ON THE POWER SEQUENCE OF A GRAPH

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ABSTRACT

Necessary and sufficient conditions for a sequence $(p_1, p_2, \ldots, p_n)$ of positive integers to be the power sequence of a connected graph on n vertices with m edges are given. The maximum power of a connected graph on n vertices with m edges and the class of all extremal graphs are also determined.

1. Introduction and definitions

We consider only finite undirected graphs without loops or multiple edges.

The power $p(x)$ of a vertex x of a connected graph G is the number of components of $G - x$. If $p_1, p_2, \cdots, p_n$ are the powers of the vertices of G, we say that G has the power sequence $(p_1, p_2, \cdots, p_n)$.

The power $p(G)$ of a connected graph G is

$$\max_{x \in G} p(x).$$

A vertex x is called a cut vertex if $p(x) \geqq 2$.

A connected graph without cut vertices is called biconnected. Thus a complete graph on two or fewer vertices is biconnected.

A maximal biconnected subgraph of a connected graph G is called a block of G.

For other definitions and notation we follow Berge [1].

In this paper we solve two problems concerning the power sequence of a graph. In §2, we obtain necessary and sufficient conditions for a sequence $(p_1, p_2, \cdots, p_n)$ of positive integers to be the power sequence of a connected graph on n vertices with m edges. In §3, we determine the maximum power of a connected graph on n vertices with m edges and the class of all extremal graphs.

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2. Graphs with given power sequence

In this section we obtain necessary and sufficient conditions for the existence of a connected graph on n vertices with m edges and with power sequence $(p_1, p_2, \cdots, p_n)$.

LEMMA 2.1 *Let $q_1, q_2, \cdots, q_n$ be positive integers. A tree with power sequence $(q_1, q_2, \cdots, q_n)$ exists if and only if $\sum_{i=1}^{n} q_i = 2(n-1)$. If $\sum_{i=1}^{n} q_i = 2(n-1)$, then any connected graph with power sequence $(q_1, q_2, \cdots, q_n)$ is a tree.*

Proof. It is evident that the power of a vertex x of a tree T coincides with the degree of x in T. So to prove the first part of the lemma, it is enough to show that if $\sum_{i=1}^{n} q_i = 2(n-1)$, then a tree T with degrees $q_1, q_2, \cdots, q_n$ exists. The existence and construction of such a tree was already obtained in [3] and [5]. Here we give a different construction. Without loss of generality we assume that $q_1 \geqq q_2 \geqq \cdots \geqq q_n$.

Take a vertex $a_{0,1}$. Then take q_1 new vertices $a_{1,1}, a_{1,2}, \cdots, a_{1,q_1}$ and join each of them to $a_{0,1}$. At the ith stage, $i \geqq 2$, take $q_i - 1$ new vertices $a_{i,1}, a_{i,2}, \cdots, a_{i,q_i-1}$ and join each of them to $a_{i-1,1}$, provided $q_i - 1 \geqq 1$. Suppose i_0 is the largest integer i such that $q_i - 1 \geqq 1$. Then it can be easily shown that

$$1 + q_1 + (q_2 - 1) + \cdots + (q_{i_0} - 1) = n,$$

so that the above construction is possible and gives a tree T with degrees $q_1, q_2, \cdots, q_n$.

To prove the second assertion of the lemma, let G be a connected graph with power sequence $(q_1, q_2, \cdots, q_n)$ and let T be a spanning tree of G. Since G and T have the same vertex set and every edge of T is an edge of G, the power of the ith vertex in $T \geqq q_i$. If $\sum_{i=1}^{n} q_i = 2(n-1)$, it follows that the power of the ith vertex in T is q_i and $G = T$. This completes the proof of the lemma.

THEOREM 2.2. *Let $p_1, p_2, \cdots, p_n$ be positive integers. Then there exists a connected graph G with power sequence $(p_1, p_2, \cdots, p_n)$ if and only if*

$$\sum_{i=1}^{n} p_i \leqq 2(n-1). \tag{2.1}$$

PROOF. Only if part follows from the proof of Lemma 2.1.

Conversely, let $p_1, p_2, \cdots, p_n$ be positive integers satisfying condition (2.1). Let $k = 2(n-1) - \sum_{i=1}^{n} p_i$. Then $0 \leqq k \leqq n-2$. Now without loss of generality we assume that $p_1 \geqq p_2 \geqq \cdots \geqq p_n$. Define a new sequence $(q_1, q_2, \cdots, q_n)$ by:

$$q_i = p_i + 1 \text{ for } i = 1, \cdots, k,$$

$$q_i = p_i \text{ for } i = k+1, \cdots, n.$$

Then $\sum_{i=1}^{n} q_i = 2(n-1)$. Let T be the tree with power sequence $(q_1, q_2, \cdots, q_n)$ constructed in the proof of Lemma 2.1.

If $k = 0$, the proof of the theorem is complete, so let $k \geqq 1$. Then it is obvious that $i_0 \geqq k$. The case $p_1 = 1$ is trivial, so we take $p_1 \geqq 2$. Let i_1 be the largest integer i such that $q_i - 1 \geqq 2$. We consider two cases now.

Case (*i*): $i_1 \geqq k$. Then join $a_{i,1}$ to $a_{i,2}$ for $i = 1, \cdots, k$.

Case (*ii*): $i_1 < k$. Then join $a_{i,1}$ to $a_{i,2}$ for $i = 1, \cdots, i_1$, and join $a_{i,1}$ to $a_{i_1,2}$ for $i = i_1 + 1, \cdots, k$.

Now it can be easily verified that the resulting graph has power sequence $(p_1, p_2, \cdots, p_n)$. This completes the proof of the theorem.

THEOREM 2.3. *Let $p_1, p_2, \cdots, p_n$ be positive integers and $m \geqq n$. Then the following two conditions together are necessary and sufficient for the existence of a connected graph on n vertices with m edges and with power sequence $(p_1, p_2, \cdots, p_n)$:*

$$\sum_{i=1}^{n} p_i < 2(n-1), \tag{2.1}$$

$$m \leqq \binom{k+2}{2} + n - k - 2, \tag{2.2}$$

where $k = 2(n-1) - \sum_{i=1}^{n} p_i$.

PROOF. The necessity of condition (2.1) was proved in Theorem 2.2. To prove the necessity of (2.2), let G be a connected graph on n vertices with m edges and with power sequence $(p_1, p_2, \cdots, p_n)$. If t is the number of blocks in G, it can be proved by induction on t that $\sum_{i=1}^{n} p_i = n + t - 1$, see [2]. Thus $k = n - t - 1$. Now from Theorem 1.2 of [4], we have

$$m \leqq \binom{n-t+1}{2} + t - 1 = \binom{k+2}{2} + n - k - 2.$$

To prove sufficiency, let conditions (2.1) and (2.2) be satisfied and let $p_1 \geqq p_2 \geqq \cdots \geqq p_n$. Then construct a graph H with power sequence $(p_1, p_2, \cdots, p_n)$ as in the proof of Theorem 2.2. If $k = 1$, then $m = n$ and H has m edges. So let $k \geqq 2$. We consider two cases.

Case (*i*). $i_1 \geqq k$. Then remove the edges incident to the vertices $a_{1,2}, a_{2,2}, \cdots, a_{k-1,2}$ and join each of these vertices to $a_{k-1,1}$ and $a_{k,1}$. The power sequence of the graph is not altered by this. Next replace the block on the $k+2$ vertices $a_{1,2}, a_{2,2}, \cdots, a_{k-1,2}, a_{k,1}, a_{k,2}, a_{k-1,1}$ by an elementary cycle C on the same vertices. The graph H_1 thus obtained has n edges. Now if we write $m = n + l$, then by (2.2), $l \leqq \binom{k+2}{2} - k - 2$, so l new edges can be added to the cycle C of H_1.

Case (*ii*). $i_1 < k$. The case $p_1 = 1$ is trivial, so let $i_1 \geqq 1$. If $i_1 = 1$, then the $k+2$ vertices $a_{0,1}, a_{1,1}, \cdots, a_{k,1}, a_{1,2}$ form a block in H. If $i_1 > 1$, then remove the edges incident to the vertices $a_{1,2}, a_{2,2}, \cdots, a_{i_1-1,2}$ and join each of these vertices to $a_{k-1,1}$ and $a_{k,1}$. Then we get a block on the $k+2$ vertices $a_{1,2}, \cdots, a_{i_1,2}, a_{i_1-1,1}, \cdots, a_{k,1}$. Now this block can be replaced by a cycle and the construction completed as in case (*i*). This completes the proof of the theorem.

3. Maximum power of a graph

In this section we determine the maximum power of a connected graph on n vertices with m edges and the class of all extremal graphs.

THEOREM 3.1. *The maximum power of a connected graph on n vertices with m edges is $r+1$, where $r = r(n,m)$ is given by*

$$r(n,m) = \max\left\{q : q \leqq n-2, m \leqq \binom{n-q}{2} + q\right\} = \left[n - \frac{3}{2} - \sqrt{2m - 2n + \frac{9}{4}}\right] \tag{3.1}$$

and $[x]$ denotes the greatest integer $\leqq x$.

PROOF. Let G be any connected graph on n vertices with m edges. If t is the number of blocks in G, obviously $p(G) \leqq t$. Now by rearranging the blocks of G in the form of a chain, we get a graph with $t-1$ cut vertices. Hence by Theorem 1.3 of [4], it follows that $t-1 \leqq r$. Thus $p(G) \leqq r+1$. To construct a graph which attains the power $r+1$, take any biconnected graph G_0 on $n-r$ vertices with $m-r$ edges, add r new vertices and join them to one vertex of G_0. This completes the proof of the theorem.

The following result can be deduced easily from the proof of Theorem 3.1: a connected graph on n vertices with m edges and with power p exists if and only if $1 \leqq p \leqq r(n,m) + 1$ and if $m = n-1$, then $p \neq 1$.

THEOREM 3.2. *Let $r = r(n,m)$ be given by (3.1). Then the connected graphs on n vertices with m edges and with power $r+1$ are the following, where (2) is possible only when* $m = \binom{n-r-1}{2} + r + 2$:

(1) *a graph consisting of $r+1$ blocks incident with a common vertex, r of the blocks being edges and the other having $n-r$ vertices and $m-r$ edges.*

(2) *a graph consisting of $r+1$ blocks incident with a common vertex, $r-1$ of the blocks being edges and the other two being complete graphs on three and $n-r-1$ vertices respectively.*

PROOF. Let G be a connected graph on n vertices with m edges and with power $r+1$ attained by a vertex x. Then x together with the vertices of any component of $G-x$ forms a block of G. Arranging these blocks in the form of a chain, we get a graph with r cut vertices, hence its structure is given by Theorem 1.8 of [4]. Now the present theorem follows easily.

We mention the following unsolved problem. Find necessary and sufficient conditions for the existence of a connected graph on n vertices with degree of the ith vertex equal to d_i and power of the ith vertex equal to p_i, $i = 1,2,\cdots,n$.

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