. Let $\mu_1 \epsilon \mathcal{J}_\mu$ and let

$\mathcal{N}=\{\xi : |\int f_i d\xi-\int f_i d\mu_1|<\epsilon_i,\ i=1, 2, \dots, k$ and f_i's are arbitrary bounded continuous functions on X, and ϵ_i's are positive numbers$\}$,

be any basic neighbourhood of μ_1. Let $\mu_1(x_i)=p_i>0$. Choose j_0 so large such that

$$p_{j_0} < \frac{\min(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)}{2\max(k_1, k_2, \ldots, k_k)}, \text{ where } |f_i| < k_i.$$

Let $$\xi(x_1) = p_1 + p_{j_0};\ \xi(x_i) = p_i, \text{ if } i \neq j_0,$$

and $\xi(x_{j_0}) = 0$. Now,

$$|\int f_i d\xi - \int f_i d\mu_1| = |[f_i(x_1) - f_i(x_{j_0})]\, p_{j_0}|$$
$$\leqslant 2k_i p_{j_0} < \epsilon_i, \text{ for all } i.$$

Therefore, $\xi \epsilon N$ and $\xi \notin \mathcal{J}_\mu$. Hence, $\mathcal{J}_\mu$ is not open. We shall next prove that if X_0 is finite and $X_0 \neq X$, then $\mathcal{J}_\mu$ will not be open. Let $X_0 = (x_1, x_2, \ldots, x_n)$. Let $\mu_1 \epsilon \mathcal{J}_\mu$. Let N be as usual any basic neighbourhood of μ_1. Let $\mu_1(x_i) = p_i > 0$, $i = 1, 2, \ldots, n$. Let $\epsilon = \min(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$. Choose k_0 so large such that $\epsilon/3k_0 < \min(p_1, p_2, \ldots, p_n)$ and $M = \max(k_0, k_1, \ldots, k_k)$, where $|f_i| < k_i$. Define $\xi(x_1) = p_1 - [\epsilon/3M]$; $\xi(x_i) = p_i$, $i = 2$ to n, $\xi(x_{n+1}) = \epsilon/3M$, where x_{n+1} is any point in X, but not in X_0. A simple verification leads us to conclude that $\xi \epsilon N$, and $\xi \notin \mathcal{J}_\mu$. Hence, $\mathcal{J}_\mu$ is not open. Thus, we have proved

1. *If X_0 is finite and $X_0 = X$, then $\mathcal{J}_\mu$ is open.*

2. *If X_0 is denumerable, then $\mathcal{J}_\mu$ is not open.*

3. *If X_0 is finite and $X_0 \neq X$, then $\mathcal{J}_\mu$ is not open.*

These three facts completely prove the lemma.

3.3 Let X be a complete, separable metric space without isolated points. Let (S, M, J) be as defined in 3.0. Let

$\mathcal{J} = \{\mu \epsilon M : \mu$ admits a thick set whose complement is also a thick set$\}$.

The following lemmas lead to the main result stated in 3.5.

3.4. Lemma 1. *(X, S, μ) is non-atomic iff every singleton has μ-measure zero.*

Proof. Since the atoms of S are singletons, if (X, S, μ) is non-atomic then every singleton has μ-measure zero. Conversely, if there exists an $E \epsilon S$, such that $\mu(E) > 0$ and for every B contained in E, $B \epsilon S$, $\mu(B) = 0$, or $\mu(E)$, then we consider E with relative σ-field S_E which is separable and the restricton of μ to E, denoted by μ_E. Let $E_1, E_2, \ldots$ be a countable base for the topology of X. Then $\{E \cap E_i,\ i = 1, 2, \ldots\}$ is a family which generates S_E. Let F be the field generated by the above family. Then

F is countable. Let $F=\{C_1, C_2, \ldots\}$. Define $D_i=C_i$ or $=$ complement of C_i according as $\mu_E(C_i)=\mu(E)$ or $=0$. Let $D=\bigcap_{i=1}^{\infty} D_i$. Then D is an atom of S_E, and $\mu_E(D)=\mu(E)=\mu(D)>0$. D being a singleton, we have a contradiction.

LEMMA 2. *If (X, S, μ) admits a thick set whose complement is also a thick set, then (X, S, μ) is non-atomic.*

Proof : Let $x \epsilon X$, then $\{x\}$ is a subset of exactly one of the above thick sets. Therefore, the measure of every singleton is zero. Hence (X, S, μ) is non-atomic by Lemma 1.

LEMMA 3. *If (X, S, μ) admits a decreasing sequence of thick sets tending to the empty set, then (X, S, μ) is non-atomic.*

Proof. If $\mu\{x\}>0$ for some $x \epsilon X$, it belongs to every thick set, and so it belongs to the intersection of the above sequence of thick sets. This is a contradiction. Consequently, every singleton has measure zero. By Lemma 1, (X, S, μ) is non-atomic.

LEMMA 4. *The converses of Lemmas* (2) *and* (3) *are also true.*

Proof. Let (X, S, μ) be a non-atomic measure space. Let $E_1, E_2, \ldots$ be a countable base for X. This base generates S. Therefore, S is separable. It can be proved without difficulty that (X, S, μ) is a proper measure space. Let (X, S^*, μ^*) be the completion of (X, S, μ). Then (X, S^*, μ^*) is also a proper measure space. For every S^* measurable function f^*, there exists an S measurable function f, such that

$$A=\{x : f^*(x) \neq f(x)\}$$

has μ^* measure zero. Therefore, there exists a set $B \epsilon S$, and containing A, with μ measure zero. Let

$$\begin{aligned} g &= f \text{ on } X-B, \\ &\equiv t \text{ on } B, \text{ for some fixed } t \epsilon f(X-B). \end{aligned}$$

Then g is an S measurable function. (X, S, μ) satisfies known conditions.[1] It follows then that there exists a Borel set C on the real line such that

$$g^{-1}(C)=X.$$

1. Blackwell (1), Th. 3, Cor. 4, p. 4.

In other words, $g(X)$ is a Borel set. Now,

$$f^*(X-B)=f(X-B)=g(X).$$

Hence, $f^*(X-B)$ is a Borel set and $\mu^*(B)=0$. Therefore, (X, S^*, μ^*) is a normal space. By a classical theorem of Halmos and Von Neumann[1], which states, 'A necessary and sufficient condition that a measure space of total measure one be point isomorphic to the unit interval, Borel σ-field, Lebesgue measure, is that it be normal', it follows that there exists a point isomorphism T. This T carries thick sets into thick sets. Since in the unit interval with the Borel σ-field and Lebesgue measure, there exists,[2] a thick set whose complement is also a thick set, it follows that there exists a thick set of (X, S, μ) whose complement is also a thick set. This proves the converse of Lemma 2. By 2.8. Example, and by a slight modification, we have a decreasing sequence of thick sets in the unit interval tending to the empty set. Due to T, it follows that the converse of Lemma 3 is true.

3.5. THEOREM. *J is of second category whose complement is of first category.*

Proof. The class of all non-atomic probability measures which gives positive mass to each open set is a dense G_δ in J, and (M, J) is a complete, separable metric space[3]. A dense G_δ in a complete metric space is a set of second category whose complement is a set of first category.[4] This completes the proof of the theorem.

We give an example to show that theorem (A) of Halmos[5] is not true in general topological spaces.

3.6. EXAMPLE. Under the assumptions stated in 3.3, let (X, S, μ) be any non-atomic measure. We can imitate the proof of example (3) of Halmos[6], since the only facts used in the example are

1. there exists a decreasing sequence of thick sets tending to the empty set,

2. the diagonal in the finite dimensional Cartesian product of X with itself is measurable in the product σ-field. (X, S, μ) satisfies condition

1. Halmos and Von- Neumann (3), see page 258.
2. Halmos (2), see Theorem 5, section 16.
3. Parthasarthy, Rao and Varadhan (5), see page 210.
4. Munroe (4), see page 69.
5. Halmos (2), see section 49.
6. Halmos (2), section 49.

1 in view of lemma 4 of 3.4. Condition 2 is satisfied in view of the following unpublished lemma of B. V. Rao.

LEMMA. The diagonal in the finite dimensional Cartesian product of X with itself is measurable iff contains a countably generated σ-field S_1 with the atoms of S_1 being singletons.

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