Representation of Fuzzy Operators Using Ordinary Sets

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***Abstract*—A different interpretation of union, intersection, and inclusion in fuzzy sets in the light of measure theory is given. The existing definitions of these operators are based only on the value of membership functions characterizing fuzzy sets. The proposed definitions take into account the nature (behavior) of the membership functions together with their values. The uniqueness of the proposed definition is established. These definitions are generalized for any arbitrary continuous function defined on any bounded closed interval. The existing definitions can also be derived from the framework proposed.**

Introduction

The first paper on fuzzy sets was published in 1965 [1]. Later, a few thousand papers appeared in various journals on different aspects of fuzzy set theory [2]. The set operations on fuzzy sets, such as union, intersection, inclusion, and complementation, were initially defined by Zadeh [1]. The mathematical foundation of these ideas was given by Bellman *et al.* [3] and Fu *et al.* [4]. Other definitions, such as bold union and bold intersection [5], were also given in this regard. Now it is more or less accepted by all that the definitions given by Zadeh [1] of union, intersection, complementation, and inclusion are the standard ones. The applications of these concepts to various practical fields, such as pattern recognition, image processing, artificial intelligence, management applications, etc., are available in the literature [2]–[10].

In this paper new definitions for those operations on fuzzy sets are proposed. The properties of the new definitions are discussed. It is also emphasized that these definitions arose only because of the different interpretation of the intuitive ideas. The proposed definitions are based on the concepts in ordinary set theory, taking for granted the definition of complementation by Zadeh [1].

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I. Preliminary Examples

In this section a few examples are discussed to put forward the intuitive ideas behind the proposed operators.

Example 1: Let Q represent the set of heights in centimeters and f and g represent, respectively, the membership functions for "tall" and "very tall." A person whose membership value for "very tall" is one has to have membership one for tall also. Observe also that the membership function value for "very tall" is $a \Rightarrow$ the membership function value for tall is at least equal to a. The nature of the two membership functions considered is similar, i.e., when one increases, the other also increases and vice versa.

Now let us look at the usual set inclusion. We say that $A \subseteq B$ if $x \in A \Rightarrow x \in B$. Therefore, by generalizing the terminology of ordinary sets to fuzzy sets it can be written in the foregoing case that $g \subseteq f$. From this definition it follows that if union and intersection are to be defined in this context, then $f \cup g$ is to equal to f and $f \cap g$ is equal to g (i.e., not based on the definitions given by Zadeh).

Extending these ideas to any two fuzzy sets A and B, it may be stated that $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ for all x,

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$
$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

where μ represents the membership function. Zadeh [1] defined the main operations as in the foregoing. Observe that the definitions depend only on the membership function values and on no other characteristics of the membership functions.

Example 2: Let f, g, and h be fuzzy membership functions defined on $Q = [0,1]$ (Fig. 1). Let $f(x) = x$, $g(x) = 1 - x$, and $h(x) = x^2$.

Let $x_0 = (\sqrt{5} - 1)/2$, $x_1 = 0.1$, $x_2 = 0.2$, and $x_3 = 0.3$. Let $Q_1 = \{x_1, x_2, x_3\}$. Let f_1, g_1, and h_1 be fuzzy membership functions f, g, and h restricted to Q_1. Then $h_1 \leq f_1 \leq g_1$ for all $x \in Q_1$. According to the definition given in Example 1, f_1 is a fuzzy subset of g_1 and h_1 is also a fuzzy subset of g_1. However, observe that the nature of g_1 is just opposite to that of f_1 and h_1. In fact, g_1 is f_1-complement. Intuitively, f_1 cannot be a subset of g_1 since f_1 is a truncation of f to Q_1 and so also is g_1, and f and g are opposite in nature.

a) If two fuzzy sets are opposite in nature as just stated, can one of them be a subset of another?

Another aspect of the same problem is stated next. Let $Q_2 = \{x_0\}$. Let f_2, g_2, and h_2 be fuzzy membership functions f, g, and h restricted to Q_2. Then observe that

$$f_2 \cap g_2 = f_2 \cap h_2 = 1 - x_0$$
$$f_2 \cup g_2 = f_2 \cup h_2 = x_0.$$

Though f_2 and g_2 are opposite in nature, f_2 and h_2 are similar in nature, the values of intersection are same, and so also the

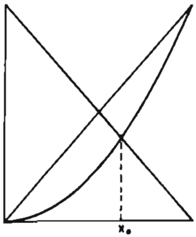

Fig. 1. Functions $f(x)$, $g(x)$, and $h(x)$.

values of union. f and h are said to be similar in nature because $\{x: f$ increases at $x\} = \{x: h$ increases at $x\}$. Observe that $g_2(x_0) = h_2(x_0)$.

b) Should it be the case that just because the membership function values for two functions g_2 and h_2 in the foregoing are same, though their natures are different, that the values of the intersection with another function are to be same?

As it transpired from the previous discussion, the nature or behavior of the membership function, if it can be quantified, would change the definitions of union and intersection very drastically. In some discrete cases, because of the lack of information, it may be impossible to get the values of union and intersection based on the behavior of membership functions. In Section II the properties of union, intersection, and inclusion are discussed on the basis of the behavior of the membership functions.

Note here that the definition of complementation given by Zadeh [1] is acceptable because the aforementioned ideas have already been incorporated there. This is explained in Section III.

Example 3: A government has taken ten measures to diffuse tension in a particular state. The ordinary people are asked to express their opinions on these measures. Let the measures be named $M_1, M_2, \cdots, M_{10}$.

An individual A supported M_1, M_2, and M_3 and thought the rest ineffective. Therefore, let $\mu_A(x) = 0.3$; μ represents the membership function and X represents the support for government. Individual B supported M_1, M_2, M_3, and M_4, i.e., $\mu_B(x) = 0.4$. Individual C supported M_5, M_6, and M_7, i.e., $\mu_C(x) = 0.3$. D supported every measure by 0.3, i.e., $\mu_D(x) = 0.3$.

[In practice, the previous representation is not followed. The usual representation is $\mu_x(A)$, $\mu_x(B), \cdots$, etc. An example is stated next to clarify the position.

Example 4: Let $\mathscr{A}$ be the set of all human beings; let $\mathscr{B}$ be the set of all likings of human beings, i.e., $\mathscr{B} = \{$music, dancing, traveling, $\cdots\}$. The membership functions can be written in two ways.

1) Information on every individual on his liking of music is recorded here:
   $\mu_{\text{music}}(a)$, $a \in \mathscr{A}$
   music $\in \mathscr{B}$.
2) For a specific human being a, his likings are recorded here:
   $\mu_a(b)$, $b \in \mathscr{B}$
   $a \in \mathscr{A}$.]

By using the definition of intersection in Example 1, it can be seen that

$$\mu_{A\cap B}(X) = 0.3 = \mu_{A\cap C}(X) = \mu_{A\cap D}(X) = \mu_{B\cap C}(X) = \mu_{B\cap D}(X) = \mu_{C\cap B}(X).$$

However, suppose the intersection is interpreted as common points of support for the government. Then

$$\mu_{A\cap B}(X) = 0.3$$

because both A and B supported M_1, M_2, M_3;

$$\mu_{A\cap C}(X) = 0$$

no common measures between A and C.

Similarly,

$$\mu_{A\cap D}(X) = 0.09,\ \mu_{B\cap C}(X) = 0,\ \mu_{B\cap D}(X) = 0.12,\ \mu_{C\cap D}(X) = 0.01.$$

The values for union can be calculated similarly as

$$\mu_{A\cup B}(X) = 0.4 \quad \mu_{A\cup C}(X) = 0.6 \quad \mu_{A\cup D}(X) = 0.51$$
$$\mu_{B\cup C}(X) = 0.7 \quad \mu_{B\cup D}(X) = 0.58 \quad \mu_{C\cup D}(X) = 0.51.$$

If the same reasoning is also followed for inclusion, then

$$\mu_A \subseteq \mu_B, \mu_A \subseteq \mu_{C^c} \quad \mu_B \subseteq \mu_{C^c} \quad \mu_C \subseteq \mu_{B^c}.$$

However, if the information is not provided about the points of support, then it is impossible to make the foregoing statements.

In both Examples 1 and 2 the behavior of one function with respect to another is taken into consideration. In Example 3 the intuitive idea behind complementation is stated. The nature or behavior of the functions, as stated in Examples 1 and 2 is essentially the same as the points of support stated in Example 3. A concise explanation of these intuitive ideas will be given in Section III.

From now on, Zadeh's union and intersection for fuzzy sets will be denoted by $(\cup, \cap)$, bold union and bold intersection will be denoted by $(\boldsymbol{\cup}, \boldsymbol{\cap})$, and the proposed union and intersection will be denoted by $(\textcircled{\cup}, \textcircled{\cap})$. The union and intersection for ordinary sets will also be represented by $\cup$ and $\cap$. It will be clear from the context whether set theoretic operations are used or Zadeh's operations are used.

II. Properties of $\textcircled{\cap}$ and $\textcircled{\cup}$

In this section properties that union and intersection must possess are discussed.

P_1:

$$\mu_{A \textcircled{\cap} A^c}(x) = 0, \quad \text{for all } x.$$

Bold intersection also satisfies this property. However, it depends only on the membership function values. Zadeh's intersection does not satisfy this property.

Example 5: Suppose $\mu_A(x) = 0.3$. Let

$$\mathscr{C} = \{C: \mu_C(x) = 0.7\}$$

$$\mu_{A\cap C}(x) = 0.3, \quad \text{for all } C \in \mathscr{C}.$$

This is one way of looking at the problem. From another view point, however, having different $\mu_{A \textcircled{\cap} C}(x)$ for different C's belonging to $\mathscr{C}$ is equally logical. This is a different interpretation and another way of visualizing the problem.

If for different C's different intersection values are to be obtained, then $\mu_{A \textcircled{\cap} C}(x) = 0$ when $C = A^c$. Similarly, the next property can be derived.

P_2:

$$\mu_{A \textcircled{\cup} A^c}(x) = 1.$$

Again, Zadeh's definition does not also satisfy this, but bold union does.

P_3 commutativity:

$$\mu_{A \circledcup B}(x) = \mu_{B \circledcup A}(x)$$

$$\mu_{A \circledcap B}(x) = \mu_{B \circledcap A}(x)$$

$\cup, \cap$, and $\dot\cup, \dot\cap$ satisfy P_3.

P_4 associativity:

$$\mu_{A \circledcup (B \circledcup C)}(x) = \mu_{(A \circledcup B) \circledcup C}(x)$$

$$\mu_{A \circledcap (B \circledcap C)}(x) = \mu_{(A \circledcap B) \circledcap C}(x)$$

$\cup, \cap$ and $\dot\cup, \dot\cap$ satisfy P_4.

P_5 idempotency:

$$\mu_{A \circledcup A}(x) = \mu_A(x)$$

$$\mu_{A \circledcap A}(x) = \mu_A(x)$$

$\cup, \cap$ satisfy P_5, but $\dot\cup$ and $\dot\cap$ do not.

P_6 distributive laws:

$$\mu_{A \circledcup (B \circledcap C)}(x) = \mu_{(A \circledcup B) \circledcap (A \circledcup C)}(x)$$

$$\mu_{A \circledcap (B \circledcup C)}(x) = \mu_{(A \circledcap B) \circledcup (A \circledcap C)}(x)$$

$\cup, \cap$ satisfy P_6, but $\dot\cup$ and $\dot\cap$ do not.

P_7 identity:

$$\mu_{A \circledcup \phi}(x) = \mu_A(x)$$

$$\mu_{A \circledcap X}(x) = \mu_A(x)$$

Here X is the universal set. $\cup, \cap$ and $\dot\cup, \dot\cap$ satisfy P_7.

P_8: This property requires the following:

a) absorption laws;
b) De Morgans' laws;
c) involution laws.

The three properties stated in P_8 are satisfied by $\cup, \cap$ as well as by $\dot\cup, \dot\cap$.

P_9:

$$0 \leq \mu_{A \circledcap B} \leq \min(\mu_A, \mu_B)$$

$$1 \geq \mu_{A \circledcup B} \geq \max(\mu_A, \mu_B).$$

$\mu_{A \circledcap B}$ cannot be greater than any one of μ_A or μ_B because the common properties of A and B cannot be greater than $\min(\mu_A, \mu_B)$ (similarly for $\mu_{A \circledcup B}$ and $\max(\mu_A, \mu_B)$). $\cap, \cup$ satisfy P_9, as do $\dot\cup$ and $\dot\cap$.

P_{10}: $\mu_{A \circledcap B}$, $\mu_{A \circledcup B}$ must be dependent not only on μ_A and μ_B, but also on their relative natures. This point was discussed in Section I. $\cup, \cap$ and $\dot\cup, \dot\cap$ do not satisfy P_{10}.

P_{11}: A fuzzy set A is said to be a subset of B if

$$\mu_{A \circledcap B} = \mu_A \quad \text{and} \quad \mu_{A \circledcup B} = \mu_B$$

This is another way of defining inclusion. In ordinary set theory $A \subseteq B$ if $x \in A \Rightarrow x \in B$ which gives $A \cap B = A$ and $A \cup B = B$. We have taken the right side of the expression as the definition of "subset." This point was also discussed in Example 1. $\cup, \cap$ satisfy P_{11}, but $\dot\cup, \dot\cap$ do not.

In the next section a definition will be proposed which would satisfy the foregoing properties.

III. Definitions of New Operators

In Example 3 the reason the membership function has the given values is clear because of the support for the governmental measures. However, in Example 2 or Example 1 it is not clear what to be quantified. An example is stated next for this purpose.

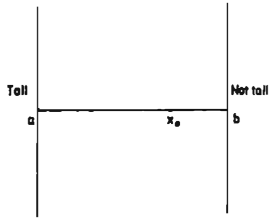

Fig. 2. Explanation for complementation.

Example 6: Let Q represent the set of heights in centimeters and let f and g, respectively, represent the membership functions for "tall" and "not tall." Therefore,

$$g(x) = 1 - f(x), \qquad \text{for all } x.$$

For an ordinary set A, A^c is the set which contains all elements except those which are in A, i.e., all the elements which do not possess the properties of elements in A. Therefore, if the definition has to be extended to fuzzy sets, when "not possessing the properties of elements in A" is to be quantified. According to Zadeh, if the membership function value for "tall" for x_0 is 0.3, then for "not tall" it is 0.7, i.e., x_0 possesses 0.3 of tall properties and 0.7 of not tall properties. One way of representing it is given in Fig. 2.

In Fig. 2 the two vertical lines represent "tall" and "not tall." Every point on the horizontal line denotes the position of an individual x. If x is at a, he possesses all properties of "tall" and so zero properties of "not tall." Therefore, let the distance between a and b be one unit (i.e., $b = a + 1$), and x_0 is at 0.7 distance from the "tall line" (i.e., x_0 is at $a + 0.7$), and so at 0.3 distance from the "not tall" line. The membership function value 0.3 can be characterized by a set of length 0.3, (i.e., $[a + 0.7, a + 1]$) or by a set of length 0.7 (i.e., $[a, a + 0.7]$). This is the idea being used while defining the operators.

Every membership function value can, therefore, be represented by a set. One may, however, get different sets for the same value. A way of choosing the right sets is mentioned in this section, but before that a few technicalities are to be taken care of.

Definition 1 (D1): Let a) the domain Q be a closed interval in $\mathbf{R}$. Let the membership functions f_1 and f_2 be such that

b) f_i: $Q \to [0,1]$ is continuous for all $i = 1, 2$;
c) $f_i(Q) = [0,1]$ for all $i = 1, 2$;
d) $f_i(x) = 0$ or 1 or undefined for all $x \in Q^c$, for all $i = 1, 2$.

It is clear from these assumptions that such a Q need not be unique [11]. Uniqueness can be achieved if

$$\mathscr{S}_{f_1, f_2} = \{ S\colon S \text{ is a domain satisfying a), b), c), and d)} \}$$

and

$$Q = \bigcap_{S \in \mathscr{S}_{f_1, f_2}} S.$$

This is the definition of Q.

Definition 2 (D2): Let f, g, and $Q = [a, b]$ be as defined in D1. Further,

i) $f\{a, b\} \subseteq \{0,1\} \qquad g\{a, b\} \subseteq \{0,1\}$.

ii) Let $a < x_0 < b$ be such that f increases at x_0. Then x_1, x_2 exists such that $a \leq x_1 < x_0 < x_2 \leq b$ such that $f(x_1) = 0$, $f(x_2) = 1$, and f is nondecreasing at all $x \in (x_1, x_2)$. Similarly, the same holds for g also.

iii) Let $a < x_0 < b$ be such that f decreases at x_0. Then x_1, x_2 exists such that $a \leqslant x_1 < x_0 < x_2 \leqslant b$, $f(x_1) = 1$, $f(x_2) = 0$, and f is nonincreasing at all $x \in (x_1, x_2)$. The same is the case with g also.

Then define the following:

$$A_x = \begin{cases} [0, f(x)], & \text{if } f \text{ is nondecreasing at } x; \\ [1 - f(x), 1], & \text{if } f \text{ is nonincreasing at } x; \\ \text{any finite set}, & \text{if } f(x) = 0; \\ [0,1], & \text{if } f(x) = 1; \end{cases}$$

$$B_x = \begin{cases} [0, g(x)], & \text{if } g \text{ is nondecreasing at } x; \\ [1 - g(x), 1], & \text{if } g \text{ is nonincreasing at } x; \\ [0,1], & \text{if } g(x) = 1; \\ \text{any finite set}, & \text{if } g(x) = 0. \end{cases}$$

Therefore, $f(x) = \lambda(A_x)$ and $g(x) = \lambda(B_x)$ where λ is the Lebesgue measure on $\mathbf{R}$.

Definition 3 (D3): Define

$$(f \circledcap g)(x) = \lambda(A_x \cap B_x)$$

$$(f \circledcup g)(x) = \lambda(A_x \cup B_x).$$

$f \subseteq g$ if 1) $A_x \subseteq B_x$ for all x, except where $g(x) = 0$ and $f(x) = 0$; 2) A_x is a finite set if $g(x) = 0$.

It is obvious from D3 that

$$(f \circledcap \bar{f})(x) = 0 \qquad \text{for all } x$$

$$(f \circledcup \bar{f})(x) = 1 \qquad \text{for all } x.$$

where $\bar{f}$ is the complement of f.

Note: 1) In the definitions of union, intersection, and inclusion the membership functions mentioned are all of the type mentioned in D2. Let $\mathscr{A} = \{f: f$ has the properties i), ii), and iii) stated in D2$\}$.

Definition: A membership function f is said to be a type I membership function if $f \in \mathscr{A}$. Every membership function need not be of type I. A theorem is proved in Section V to make D3 applicable to all $f \notin \mathscr{A}$.

2) Observe that a) if $f \in \mathscr{A}$, then every $f(x)$ can be represented by a set from D2; b) f may also be expressed as union or intersection of two f_i, $f_i \in \mathscr{A}$ for all $i = 1, 2$. This may result in different sets for the same $f(x)$'s. A few theorems are proved in Section IV to show that definition D3 is unambiguous.

3) A comparison of D3 with the earlier definitions can be found in Section VI.

IV. Proof of Uniqueness

Theorem 1: Let f be a type I membership function of the form shown in Fig. 3, i.e., $x_1 < x_2 \Rightarrow f(x_1) \leqslant f(x_2)$, $f(b) = 1$ and $f(a) = 0$, and $0 < f(x) < 1$ for all $x \in (a, b)$. Let g and h be two membership functions, g, $h \in \mathscr{A}$ and $(g \circledcap h)(x) = f(x)$. Let B_x, C_x be sets such that $\lambda(B_x) = g(x)$ for all $x \in [a, b]$ and $\lambda(C_x) = h(x)$ for all $x \in [a, b]$, where B_x and C_x are obtained according to D2. Let A_x, $x \in [a, b]$ be sets such that $\lambda(A_x) = f(x)$ for all x and A_x are obtained according as D2. Then $B_x \cap C_x = A_x$ for all $x [a, b]$.

Proof: Observe that a point x_0 will always exist at which $g(x_0) = 1$. x_0 can be 1) either a, 2) or b, or 3) $a < x_0 < b$.

Case 1: $x_0 = a$, i.e., $g(a) = 1$.

Claim 1: No point $x \in (a, b]$ exists such that $g(x) = 0$.

Proof of claim 1: If one such x_1 exists, then $(g \circledcap h)(x_1) = 0$, i.e., $f(x_1) = 0$. However, $f(x) > 0$ for all $x > a$.

Claim 2: $g(x) = 1$ for all $x \in [a, b]$.

Proof of claim 2: If an x_1 exists such that $g(x_1) < 1$, then an $x_2 > x_1$ exists such that $g(x_2) = 0$ (from D2). Therefore, it is proved that $g(x) = 1$ for all $x \in [a, b]$ if $g(a) = 1 \Rightarrow B_x = [0, 1]$ for all $x \in [a, b]$.

Claim 3: $h(x) = f(x)$ for all $x \in [a, b]$.

Proof of claim 3: Let $x \in [a, b]$ and $h(x) = \lambda(C_x)$. Now $f(x) = \lambda(C_x \cap B_x) = \lambda(C_x \cap [0, 1]) = \lambda(C_x) = h(x)$. Therefore, in case 1) it is proved that $C_x \cap B_x = A_x$.

Case 2: $x_0 = b$, i.e., $g(b) = 1$. Now since $f(b) = 1$, $h(b)$ has to be equal to 1. Let us assume that $g(x) = 1$ for all $x \in [a, b]$ is not possible since it has already been tackled in case 1). Therefore, $\exists x_1 < b$ such that $g(x_1) = 1$.

Observe that if $x_1 > a$, then $g(x_1) = 0 \Rightarrow f(x_1) = 0$, which is not true. Therefore, $g(a) = 0$ and $g(x) > 0$ for all $x > a$.

Observe also that if $g(x_2) < f(x_2)$ for an x_2, $a < x_2 < b$, then

$$f(x_2) = \lambda(B_{x_2} \cap C_{x_2}) \leqslant \lambda(B_{x_2}) < f(x_2)$$

which is a contradiction, i.e., $g(x) \geqslant f(x)$ for all $x \in [a, b]$. Let $x_4 \in (a, b]$ be such that $g(x_4) = 1$ and $g(x) < 1$ for all $x < x_4$. Observe that if an x_5 exists such that $b > x_5 > x_4$ and $g(x_5) = 0$, then $f(x_5)$ has to be equal to zero which cannot happen since $f(x) > 0$ for all $x > a$, i.e., $f(x) = 1$ for all $b \geqslant x \geqslant x_4$.

Therefore, g is a continuous nondecreasing function from $[a, b]$ to $[0, 1]$, and $B_x = [0, g(x)]$ for all $x \in [a, b]$ and $g(x) \geqslant f(x)$. By following the foregoing argument, it can also be proved that $C_x = [0, h(x)]$ for all $x \in [a, b]$ and $h(x) \geqslant f(x)$. In addition, $\lambda(B_x \cap C_x) = \min(g(x), h(x)) = f(x)$; i.e., $B_x \cap C_x = [0, \min(g(x), h(x))] = [0, f(x)]$ for all $x \in [a, b]$.

Case 3: $a < x_0 < b$. Observe that this case boils down to either case 1 or case 2.

Theorem 2: Let f, g, and h be as defined in the previous theorem, except that $(g \circledcup h)(x) = f(x)$. Then $(B_x \cup C_x) = A_x$ for all $x \in [a, b]$.

Proof: $\exists$ a point x_0 such that $g(x_0) = 0$, then either 1) $x_0 = b$, 2) $x_0 = a$, or 3) $a < x_0 < b$.

Case 1: $x_0 = b$. It can be proved that $g(x) = 0$ for all $x \in [a, b]$ $\Rightarrow h(x) = f(x)$ for all $x \in [a, b]$.

Case 2: Let x_4 be such that $g(x_4) = 0$ and $g(x) > 0$ for all $x > x_4$ (see Fig. 4). Then it can be proved that $g(x) = 0$ for all $a \leqslant x \leqslant x_4$.

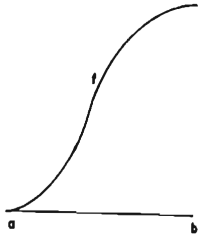

Fig. 3. Type I membership function.

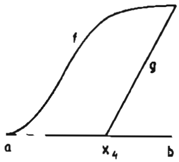

Fig. 4. Type I membership functions.

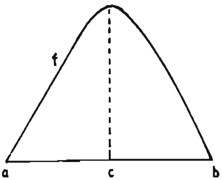

Fig. 5. Type I membership function.

By following the reasoning shown in case 2 of Theorem 1, making modifications for ⓤ, it can be proved that g is nondecreasing in $[a,b]$ and $g(x) \leqslant f(x)$ for all $x \in [a,b]$ and $h(x) \leqslant f(x)$ and h is nondecreasing, i.e.,

$$B_x = [0, g(x)]$$
$$C_x = [0, h(x)]$$
$$A_x = [0, f(x)]$$
$$B_x \cup C_x = [0, \max(g(x), h(x))]$$
$$\lambda(B_x \cup C_x) = \max(g(x), h(x)) = f(x).$$

Therefore, $A_x = [0, f(x)] = B_x \cup C_x$.

Case 3: $a < x_0 < b$; this follows from case 2.

Theorem 3: Let f be as defined follows:

$$f: [a,b] \to [0,1]$$
$$f(a) = 1 \qquad f(b) = 0, \qquad 0 < f(x) < 1 \text{ for all } x \in (a,b)$$
$$x_1 < x_2 \Rightarrow f(x_1) \geqslant f(x_2), \qquad \text{for all } x_1, x_2 \in [a,b].$$

Let g and h be membership functions of type I. Let A_x, B_x, and C_x be sets defined as in D2 for f, g, and h, respectively. Therefore, $\lambda(A_x) = f(x)$, $\lambda(B_x) = g(x)$, and $\lambda(C_x) = h(x)$. Then 1) $A_x = B_x \cup C_x$ for all x if $f = g ⓤ h$, or 2) $A_x = B_x \cap C_x$ if $f = g ⓝ h$.

Proof: The proof will be along the same lines as those of Theorems 1 and 2.

Theorem 4: Let f be a membership function as shown in Fig. 5, i.e., $f(a) = f(b) = 0$, $f(c) = 1$ where $a < c < b$. f is nondecreasing in the interval (a,c), and f is nonincreasing in the interval (c,b). In addition,

$$0 < f(x) < 1, \qquad \text{for all } x \in (a,c) \cup (c,b)$$
$$A_x = \begin{cases} [0, f(x)], & \text{if } x \in [a,c] \\ [1 - f(x), 1)], & \text{if } x \in (c,b). \end{cases}$$

Let $g, h \in \mathscr{A}$. Therefore, $\exists B_x, C_x$ for every $x \in [a,b]$ corresponding to g and h, respectively, such that $\lambda(B_x) = g(x)$ and $\lambda(C_x) = h(x)$. Let $g ⓝ h = f$. Then $B_x \cap C_x = A_x$ for all x.

Proof: Observe that $g(c) = h(c) = 1$. Then the Theorem 4 boils down to getting g and h in the intervals $[a,c]$ and $[c,b]$. This procedure is shown in Theorems 1 and 3.

Theorem 5: Let f, g, and h satisfy the properties stated in the hypothesis of Theorem 4, except that $g ⓤ h = f$. Then $B_x \cup C_x = A_x$ for all x.

Proof: Observe that $g(a) = h(a) = g(b) = h(b) = 0$. If $g(x_0) > 0$ for an x_0, then $\exists x_1 \in (a,b)$ such that $g(x_1) = 1$ (since $g \in \mathscr{A}$). We have $g(x_1) = 1 \Rightarrow f(x_1) = 1$. However, exactly one point exists at which f is equal to one. Therefore, $g(x) > 0$ for an $x \Rightarrow g$ is nondecreasing in the interval (a,c) and nonincreasing in the interval (c,b). This is the case with h as well. Then from Theorems 2 and 3 the result follows.

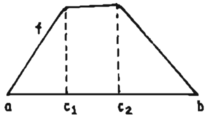

Fig. 6. Type I membership function.

Theorem 6: Let f be a membership function as shown in Fig. 6, i.e.,

$$f(a) = f(b) = 0$$
$$f(x) = 1, \qquad \text{for all } x \in [c_1, c_2] \text{ where } a < c_1 < c_2 < b.$$
$$f(x) > 0, \qquad \text{for all } x \in (a,b).$$

f is nondecreasing in the interval (a, c_1) and nonincreasing in the interval (c_2, b). Let $g, h \in \mathscr{A}$ be such that $g ⓝ h = f$. Let A_x, B_x, and C_x be defined as in D2 for f, g, and h, respectively. Then $B_x \cap C_x = A_x$ for all x.

Proof: Observe that $g(x) = h(x) = 1$, for all $x \in [c_1, c_2]$. Then from Theorems 1 and 3 the proof is obvious.

Theorem 7: Let f, g, h, A_x, B_x, and C_x be as defined in Theorem 6, except that $g ⓤ h = f$. Then $B_x \cup C_x = A_x$, for all x.

Proof: Observe that $g(a) = g(b) = h(a) = h(b) = 0$. Observe also that $g(x) = h(x) = 0$ for all $x \in [a,b]$ is not possible, i.e., there exists an x and a function among g and h (without loss of generality g) such that $g(x) > 0$. Since $f(x) = 1$ for all $x \in [c_1, c_2]$, $\exists x_1 \in [c_1, c_2]$ such that $g(x_1) = 1$. Let x_2 and x_3 be such that $c_1 \leqslant x_2 \leqslant x_3 \leqslant c_2$, $g(x_2) = g(x_3) = 1$, and $g(x) < 1$ for all $x < x_2$ and $x > x_3$. Note that $g(x) \leqslant f(x)$ for all $x \in [a,b]$.

If $h(x) = 0$ for all x, then $g(x) = f(x)$ for all $x \in [a,b]$. Therefore, let $h(x) > 0$ for at least one $x \in (a,b)$. Let x_4 and x_5 be such that $c_1 \leqslant x_4 \leqslant x_5 \leqslant c_2$, $h(x_4) = h(x_5) = 1$, and $h(x) < 1$ for all $x < x_4$ and $x > x_5$. Then

$$A_x = \begin{cases} [0, g(x)], & \text{for all } x \in [a, x_2] \\ [1 - g(x), 1], & \text{for all } x \in [x_3, b] \end{cases}$$

$$B_x = \begin{cases} [0, h(x)], & \text{for all } x \in [a, x_4] \\ [1 - h(x), 1], & \text{for all } x \in [x_5, 1]. \end{cases}$$

Observe that $A_x \cup B_x = [0,1]$ for all $x \in [\min(x_2, x_4), \max(x_3, x_5)]$. This is true for the following reasons.

1) $\lambda(A_x \cup B_x) = 1$.
2) A_x and B_x are closed intervals, so $A_x \cup B_x$ is closed.
3) Let $x_0 \notin A_x \cup B_x$ and $x_0 \in (0,1)$. Let

$$\left.\begin{array}{l} A_x \subseteq [0, x_0) \\ B_x \subseteq (x_0, 1] \end{array}\right\} \quad \text{without loss of generality.}$$

i.e.,

$$A_x = [0, x_0 - \tau_1], \qquad \tau_1 > 0$$
$$B_x = [x_0 + \tau_2, 1], \qquad \tau_2 > 0.$$

Then $\lambda(A_x \cup B_x) = 1 - \tau_1 - \tau_2 < 1 = f(x)$, so $x_0 \in (0,1) \Rightarrow x_0 \in A_x \cup B_x$.

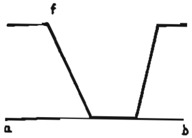

Fig. 7. Type I membership function.

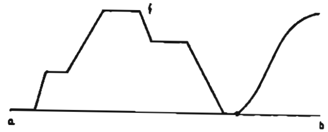

Fig. 8. Type I membership function.

Let

$$a_1 = \min(x_2, x_4)$$
$$a_2 = \max(x_3, x_5).$$

To get the sets in the intervals $[a, a_1]$ and $[a_2, b]$, one has to follow reasoning similar to that mentioned in Theorems 2, 3, and 5, hence the theorem.

Similar proofs can be given if the membership functions are of the types shown in Figs. 7 and 8.

V. Extension to Any Membership Function

In the previous sections the union and intersection operators were defined only for type I membership functions and combinations of them. However, to calculate the same for any two arbitrary continuous functions, it is initially necessary to express them as a suitable combination of unions and intersections of type I functions. To do this, it must be proved that a collection of type I functions always exists whose combination gives rise to any such arbitrary continuous function. A theorem to that effect is proved in this section.

Definition 4 (D4): Let f be a continuous membership function from $[a, b]$ to $[0,1]$ such that f satisfies D1. Then

$$f = (\cdots((g_1 \circ g_2) \circ g_3) \circ \cdots \circ g_n) \qquad (1)$$

where $g_i \in \mathscr{A}$ for all $i = 1, \cdots, n$, and n is a positive integer and $\circ = \textcircled{\cap}$ or $\textcircled{\cup}$ if $f(x) = \lambda\ (\cdots((B_{1x} \circ B_{2x}) \circ B_{3x}) \circ \cdots \circ B_{nx})$ for all $x \in [a, b]$ where B_{ix} for all $x \in [a, b]$ are the sets obtained from g_i for all $i = 1, \cdots, n$ and $\circ = \cap$ or $\cup$ as the notation $\circ$ stands in (1).

Assumption: For a continuous membership function f defined on $[a, b]$, $M_f = \{x: x \in (a, b), x$ is a maximum or a minimum of $f\}$. Then M_f is either finite, or the set of connected components of M_f is finite [13].

The previous assumption is made to avoid membership functions which do not appear in real life.

Theorem 8: Let f be a continuous membership function from $[a, b]$ to $[0,1]$ such that f satisfies D1 and M_f is finite. Then $\exists g_1, g_2, \cdots, g_n \in \mathscr{A}$ such that $f = (\cdots((g_1 \circ g_2) \circ g_3) \circ \cdots \circ g_n)$, where $\circ = \textcircled{\cap}$ or $\textcircled{\cup}$.

Proof: From D1 the function f takes either 0 or 1 at a and b. We have the following cases:

1) $f(a) = 0 \quad f(b) = 1$
2) $f(a) = 0 \quad f(b) = 0$
3) $f(a) = 1 \quad f(b) = 0$
4) $f(a) = 1 \quad f(b) = 1.$

Proof for case 1: Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ be such that $x_i < y_i$ for all $i = 1, \cdots, n$ and $y_i < x_{i+1}$ for all $i = 1, \cdots, (n-1)$, and the x_i are local maxima and the y_i are local minima.

Let

$$g_{2i-1}(x) = \begin{cases} 1, & \text{for all } x \leqslant x_i,\ \forall i = 1, \cdots, n \\ 0, & \text{for all } x \geqslant x_{i+1},\ i = 1, \cdots, n\ (x_{n+1} = b) \\ f(y_i) + \dfrac{[f(x) - f(y_i)][1 - f(y_i)]}{f(x_i) - f(y_i)}, & x_i \leqslant x \leqslant y_i \\ \left(\dfrac{-x + x_{i+1}}{x_{i+1} - y_i}\right) f(y_i), & y_i \leqslant x \leqslant x_{i+1}; \end{cases}$$

$$g_{2i}(x) = \begin{cases} 0, & \text{if } a \leqslant x \leqslant y_i \text{ for all } i = 0, \cdots, n,\ y_0 = a \\ f(x), & \text{if } x \in [a, x_1] \text{ and } i = 0 \\ 1 + f(x) - g_1(x), & \text{if } x \in [x_1, y_1] \text{ and } i = 0 \\ 1, & \text{if } x \geqslant y_1 \text{ and } i = 0 \\ 0, & \text{if } x \leqslant y_i \text{ for all } i = 1, \cdots, n \\ 1, & \text{if } x \geqslant y_{i+1} \text{ for all } i = 1, \cdots, (n-1) \\ f(x) - g_{2i-1}(x), & \text{for all } i = 1, \cdots, n,\ x \in [y_i, x_{i+1}] \\ 1 + f(x) - g_{2i+1}(x), & \text{for all } i = 1, \cdots, (n-1),\ x \in [x_{i+1}, y_{i+1}]. \end{cases}$$

Let

$$h_1 = (g_0 \textcircled{\cap} g_1)$$
$$h_{2i} = h_{2i-1} \textcircled{\cup} g_{2i}, \qquad \text{for all } i = 1, 2, \cdots, n$$
$$h_{2i+1} = h_{2i} \textcircled{\cap} g_{2i+1}, \qquad i = 1, \cdots, (n-1)$$

Let $B_{0x}, B_{1x}, \cdots, B_{2nx}$ be the sets obtained from $g_0, g_1, \cdots, g_{2n}$. Let

$$A_{1x} = B_{0x} \cap B_{1x}, \qquad \text{for all } x$$
$$A_{2ix} = A_{(2i-1)x} \cup B_{2ix}, \quad \text{for all } i = 1, 2, \cdots, n, \text{ for all } x$$
$$A_{(2i+1)x} = A_{2ix} \cap B_{(2i+1)x}, \quad \text{for all } i = 1, 2, \cdots, (n-1), \text{ for all } x.$$

Observe that $\lambda(A_{2nx}) = f(x)$ for all x. The proofs for other cases are similar.

A similar proof holds when the set of connected components of M_f is finite. An example is given next to make the aforementioned theorem clear.

Example 7: Let f: $[0,1] \to [0,1]$ be as shown in Fig. 9, i.e.,

$$f(0) = 0$$
$$f(x_1) = a < 1$$
$$f(y_1) = b < a$$
$$f(1) = 1.$$

f is nondecreasing in $[0, x_1]$, nonincreasing in $[x_1, y_1]$, and

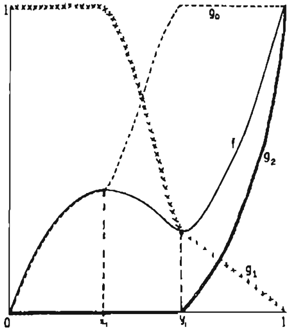

Fig. 9. Membership function f is represented as combination of type 1 functions g_0, g_1, and g_2.

nondecreasing in $[y_1, 1]$. Let

$$g_0(0) = 0$$

$$g_0(x) = \begin{cases} f(x), & \text{for all } x \leq x_2 \\ 1 + f(x) - g_1(x), & \text{if } x \in [x_1, y_1] \\ 1, & \text{if } x \geq y_1. \end{cases}$$

Let

$$g_1(x) = \begin{cases} 1, & \text{if } x \leq x_1 \\ f(y_1) + \dfrac{(f(x) - f(y_1))(1 - f(y_1))}{f(x_1) - f(y_1)}, & \text{if } x_1 \leq x \leq y_1 \\ \dfrac{1-x}{1-y_1} f(y_1), & \text{if } x \in [y_1, 1]. \end{cases}$$

Let

$$g_2(x) = \begin{cases} 0, & \text{if } x \leq y_1 \\ f(x) - g_1(x), & \text{if } x \in [y_1, 1]. \end{cases}$$

Therefore,

$$B_{0x} = \begin{cases} [0, f(x)], & \text{if } 0 \leq x \leq x_1 \\ [0, 1 + f(x) - g_1(x)], & \text{if } x \in [x_1, y_1] \\ [0, 1], & \text{if } x \geq y_1; \end{cases}$$

$$B_{1x} = \begin{cases} [0, 1], & \text{if } x \leq x_1 \\ [1 - g_1(x), 1], & \text{if } x_1 \leq x \leq 1; \end{cases}$$

$$B_{2x} = \begin{cases} \varphi, & \text{if } x \leq y_1 \\ [0, f(x) - g_1(x)], & \text{if } y_1 \leq x \leq 1; \end{cases}$$

$$B_{0x} \cap B_{1x} = \begin{cases} [0, f(x)], & \text{for all } x \leq x_1 \\ [1 - g_1(x), 1 + f(x) - g_1(x)], & \text{if } x_1 \leq x \leq y_1 \\ [1 - g_1(x), 1], & \text{if } x \geq y_1; \end{cases}$$

$$(B_{0x} \cap B_{1x}) \cup B_{2x} = \begin{cases} [0, f(x)], & x \leq x_1 \\ [1 - g_1(x), 1 + f(x) - g_1(x)], & \text{if } x_1 \leq x \leq y_1 \\ [1 - g_1(x), 1] \cup [0, f(x) - g_1(x)], & \text{for all } x \geq y_1. \end{cases}$$

Observe that

$$\lambda[(B_{0x} \cap B_{1x}) \cup B_{2x}] = f(x), \quad \text{for all } x \in [0,1],$$

i.e.,

$$f = \left[\left(g_0 \circledcap g_1 \right) \circledcup g_2 \right].$$

VI. Comparison Between the Definitions

The properties ($P1$–$P11$) mentioned in Section II of $\circledcap$, $\circledcup$, and inclusion are trivial to prove because the operations are ordinary set operations, and the Lebesgue measure satisfies similar properties in connection with ordinary sets [12]. Before making the comparison of the various definitions, let us define in some other way the union and intersection operations.

Definition 5 (D5): Define the following for any two functions f and g on any domain Q:

$$A_x = [0, f(x)] \qquad B_x = [0, g(x)] \qquad \text{for all } x \in \Omega.$$

Define

$$f \subseteq g, \quad \text{if } A_x \subseteq B_x \text{ for all}$$

$$f \text{ union } g = \lambda(A_x \cup B_x)$$

$$f \text{ intersection } g = \lambda(A_x \cap B_x).$$

Observe that Zadeh's definitions are obtained from D5.

Definition 6 (D6): Define the following for any two functions f and g on any domain Q:

$$A_x = [0, f(x)], \quad B_x = [1 - g(x), 1] \qquad \text{or} \qquad A_x = [1 - f(x), 1], \quad B_x = [0, g(x)].$$

Define

$$f \text{ union } g = \lambda(A_x \cup B_x)$$

$$f \text{ intersection } g = \lambda(A_x \cap B_x).$$

Observe that bold union and bold intersection follow from D6.

From D5 and D6 it can be concluded that D2, D3, and D4 provide other ways of defining union and intersection mathematically. From definitions D2 to D6 it can be seen that the proposed definitions $\circledcup$ and $\circledcap$ are a mixture of Zadeh's definitions ($\cup, \cap$) and bold union and intersection ($\boldsymbol{\cup}, \boldsymbol{\cap}$) for type I functions. Tables I–III show the same results.

Note: 1) If two membership functions are defined on the same domain, then one can always consider intersection and union of those membership functions. 2) Nothing has been mentioned about the membership functions defined on a finite domain. If information is available about the factors which influence the membership function values (Example 3), then unions and intersections can be calculated.

Acknowledgment

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TABLE I
ZADEH'S UNION AND INTERSECTION[a]

f	g	h	$\bar{g}$	$f\cap g$	$f\cap h$	$f\cap\bar{g}$	$g\cap h$	$g\cap\bar{g}$	$h\cap\bar{g}$	$f\cup g$	$f\cup h$	$f\cup\bar{g}$	$g\cup h$	$g\cup\bar{g}$	$h\cup\bar{g}$
0.5	0.25	0.5	0.75	0.25	0.5	0.5	0.25	0.25	0.5	0.5	0.5	0.5	0.5	0.75	0.75
0.8	0.64	0.2	0.36	0.64	0.2	0.36	0.2	0.36	0.2	0.8	0.8	0.8	0.64	0.64	0.36
0.3	0.09	0.7	0.91	0.09	0.3	0.3	0.09	0.09	0.7	0.3	0.7	0.91	0.7	0.91	0.91
1.0	1.0	0	0	1.0	0.0	0	0	0	0	1.0	1.0	1.0	1.0	1.0	0
0	0	1.0	1.0	0	0	0	0	0	1.0	0	1.0	1.0	1.0	1.0	1.0

[a] $f(x)=x$, $g(x)=x^2$, $h(x)=1-x$, $\bar{g}(x)+1-x^2$ for all $x\in[0,1]$.

TABLE II
BOLD UNION AND BOLD INTERSECTION[a]

f	g	h	$\bar{g}$	$f\Cap g$	$f\Cap h$	$f\Cap\bar{g}$	$g\Cap h$	$g\Cap\bar{g}$	$h\Cap\bar{g}$	$f\Cup g$	$f\Cup h$	$f\Cup\bar{g}$	$g\Cup h$	$g\Cup\bar{g}$	$h\Cup\bar{g}$
0.5	0.25	0.5	0.75	0	0	0.25	0	0	0.25	0.75	1.0	1.0	0.75	1.0	1.0
0.8	0.64	0.2	0.36	0.44	0	0.16	0	0	0	1.0	1.0	1.0	0.84	1.0	0.56
0.3	0.09	0.7	0.91	0	0	0.21	0	0	0.61	0.39	1.0	1.0	0.79	1.0	1.0
1.0	1.0	0	0	1.0	0	0	0	0	0	1.0	1.0	1.0	1.0	1.0	0
0	0	1.0	1.0	0	0	0	0	0	1.0	0	1.0	1.0	1.0	1.0	1.0

[a] $f(x)=x$, $g(x)=x^2$, $h(x)=1-x$, $\bar{g}(x)=1-x^2$ for all $x\in[0,1]$.

TABLE III
PROPOSED UNION AND INTERSECTION[a]

f	g	h	$\bar{g}$	$f \textcircled{$\cap$} g$	$f \textcircled{$\cap$} h$	$f \textcircled{$\cap$} \bar{g}$	$g \textcircled{$\cap$} h$	$g \textcircled{$\cap$} \bar{g}$	$h \textcircled{$\cap$} \bar{g}$	$f \textcircled{$\cup$} g$	$f \textcircled{$\cup$} h$	$f \textcircled{$\cup$} \bar{g}$	$g \textcircled{$\cup$} h$	$g \textcircled{$\cup$} \bar{g}$	$h \textcircled{$\cup$} \bar{g}$
0.5	0.25	0.5	0.75	0.25	0.0	0.25	0.0	0.0	0.5	0.5	1.0	1.0	0.75	1.0	0.75
0.8	0.64	0.2	0.36	0.64	0.0	0.16	0.0	0.0	0.2	0.8	1.0	1.0	0.84	1.0	0.32
0.3	0.09	0.7	0.91	0.09	0.0	0.21	0.0	0.0	0.7	0.3	1.0	1.0	0.79	1.0	0.91
1.0	1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	1.0	1.0	1.0	1.0	1.0	0.0
0.0	0.0	1.0	1.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	1.0	1.0	1.0	1.0	1.0

[a] $f(x)=x$, $g(x)=x^2$, $h(x)=1-x$, $\bar{g}(x)=1-x^2$ for all $x\in[0,1]$. The definitions of $\cup$, $\cap$, $\Cup$, and $\Cap$ are as follows: $(f_1\cup g_1)(x)=\max(f_1(x),g_1(x))$; $(f_1\cap g_1)(x)=\min(f_1(x),g_1(x))$; $(f_1\Cup g_1)(x)=\min(1,f_1(x)+g_1(x))$; $(f_1\Cap g_1)(x)=\max(0,f_1(x)+g_1(x)-1)$.

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