Sankhyā : The Indian Journal of Statistics
1990, Volume 52, Series A, Pt. 1, pp. 131-141.

ON TWO CONJECTURES ABOUT TWO-STAGE SELECTION PROCEDURES

By SUBIR KUMAR BHANDARI
Indian Statistical Institute
and
ATASI RAY CHAUDHURI
Indian Statistical Institute

SUMMARY. For given k normal populations with unknown means and common known variance, Alam (1970) suggested a two-stage procedure to select the population having the largest mean. He conjectured that under this procedure, the least favourable configuration (L.F.C.) would be the slippage configuration. This procedure has been subsequently studied by Tamhane and Bechhofer (1977, 1979) and Miescke and Sehr (1980) while in the latter another two-stage procedure has been given and a similar conjecture is made about the LFC. In this paper both these conjectures have been settled and both are found to be true. Though the conjectures here were made for normal distribution, the proofs given in this paper hold for any distribution whose sample mean has MLR property.

1. Introduction

Let $\pi_1, \pi_2, \ldots, \pi_k$ denote k normal populations with unknown means $\mu_1, \mu_2, \ldots, \mu_k$ respectively and a common known variance $\sigma^2 > 0$. Let $\mu_{[1]} \leqslant \mu_{[2]} \leqslant \ldots, \leqslant \mu_{[k]}$ denote the ordered set of values of the means. The problem is to select the population with the largest mean $\mu_{[k]}$.

For given sample size n_1,let $(X_{i1}, \ldots, X_{in_1})$, $i = 1, 2, \ldots, k$ denote k independent samples from $\pi_1, \pi_2, \ldots, \pi_k$ respectively. Define $X_i = n_1^{-1}(X_{i1} + \ldots + X_{in_1})$, $i = 1, 2, \ldots, k$. Bechhofer's fixed sample procedure ($\mathcal{J}$) is to choose the cell corresponding to the maximum of X_i for $i = 1, 2, \ldots, k$.

Let PCS $(\mu, \mathcal{J})$ be the probability of correct selection under $\mathcal{J}$ with the true mean $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)$ such that $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_{k-1} < \mu_k$. Noting

$$\text{PCS}(\mu : \mathcal{J}) = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \phi\left(\frac{\sqrt{n_1}(x-\mu_i)}{\sigma}\right) d\phi\left(\frac{\sqrt{n_1}(x-\mu_k)}{\sigma}\right)$$

AMS (1980) *subject classification :* 62F07.

Key words and phrases : Two-stage selection procedure ; Least favourable configuration ; Logconcavity.

where ϕ (·) is the c.d.f. of standard normal variate, one can show the monotonicity of PCS ($\boldsymbol{\mu}$: $\mathcal{J}$) in $\mu_1 \mu_2, \ldots, \mu_{k-1}$ by showing

$$\frac{\partial \, \text{PCS} \, (\boldsymbol{\mu} \, : \, \mathcal{J})}{\partial \mu_i} \leqslant 0 \; \forall \; i = 1, 2, \ldots, k-1, \qquad \ldots \quad (1.1)$$

Alam (1970) proposed the following two-stage procedure P_1 :

Stage 1 : Take k independent samples $(X_{i1}, \ldots, X_{in_1})$ of size n_1, $i = 1, 2, \ldots, k$, from $\pi_1, \pi_2, \ldots, \pi_k$ and compute $X_i = n_1^{-1} (X_{i1} + \ldots + X_{in_1})$ for $i = 1, 2, \ldots k$. Select all population π_i with $X_i \geqslant \max \{X_j : j = 1, 2, \ldots k,\}$ $-c$ where c is a fixed positive real number. If only one population is selected, stop and assert that, this one has the largest mean, otherwise proceed to stage 2.

Stage 2 : Take additional independent samples $(Y_{i1}, \ldots, Y_{in_2})$ of size n_2 from the populations selected in stage 1 and compute $Y_i = n_2^{-1}$ $(Y_{i1} + \ldots + Y_{in_2})$ for them. Then select the population giving the maximum of $(n_1 X_i + n_2 Y_i)$.

Thus procedure P_1 is a combination of two classical one-stage procedures where the first one (in stage 1) is due to Gupta (1956) and the second one is due to Bechhofer (1954). In Alam (1970) the following conjecture was made.

Conjecture I : Let $\delta_0 > 0$ be fixed. Consider

$$\Omega_{\delta_0} = \{\boldsymbol{\mu} \in R^k : \mu_{[k-1]} \leqslant \mu_{[k]} - \delta_0\}$$

where for $\boldsymbol{\mu} \in R^k$, $\mu_{[1]} \leqslant \ldots \leqslant \mu_{[k]}$ denote the ordered co-ordinates of $\boldsymbol{\mu}$. Then for every $t \in R$,

$$\inf_{\boldsymbol{\mu} \in \Omega_{\delta_0}} \text{PCS} \; (\boldsymbol{\mu} : P_1) = \text{PCS}((t, t, \ldots, t+\delta_0) : P_1).$$

Another procedure P_2 was given by Miescke and Sehr (1980). P_2 differs from P_1 only in stage 2 where the final decision is made in terms of the Y_i's instead of $(n_1 X_i + n_2 Y_i)$'s. A similar conjecture is made here, conjecture II (say) :

$$\inf_{\boldsymbol{\mu} \in \Omega_{\delta_0}} \text{PCS} \; (\boldsymbol{\mu} : P_2) = \text{PCS}((t, t, \ldots, t+\delta_0) : P_2) \; \forall \; t \in R.$$

From now onwards we shall denote

$$\text{PCS}(\boldsymbol{\mu} : P_j) = \text{PCS}_j(\boldsymbol{\mu}) \text{ for } j = 1, 2.$$

Remark 1 : Gupta and Miescke (1984) has shown that procedure P_2 is inferior to P_1. Procedure P_2 is only reasonable if the data from stage 1 are lost and only the information about the subset decision is available at stage 2.

It can be seen that both the conjectures I and II hold for independent samples from populations $\pi_1, \pi_2, \ldots, \pi_k$ where π_i has density $g(x-\mu_i)$, $1 \leqslant i \leqslant k$ for some g, such that the sample mean of π_i is MLR in μ_i. This can be verified by noting that in both the proofs we have used only the MLR property of X_i and Y_i in μ_i and the fact that μ_i is location parameter. In particular equation (1.1) is also valid for such distributions since MLR in μ_i implies stochastic ordering in μ_i.

Remark 2 : $g(x-\mu)$ is MLR in μ if and only if g is logconcave (Karlin, 1968). Also note that logconcavity is closed under convolution (Dasgupta, 1980).

We first give the proof of conjecture II for $k \geqslant 2$ in Section 2. The proof of conjecture I is similar to that of conjecture II and is given in Section 3.

2. Proof of conjecture II

The main idea of the proof is to introduce a function $\text{PCS}_2^*(\boldsymbol{\mu})$ (need not be probability) such that $\text{PCS}_2(\boldsymbol{\mu}) \geqslant \text{PCS}_2^*(\boldsymbol{\mu})\ \forall\ \boldsymbol{\mu}$ and then to show $\inf\limits_{\mu \in \Omega_{\delta_0}} \text{CSP}_2^*(\boldsymbol{\mu}) = \text{PCS}_2(t, t, \ldots, t+\delta_0)\ \forall\ t \in R$.

Let us now define, $\text{PCS}_2(\boldsymbol{\mu} \mid \boldsymbol{x})$ = Probability of correct selection given that $\boldsymbol{x} = (x_1, \ldots, x_k)$ is observed in the first stage,

$$\text{PCS}_2(\boldsymbol{\mu}) = \int_{\boldsymbol{x} \in R^k} \text{PCS}_2(\boldsymbol{\mu} \mid \boldsymbol{x}) f_{\boldsymbol{\mu}}(\boldsymbol{x}) d\boldsymbol{x} \quad \ldots \quad (2.1)$$

where $f_{\boldsymbol{\mu}}(\boldsymbol{x})$ is the density of $(X_1, X_2, \ldots, X_k)$. Let $\boldsymbol{\mu}_0 = (0, 0, \ldots, 0, \delta_0)$.

Now, to prove conjecture II it is sufficient to show

$$\text{PCS}_2(\boldsymbol{\mu}) \geqslant \text{PCS}_2(\boldsymbol{\mu}_0)\ \forall\ \boldsymbol{\mu} \leqslant \boldsymbol{\mu}_0, \text{ with } \mu_{[k]} = \delta_0 \quad \ldots \quad (2.2)$$

as the PCS_2 is invariant under translation i.e.

$$\text{PCS}_2(\mu_1, \mu_2, \ldots, \mu_k) = \text{PCS}_2(\mu_1+a, \mu_2+a, \ldots, \mu_k+a)\ \forall\ a \in R.$$

Without loss of generality we consider $\mu_k = \mu_{[k]}$, where $\mu = (\mu_1, \ldots, \mu_k)$.

Define $$\text{PCS}_2^*(\boldsymbol{\mu}) = \int_{\boldsymbol{x} \in R^k} \text{PCS}_2(\boldsymbol{\mu}_0 \mid \boldsymbol{x}) f_{\boldsymbol{\mu}}(\boldsymbol{x}) d\boldsymbol{x}. \quad \ldots \quad (2.3)$$

As in P_2, the final decision is based only on Y_i's, a result similar to (1.1) holds and we have for all $\boldsymbol{\mu} \leqslant \boldsymbol{\mu}_0$, with $\mu_k = \delta_0$

$$\mathrm{PCS}_2(\boldsymbol{\mu} \,|\, \boldsymbol{x}) \geqslant \mathrm{PCS}_2(\boldsymbol{\mu}_0 \,|\, \boldsymbol{x}). \qquad \ldots \quad (2.4)$$

Hence from (2.1) and (2.3),

$$\mathrm{PCS}_2(\boldsymbol{\mu}) \geqslant \mathrm{PCS}_2^*(\boldsymbol{\mu}). \qquad \ldots \quad (2.5)$$

Again as $\mathrm{PCS}_2(\boldsymbol{\mu}_0) = \mathrm{PCS}_2^*(\boldsymbol{\mu}_0)$, to show (2.2), it is sufficient to show

$$\mathrm{PCS}_2^*(\boldsymbol{\mu}) \geqslant \mathrm{PCS}_2^*(\boldsymbol{\mu}_0), \qquad \ldots \quad (2.6)$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k) \leqslant \boldsymbol{\mu}_0 = (0, 0, \ldots, 0, \delta_0)$ with $\mu_k = \delta_0$. Now to show (2.6), without loss of generality we consider $\mu_1 \leqslant \mu_2 \ldots \leqslant \mu_{k-1} < \mu_k = \delta_0$. For this we may have any of the following configuration for $1 \leqslant r \leqslant k-1$,

$$\mu_1 = \mu_2 = \ldots = \mu_r < \mu_{r+1} \leqslant \ldots \leqslant \mu_{k-1} < \mu_k = \delta_0.$$

Hence (2.6) (i.e., conjecture II), follows from the following result by considering directional derivatives in the direction (1, 1, ..., 1, 0, 0, ..., 0) (with r many 1's).

Result :
$$\frac{\partial\, \mathrm{PCS}_2^*(\boldsymbol{\mu})}{\partial \mu_1} \leqslant 0. \qquad \ldots \quad (2.7)$$

Now for fixed $\varepsilon > 0$ and $c(>0$, used in stage 1 of P_2), define

$$C_\varepsilon = \{(x_2, x_3, \ldots, x_k) : |x_i - x_j| \geqslant \varepsilon,\ |x_i - x_j + c| \geqslant \varepsilon\ \forall\ i, j = 2, 3, \ldots, k\}$$

$$A_\varepsilon = R \times C_\varepsilon.$$

Clearly, $A_\varepsilon \subset R^k$.

Let
$$\mathrm{PCS}_2^*(\boldsymbol{\mu} : S) = \int_{\boldsymbol{x} \in S} \mathrm{PCS}_2(\boldsymbol{\mu}_0 \,|\, \boldsymbol{x}) f_{\boldsymbol{\mu}}(\boldsymbol{x}) d\boldsymbol{x}.$$

Consider $\mathrm{PCS}_2^*(\boldsymbol{\mu} : A_\varepsilon)$,

$$\lim_{\varepsilon \to 0} \frac{\partial\, \mathrm{PCS}_2^*(\boldsymbol{\mu} : A_\varepsilon)}{\partial \mu_1}$$

$$= \lim_{\varepsilon \to 0} \int_{x \varepsilon A_\varepsilon} \mathrm{PCS}_2(\boldsymbol{\mu}_0 \,|\, \boldsymbol{x}) \left(\frac{\partial}{\partial \mu_1} f_{\boldsymbol{\mu}}(\boldsymbol{x}) \right) d\boldsymbol{x}$$

$$= \frac{\partial}{\partial \mu_1} \mathrm{PCS}_2^*(\boldsymbol{\mu}) \ [\text{As } A_\varepsilon \uparrow R^k, \text{ as } \varepsilon \to 0].$$

Hence if we prove the following Lemma (2.1) the above result will be proved and thereby proving conjecture II completely.

Lemma 2.1 : $\dfrac{\partial\, \mathrm{PCS}_2^*(\boldsymbol{\mu} : A_\varepsilon)}{\partial \mu_1} \leqslant 0\ \forall\ \varepsilon > 0$ *fixed*. ... (2.8)

Proof of Lemma 2.1 : Fix $0 < \delta \ll \varepsilon$ i.e. δ is very small compared to ε. Let $e_1 = (1, 0, 0, \ldots, 0) \epsilon R^k$ and

$$u(\delta) = \mathrm{PCS}_2^*((\boldsymbol{\mu}+\delta e_1) : A_\varepsilon)$$

$$u(0) = \mathrm{PCS}_2^*(\boldsymbol{\mu} : A_\varepsilon).$$

Note that
$$u'(0) = \frac{\partial \, \mathrm{PCS}_1^*(\boldsymbol{\mu} : A_\varepsilon)}{\partial \mu_1}.$$

For $2 \leqslant i_0 \leqslant k$, define

$$W_{i_0}^+(\delta) = A_\varepsilon \cap \{\boldsymbol{x} \,\epsilon\, R^k : x_1 \geqslant x_i \,\forall\, i \neq 1$$

and
$$(x_{i_0}+c-\delta) < x_1 \leqslant (x_{i_0}+c)\}.$$

$$W_{i_0}^-(\delta) = A_\varepsilon \cap \{\boldsymbol{x} \,\epsilon\, R^k : x_{i_0} \geqslant x_i \,\forall\, i \neq i_0$$

and
$$x_{i_0}-c > x_1 \geqslant x_{i_0}-c-\delta\}.$$

Note that $W_{i_0}^+(\delta)$, $i_0 = 2, \ldots, k$ and $W_{i_0}^-(\delta)$, $i_0 = 2, \ldots, k$ are all disjoint, for the structure of A_ε and the fact that $\delta \ll \varepsilon$.

Now
$$u(\delta) = \mathrm{PCS}_2^*((\mu+\delta e_1) : A_\varepsilon)$$

$$= \int_{A_\varepsilon} \mathrm{PCS}_2(\boldsymbol{\mu} \,|\, \boldsymbol{x}) f_{\boldsymbol{\mu}+\delta e_1}(\boldsymbol{x}) d\boldsymbol{x}$$

$$= \int_{A_\varepsilon} \mathrm{PCS}_2(\boldsymbol{\mu}_0 \,|\, \boldsymbol{x}) f_{\boldsymbol{\mu}}(x_1-\delta, x_2, \ldots, x_k)\, d\boldsymbol{x}.$$

$$= \int_{A_\varepsilon} \mathrm{PCS}_2(\boldsymbol{\mu}_0 \,|\, (x_1+\delta, x_2, \ldots x_k)) f_{\boldsymbol{\mu}}(\boldsymbol{x}) d\boldsymbol{x}$$

(by change of variable and the fact that $A_\varepsilon = R \times C_\varepsilon$ does not change with any location change of x_1)

$$= \sum_{i_0=2}^{k} \int_{W_{i_0}^+(\delta)} \mathrm{PCS}_2(\boldsymbol{\mu}_0 \,|\, I_{i_0}(\gamma_{i_0}(\boldsymbol{x})+c.e_1) \cup \{1\}) f_{\boldsymbol{\mu}}(\boldsymbol{x}) d\boldsymbol{x}$$

$$+ \sum_{i_0=2}^{k} \int_{W_{i_0}^-(\delta)} \mathrm{PCS}_2(\boldsymbol{\mu}_0 \,|\, I_{i_0}(\gamma_{i_0}(\boldsymbol{x})-c.e_1) \cup \{1, i_0\}) f_{\boldsymbol{\mu}}(\boldsymbol{x}) d\boldsymbol{x}$$

$$+ \int_{A_\varepsilon - W(\delta)} \mathrm{PCS}_2(\boldsymbol{\mu}_0 \,|\, x_1+\delta, x_2, \ldots, x_k) f_{\boldsymbol{\mu}}(\boldsymbol{x}) d\boldsymbol{x}$$

$$= \sum_{i_0=2}^{k} A_{i_0, \delta}^+(\delta) + \sum_{i_0=2}^{k} A_{i_0, \delta}^-(\delta) + B_\delta(\delta) \text{ (say)} \qquad \ldots \quad (2.9)$$

where $$W(\delta) = \bigcup_{i_0=2}^{k} (W^+_{i_0}(\delta) \cup W^-_{i_0}(\delta)),$$

$$\gamma_{i_0}(x) = (x_{i_0}, x_2, \ldots, x_k),$$

$I_{i_0}(x) =$ (subset selected in stage 1 when $\boldsymbol{x}$ is observed)$-\{1, i_0\}$

and

$\mathrm{PCS}_2\,(\mu_0 \mid J(\boldsymbol{x})) =$ Probability of correct selection given that subset $J(\boldsymbol{x})$ is selected in stage 1, when $\boldsymbol{x}$ is observed. Here $J(\boldsymbol{x}) \subsetneq \{1, 2, \ldots, k\}$.

Note that $\mathrm{PCS}_2(\boldsymbol{\mu}_0 \mid J(\boldsymbol{x})) = \mathrm{PCS}_2(\boldsymbol{\mu}_0 \mid \boldsymbol{x})$.

Again,

$$u(0) = \sum_{i_0=2}^{k} \int_{W^+_{i_0}(\delta)} \mathrm{PCS}_2(\boldsymbol{\mu}_0 \mid I_{i_0}(\gamma_{i_0}(\boldsymbol{x})+c.e_1) \cup \{1, i_0\}) f_{\boldsymbol{\mu}}(\boldsymbol{x})d\boldsymbol{x}$$

$$+ \sum_{i_0=2}^{k} \int_{W^-_{i_0}(\delta)} \mathrm{PCS}_2(\boldsymbol{\mu}_0 \mid I_{i_0}(\gamma_{i_0}(\boldsymbol{x})-c.e_1) \cup \{i_0\}) f_{\boldsymbol{\mu}}(\boldsymbol{x})d\boldsymbol{x}$$

$$+ \int_{A-W(\delta)} \mathrm{PCS}_2(\boldsymbol{\mu}_0 \mid \boldsymbol{x}) f_{\boldsymbol{\mu}}(\boldsymbol{x})d\boldsymbol{x}$$

$$= \sum_{i_0=2}^{k} A^+_{i_0,0}(\delta) + \sum_{i_0=2}^{k} A^-_{i_0,0}(\delta) + B_0(\delta) \text{ (say).} \qquad \ldots \quad (2.10)$$

Note that

$$\lim_{\delta \to 0} \frac{1}{\delta} \left[A^+_{i_0,\delta}(\delta) - A^+_{i_0,0}(\delta) \right]$$

$$= \lim_{\delta \to 0} \int_{C_\epsilon \cap \{(x_2, \ldots x_k) : x_{i_0}+c \geqslant x_i \,\forall\, i \neq 1\}}$$

$$[\mathrm{PCS}_2(\boldsymbol{\mu}_0 \mid I_{i_0}(\gamma_{i_0}(\boldsymbol{x})+c.e_1) \cup \{1\}) - \mathrm{PCS}_2(\boldsymbol{\mu}_0 \mid I_{i_0}(\gamma_{i_0}(\boldsymbol{x})+c.e_1) \cup \{1, i_0\})]$$

$$f_{\mu_2, \ldots \mu_k}(x_2, \ldots x_k) . \left[\frac{1}{\delta} \int_{x_{i_0}+c-\delta}^{x_{i_0}+c} f_{\mu_1}(x_1)dx_1 \right] dx_2 \ldots dx_k. \qquad \ldots \quad (2.11)$$

[Since from definition of A_ϵ, $W^+_{i_0}(\delta) = A_\epsilon \cap \{\boldsymbol{x} : x_{i_0}+c \geqslant x_i \,\forall\, i \neq 1$ and $x_{i_0}+c-\delta < x_1 \leqslant x_{i_0}+c\}$ and as $\gamma_{i_0}(\boldsymbol{x})$ does not depend on x_1, (2.11) is well-defined].

$$= \int_{C_\epsilon \cap \{(x_2, \ldots, x_k) : x_{i_0}+c \geqslant x_i \,\forall\, i \neq 1\}} [\mathrm{PCS}_2(\boldsymbol{\mu}_0 \mid I_{i_0}(\gamma_{i_0}(\boldsymbol{x})+c.e_1) \cup \{1\})$$

$$-\mathrm{PCS}_2(\boldsymbol{\mu}_0 \mid I_{i_0}(\gamma_{i_0}(\boldsymbol{x})+c.e_1) \cup \{1, i_0\})] f_{\boldsymbol{\mu}}(x_{i_0}+c, x_2, x_3 \ldots x_k)dx_2 \ldots dx_k$$

$$= D^+_{i_0} \text{ (say).} \qquad \ldots \quad (2.12)$$

[(2.12) follows by noting that $\lim\limits_{\delta \to 0} \frac{1}{\delta} \left[\int\limits_{x_{i_0}+c-\delta}^{x_{i_0}+c} f_{\mu_1}(x_1)dx_1 \right] = f_{\mu_1}(x_{i_0}+c)$ and then by Dominated Convergence Theorem].

Similarly,

$$\lim_{\delta \to 0} \frac{1}{\delta} \left[A^-_{i_0,\delta}(\delta) - A^-_{i_0,0}(\delta) \right]$$

$$= -\int_{C \in \cap \{(x_2, \dots x_k) : x_{i_0} \geqslant x_i \,\forall\, i \neq i_0\}} \left[\mathrm{PCS}_2\left(\mu_0 \mid I_{i_0}\left(\gamma_{i_0}(x) - c.e_1\right) \cup \{i_0\}\right) \right.$$

$$\left. - \mathrm{PCS}_2\left(\mu_0 \mid I_{i_0}(\gamma_{i_0}(x) - c.e_1) \cup \{1, i_0\}\right) \right] f_\mu\left(x_{i_0} - c, x_2, \dots x_k\right) dx_2 \dots dx_k$$

$$= -D^-_{i_0} \text{ (say)}. \qquad \dots (2.13)$$

Let $\quad x_{i_0} = z_{i_0} + c$ and $z_i = x_i \;\forall\; i \neq i_0$.

Then from (2.13), for all $i_0 \neq 1, k$.

$$D^-_{i_0} = \int_{C \in \cap \{(z_2, \dots z_k) : z_{i_0} + c \geqslant z_i \forall i \neq 1\}} [\mathrm{PCS}_2(\mu_0 \mid I_{i_0}(\gamma_i(z) + c.e_1) \cup \{1\})$$

$$- \mathrm{PCS}_2(\mu_0 \mid I_{i_0}(\gamma_{i_0}(z) + c.e_1) \cup \{1, i_0\})] .\, f_\mu(z_{i_0}, z_2 \dots z_{i_0 - 1},$$

$$z_{i_0} + c, z_{i_0+1}, \dots z_k)\, dz_2\, dz_3 \dots dz_k \qquad \dots (2.14)$$

[(2.14) follows by noting that in the relevant region for all $i_0 \neq 1, k$,

$$\mathrm{PCS}_2(\mu_0 \mid I_{i_0}(\gamma_{i_0}(x) - c.e_1) \cup \{i_0\})$$

$$= \mathrm{PCS}_2(\mu_0 \mid I_{i_0}(\gamma_{i_0}(z) + c.e_1) \cup \{1\})$$

and $\quad \mathrm{PCS}_2(\mu_0 \mid I_{i_0}(\gamma_{i_0}(x) - c.e_1) \cup \{1, i_0\})$

$$= \mathrm{PCS}_2(\mu_0 \mid I_{i_0}(\gamma_{i_0}(z) + c.e_1) \cup \{1, i_0\})]$$

Now note that $B_0(\delta) = B_\delta(\delta)$ as

$$\mathrm{PCS}_2(\mu_0 \mid x_1, x_2 \dots x_k) = \mathrm{PCS}_2(\mu_0 \mid x_1 + \delta, x_2, x_3, \dots x_k) \;\forall\; x \in A_\varepsilon - W(\delta) \quad \dots (2.15)$$

This is because the subset $J(x_1, x_2, \dots, x_k)$ differs from $J(x_1 + \delta, x_2, \dots, x_k)$ for $x \in A_\varepsilon$, only if x_1 lies close to $x_{\max}$ or $x_{\max} - c$. These causes have been taken into account in $W(\delta)$. Also note that, by the structure of A_ε, (2.15) holds for $x_{\max} - \delta \leqslant x_1 < x_{\max}$.

$$\text{Now } \frac{\partial \text{PCS}_2^*(\boldsymbol{\mu} : A_\varepsilon)}{\partial \mu_1} = u^1(0)$$

$$= \sum_{i_0=2}^{k} D^+_{i_0} - \sum_{i_0=2}^{k} D^-_{i_0}$$

$$= D_k^+ - D_k^- + \sum_{i_0=2}^{k-1} (D^+_{i_0} - D^-_{i_0})$$

$\leqslant 0$, by the following observations :

(i) $D_k^- \geqslant 0$ [follows from (2.13)]

(ii) $D_k^+ \leqslant 0$ [In (2.12), for $i_0 = k$, $\text{PCS}_2(\boldsymbol{\mu}_0 \,|\, I_{i_0}(\gamma_{i_0}(\boldsymbol{x}) + c.e_1) \bigcup \{1\}) = 0$ as the set $(I_{i_0}(\gamma_{i_0}(\boldsymbol{x}) + c.e_1) \bigcup \{1\})$ does not contain k]

(iii) $D^+_{i_0} \leqslant D^-_{i_0}$ for $i \neq 1, k$, follows from (2.12) and (2.14), by noting that

$$\frac{f_{\boldsymbol{\mu}}(z_{i_0}, z_2, \ldots, z_{i_0-1}, z_{i_0 0}+c, z_{i_0+1}, \ldots z_k)}{f_\mu(z_{i_0}+c, z_2, z_3, \ldots, z_k)}$$

$$= \frac{f_{\mu_1}(z_{i_0}) \cdot f_{\mu_{i_0}}(z_{i_0}+c)}{f_{\mu_1}(z_{i_0}+c) \cdot f_{\mu_{i_0}}(z_{i_0})} \geqslant 1$$

as f is MLR (or, totally positive of order 2), $\mu_1 \leqslant \mu_{i0}$ and $c > 0$.

This proves Lemma 2.1.

3. Proof of conjecture I

As the proof of conjecture I is exactly similar to that of conjecture II, only the important steps are given here. Here also we consider $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_{k-1} < \mu_k$ without loss of generallity.

Observe that

$$\text{PCS}_1(\boldsymbol{\mu} \,|\, \boldsymbol{x}) = \text{PCS}_2\Big(\Big(\boldsymbol{\mu} + \frac{n_1}{n_2}\,\boldsymbol{x}\Big) \,|\, \boldsymbol{x}\Big),$$

where $\text{PCS}_2\Big(\Big(\boldsymbol{\mu} + \frac{n_1}{n_2}\,\boldsymbol{x}\Big) \,|\, \boldsymbol{x}\Big)$ means the probability of selecting the k-th population by choosing the population corresponding to the maximum observation (maximum among the populations given by $J(\boldsymbol{x})$). Here for given $\boldsymbol{x}$, the observations follow

$$N_k\left(\boldsymbol{\mu} + \frac{n_1}{n_2}\,\boldsymbol{x}, \frac{\sigma^2}{n_2} I_k\right).$$

Hence $$\text{PCS}_1(\boldsymbol{\mu}) = \int_{x \in R^k} \text{PCS}_2\left(\left(\boldsymbol{\mu}+\frac{n_1}{n_2}\,x\right) \mid x\right) f_{\boldsymbol{\mu}}(x)dx \quad \ldots \quad (3.1)$$

As in the earlier proof, define

$$\text{PCS}_1^*(\boldsymbol{\mu}) = \int_{x \in R^k} \text{PCS}_2\left(\left(\boldsymbol{\mu}_0+\frac{n_1}{n_2}\,x\right) \mid x\right) f_{\boldsymbol{\mu}}(x)dx \quad \ldots \quad (3.2)$$

Now by analogous argument, to prove conjecture I, it is sufficient to prove the following lemma.

Lemma 3.1 : $$\frac{\partial\, \text{PCS}_1^*(\boldsymbol{\mu} : A_\epsilon)}{\partial \mu_1} \leqslant 0 \;\forall\; \textit{fixed}\; \epsilon > 0,$$

where

$$\text{PCS}_1^*(\boldsymbol{\mu} : A_\varepsilon) = \int_{x \in A_\epsilon} \text{PCS}_2\left(\left(\boldsymbol{\mu}_0+\frac{n_1}{n_2}\,x\right) \mid x\right) f_{\boldsymbol{\mu}}(x)dx \quad \ldots \quad (3.3)$$

Proof of Lemma 3.1 : As the proof of Lemma 3.1 is similar to that of Lemma 2.1, only the main steps have been shown here.

$$\text{PCS}_1^*\left((\boldsymbol{\mu}+\delta e_1) : A_\varepsilon\right)$$

$$= \int_{A_\epsilon} \text{PCS}_2\left(\left(\boldsymbol{\mu}_0+\frac{n_1}{n_2}(x_1+\delta, x_2, \ldots x_k)\right) \mid x_1+\delta, x_2, \ldots x_k\right) f_{\boldsymbol{\mu}}(x)dx$$

$$\leqslant \sum_{i_0=2}^{k} \int_{W_{i_0}^+(\delta)} \text{PCS}_2\left(\boldsymbol{\mu}_0+\frac{n_1}{n_2}(\gamma_{i_0}(x)+c.e_1) \mid I_{i_0}(\gamma_{i_0}(x)+c.e_1) \cup \{1\}\right) f_{\boldsymbol{\mu}}(x)dx$$

$$+ \sum_{i_0=2}^{k} \int_{W_{i_0}^-(\delta)} \text{PCS}_2\left(\boldsymbol{\mu}_0+\frac{n_1}{n_2}(\gamma_{i_0}(x)-c.e_1) \mid I_{i_0}(\gamma_{i_0}(x)-c.e_1) \cup \{1, i_0\}\right) f_{\boldsymbol{\mu}}(x)dx$$

$$+ \int_{A_\epsilon^- W(\delta)} \text{PCS}_2\left(\boldsymbol{\mu}_0+\frac{n_1}{n_2}(x_1, x_2, \ldots x_k) \mid x_1+\delta, x_2, \ldots x_k\right) f_{\boldsymbol{\mu}}(x)dx \quad \ldots \quad (3.4)$$

[(3.4) follows in exactly the same way as (2.9) and the inequality is due to the fact that, for $\eta > 0$, $\text{PCS}_2(\boldsymbol{\mu}+x+\eta.e_1 \mid J(x)) \leqslant \text{PCS}_2(\boldsymbol{\mu}+x \mid J(x))$ as in (1.1)]

$$= \sum_{i_0=2}^{k} \tilde{A}_{i_0,\delta}^+(\delta) + \sum_{i_0=2}^{k} \tilde{A}_{i_0,\delta}^-(\delta) + \tilde{B}_\delta(\delta) \quad \text{(say)}$$

$$= \tilde{u}(\delta) \quad \text{(say).}$$

Now

$$\text{PCS}_1^*(\boldsymbol{\mu} : A_\epsilon) \geqslant \sum_{i_0=2}^{k} \int_{W_{i_0}^+(\delta)}$$

$$\text{PCS}_2(\boldsymbol{\mu}_0 + \frac{n_1}{n_2}(\gamma_{i_0}(\boldsymbol{x}) + c.e_1) \mid I_{i_0}(\gamma_{i_0}(x) + c.e_1) \cap \{1, i_0\}) f_{\boldsymbol{\mu}}(\boldsymbol{x}) d\boldsymbol{x}$$

$$+ \sum_{i_0=2}^{k} \int_{W_{i_0}^-(\delta)} \text{PCS}_2(\mu_0 + \frac{n_1}{n_2}(\gamma_{i_0}(\boldsymbol{x}) - c.e_1) \mid I_{i_0}(\gamma_{i_0}(\boldsymbol{x}) - c.e_1) \cup \{i_0\}) f_\mu(\boldsymbol{x}) d\boldsymbol{x}$$

$$+ \int_{A_\epsilon - W(\delta)} \text{PCS}_2(\boldsymbol{\mu}_0 + \frac{n_1}{n_2}\boldsymbol{x} \mid \boldsymbol{x}) f_\mu(\boldsymbol{x})\, d\boldsymbol{x} \qquad \ldots(3.5)$$

[(3.5) is defined like (2.10) in exactly the same way as (3.4)]

$$= \sum_{i_0=2}^{k} \tilde{A}_{i_0,0}^{+}(\delta) + \sum_{i_0=2}^{k} \tilde{A}_{i_0}^{-}(\delta) + \tilde{B}_0(\delta) \text{ (say)}$$

$$= \tilde{u}(0) \text{ (say)}.$$

As in Lemma 2.1 $\tilde{B}_\delta(\delta) = \tilde{B}_0(\delta)$ as $\text{PCS}_2(\boldsymbol{\mu}_0 + \frac{n_1}{n_2}(x_1, \ldots, x_k) \mid (x_1 + \delta, x_2 \ldots, x_k))$

$$= \text{PCS}_2(\mu_0 + \frac{n_1}{n_2}(x_1, \ldots, x_k), \mid (x_1, \ldots, x_k)), \ \forall\, x \in A_\epsilon - W(\delta).$$

Now

$$\frac{\partial\, \text{PCS}_1^*(\mu : A_\epsilon)}{\partial \mu_1}$$

$$= \lim_{\delta \to 0} \delta^{-1}[\text{PCS}_1^*(\mu + \delta\cdot e_1 : A_\epsilon) - \text{PCS}_1^*(\mu : A_\epsilon)] \leqslant \lim_{\delta \to 0} \delta^{-1}[\tilde{u}(\delta) - \tilde{u}(0)]$$

$$= \sum_{i_0=2}^{k} \lim_{\delta \to 0} \delta^{-1}[\tilde{A}_{i_0,\delta}^{+}(\delta) - \tilde{A}_{i_0,0}^{+}(\delta)] + \sum_{i_0=2}^{k} \lim_{\delta \to 0} \delta^{-1}[\tilde{A}_{i_0,\delta}^{-}(\delta) - \tilde{A}_{i_0,0}^{-}(\delta)]$$

$$= \sum_{i_0=2}^{k} \tilde{D}_{i_0}^{+} - \sum_{i_0=2}^{k} \tilde{D}_{i_0}^{-} \qquad \text{(say)}$$

$\leqslant 0$, [since $\tilde{D}_k^- \geqslant 0$, $\tilde{D}_k^+ \leqslant 0$ as in Section 2.

Also $\tilde{D}_{i_0}^{+} \leqslant \tilde{D}_{i_0}^{-}$, $\forall\, i_0 \neq 1, k$ by deriving equations analogous to (2.12), (2.13) and (2.14).]

Thus the proof of Lemma 3.1 follows.

Acknowledgement. The authors are grateful to Professor Somesh Das Gupta for helpful suggestions.

REFERENCES

ALAM, K. (1970) : A two-sample procedure for selecting the population with the largest mean from k normal populations. *Ann. Inst. Statist. Math.*, **22**, 127-136.

BECHHOFER, R. E. (1954) : A single-sample multiple decision procedure for ranking means of normal populations with known variances. *Ann. Math. Statist.*, **25**, 16-39.

BHANDARI, S. K. and BOSE, A. (1987) : On selecting the most likely event. *Journal of Statistical Planning and Inference.* (to appear.).

DASGUPTA, S. (1980) : Brunn-Minkowski inequality and its aftermath. *J. of Mult. Analysis*, **10**, 296-318.

GUPTA, S. S. (1956) : *On a Decision Rule for a Problem in Ranking Means.* Mimeo. Series No. 150, Inst. of Statist., Univ. of North Carolina, Chapel Hill, North Carolina.

GUPTA, S. S. and MIESCKE, K. J. (1984) : Sequential selection procedures—A decision-theoretic approach. *Ann. of Stat.*, **12**, 336-350.

KARLIN, S. (1968) : *Total Positivity*, Stanford University Press, Stanford, California.

MIESCKE, K. J. and SEHR, J. (1980) : On a conjecture concerning the least favourable confiugration of certain two-stage selection procedures. *Commun. Statist.*, A **9**, 1609-1617.

TAMHANE, A. C. and BECHHOFER, R. E. (1977) : A two-stage minimax procedure with screening for selecting the largest normal mean. *Commun. Statist.*, A**6**, 1003-1033.

TAMHANE, A. C. and BECHHOFER, R. E. (1979): A two-stage minimax procedure with screening for selecting the largest normal mean. (II) : An improved lower bound on the PCS and associated tables. *Commun. Statist.* A**8**, 337-358.

Paper received : January, 1987.

Revised : May, 1988.