Econometrica, Vol. 32, No. 4 (October, 1964)

A CONSISTENT METHOD OF ESTIMATING THE ENGEL CURVE FROM GROUPED SURVEY DATA[1]

BY N. SREENIVASA IYENGAR

1. INTRODUCTION

IN AN EARLIER paper [9], a simple graphical method was developed for computing Engel elasticities from concentration curves. This method, which has since been used in some of the Indian studies on consumer behaviour [10, 12] rests on two basic assumptions—the log-normality of the income (or total expenditure) distribution and the constancy of the Engel elasticity—which admit empirical testing. It has been used with advantage in empirical work involving the calculation of a large number of elasticities from the available National Sample Survey data, which usually provide aggregate consumption patterns either in fixed size classes of income (total consumer expenditure per capita) or in fixed fractile classes.[2] The latter type of tabulation has certain advantages in economic analyses [15] and provides the primary data for the application of our method.

Perhaps the question that has not yet been adequately examined is the following: Does this method, apparently so simple and probably less expensive, yield in any statistical sense a better estimate of the Engel curve than the one provided by the regression method under similar assumptions? The present paper seeks to answer this question and shows that our procedure is consistent. An expression for the asymptotic variance of our estimate, which may be computed from the given data, is also worked out for the slope of the double-log Engel curve. The regression estimate of the elasticity computed from group means under the double-log hypothesis is shown to be asymptotically biased with the bias increasing with the "true" elasticity. Under certain conditions, which arise in actual practice, it is shown that our method yields asymptotically more efficient estimates than Wald's in the double-log case, at least for relative luxuries.

In Section 2, the basic notation is developed and some empirical tests are proposed for verifying the basic assumptions. An alternative procedure of estimation, based on the concepts of specific and Lorenz concentration ratios, is considered in Section 3. The classical method of least squares is compared with our method in Section 4. In Section 5 we consider alternative hypotheses for the distribution of

[1] The studies reported in this paper were initiated at the Indian Statistical Institute and completed at the Harvard Economic Research Project under a Rockefeller Foundation Fellowship. The author gratefully acknowledges the help and support received from these institutions. Thanks are due to Professor H. S. Houthakker for taking kind interest and offering several useful comments on the original manuscript, to Dr. J. Roy and Dr. J. Sethuraman for giving valuable suggestions in the initial stages, and to Professor P. C. Mahalanobis for providing stimulating guidance. Thanks are also due to Mrs. Joan Cochran for typographical assistance.

[2] These studies are yet unpublished.

income and the Engel curve and suggest appropriate estimation procedures. Finally, we make a few general remarks in the last section. Some of the derivations, tables, and charts are given in the Appendix.

2. SOME METHODOLOGICAL CONSIDERATIONS

This section deals with the estimation of the parameters of the double-log Engel curve defined by

$$\Psi(x) = \mathscr{E}(y|x) = Ax^{\eta}, \tag{2.1}$$

in which y and x represent respectively the household expenditure on the specific commodity and income (or total expenditure). The latter is supposed to be log-normally distributed with the parameters (θ, λ). That is to say, the random variable x has the density function $g(x)$ given by

$$g(x) = \frac{1}{x\lambda\sqrt{2\pi}} \exp\left\{-\tfrac{1}{2}\left(\frac{\log x - \theta}{\lambda}\right)^2\right\}, \qquad x > 0. \tag{2.2}$$

Equations (2.1) and (2.2) constitute our basic assumptions and may be easily tested given grouped survey data; this has been done at the Indian Statistical Institute using the National Sample Survey data [3].

Under the assumptions stated in (2.1) and (2.2), we propose the following procedure: Let

$$q = \frac{\bar{x}_1}{\bar{x}_1 + \bar{x}_2} \quad \text{and} \quad Q = \frac{\bar{y}_1}{\bar{y}_1 + \bar{y}_2}. \tag{2.3}$$

In (2.3) the $\bar{x}$'s represent the mean incomes, and $\bar{y}$'s the mean specific expenditures corresponding to two fractile classes of income [15]. In other words, we divide the households into two equal groups on the basis of income and compute the proportionate shares of total income and consumption accruing to the lower income group. Such proportions may be obtained directly from the fractile data. But, in the fixed interval case, these have to be computed by interpolation from concentration curves. The elasticity η is estimated according to [9]:

$$\hat{\eta} = \frac{t_Q}{t_q}, \tag{2.4}$$

where t denotes the standard normal deviation defined by

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{t^2}{2}\right) dt. \tag{2.5}$$

The remaining parameters in the equations (2.1) and (2.2) may be estimated as follows:

$$\hat{\lambda}=-t_q\,,$$
$$(2.6)\qquad \hat{\theta}=\log\bar{x}-\tfrac{1}{2}t_q^2\,,$$
$$\hat{A}=\bar{y}(\bar{x})^{-t_Q/t_q}\exp\{\tfrac{1}{2}t_Q(t_q-t_Q)\}\,,$$

where $\bar{x}$ and $\bar{y}$ are the observed mean income and mean specific expenditure of households for both income groups combined.

It now remains to be shown that the above estimates are consistent for their respective parameters. In order to do this, we shall use a basic property concerning the asymptotic joint distribution of the fractile means [21].

Let $(y_1, x_1), (y_2, x_2), \ldots, (y_n, x_n)$ be n independent observations on the random variable (y, x) defined in (2.1) and let us rearrange the sample as follows:

$$\{y_{(1)}, x_{(1)}\}, \{y_{(2)}, x_{(2)}\}, \ldots, \{y_{(n)}, x_{(n)}\}$$
$$x_{(1)}\leqslant x_{(2)}\leqslant\ldots\leqslant x_{(n)}.$$

Assuming $n=2m$, we define the sample fractile means thus:

$$(2.7)\qquad \begin{aligned}\bar{x}_1 &= \frac{1}{m}(x_{(1)}+\ldots+x_{(m)})\,, & \bar{x}_2 &= \frac{1}{m}(x_{(m+1)}+\ldots+x_{(n)})\\ \bar{y}_1 &= \frac{1}{m}(y_{(1)}+\ldots+y_{(m)})\,, & \bar{y}_2 &= \frac{1}{m}(y_{(m+1)}+\ldots+y_{(n)})\,.\end{aligned}$$

Let us normalise the above fractile means and write

$$(2.8)\qquad \begin{aligned}u_i&=\sqrt{m}(\bar{x}_i-\mu_i)\,, & i&=1,2\,,\\ v_i&=\sqrt{m}(\bar{y}_i-\nu_i)\,, & i&=1,2\,,\end{aligned}$$

where the corresponding parameters (μ_i, ν_i) are defined as the truncated means,

$$(2.9)\qquad \begin{aligned}\mu_1&=\mathscr{E}(x|x\leqslant C), & \mu_2&=\mathscr{E}(x|x>C)\,,\\ \nu_1&=\mathscr{E}(y|x\leqslant C), & \nu_2&=\mathscr{E}(y|x>C)\,,\end{aligned}$$

C being the median of x. It may be shown that the joint distribution of $\{u_1, u_2\vdots v_1, v_2\}$ tends to a multivariate normal distribution with zero mean and variance-covariance matrix

$$(2.10)\qquad \left(\begin{array}{c|c}\Sigma & E\\ \hline & T\end{array}\right).$$

The partitioned variance-covariance matrices are systematically evaluated and presented below:

$$(2.11)\qquad \begin{aligned}&\mu=\mathscr{E}(x)=\exp(\theta+\tfrac{1}{2}\lambda^2)\,,\\ &\nu=\mathscr{E}(y)=\mathscr{E}_x\mathscr{E}(y|x)=A\exp(\eta\theta+\tfrac{1}{2}\lambda^2\eta^2)\,,\\ &\mu_1=2\mu\Phi(-\lambda),\quad \mu_2=2\mu\Phi(\lambda)\,,\\ &C=\text{median}=\exp(\theta)\,,\\ &\nu_1=2\nu\Phi(-\lambda\eta),\quad \nu_2=2\nu\Phi(\lambda\eta)\,,\\ &\xi=\Psi(C)=AC^{\eta}\,.\end{aligned}$$

Let us now introduce a set of deviations of the fractile means from their respective median positions.

$$(2.12)\qquad \begin{aligned} M_1&=M_1^0=C-\mu_1\,, & M_2&=M_2^0=\mu_2-C\,,\\ N_1&=N_1^0=\xi-\nu_1\,, & N_2&=N_2^0=\nu_2-\xi\,. \end{aligned}$$

The next step will be to derive the truncated variances and covariances in the given fractile classes:

$$(2.13)\qquad \begin{aligned} \sigma_1^2&=\mathrm{Var}(x|x\leqslant C)=2\mu^2\exp(\lambda^2)\,\Phi(-2\lambda)-\mu_1^2\,,\\ \sigma_2^2&=\mathrm{Var}(x|x>C)=2\mu^2\exp(\lambda^2)\,\Phi(2\lambda)-\mu_2^2\,,\\ \tau_1^2&=\mathrm{Var}(y|x\leqslant C)=2\nu^2\exp(\lambda^2\eta^2)\,\Phi(-2\lambda\eta)-\nu_1^2\,,\\ \tau_2^2&=\mathrm{Var}(y|x>C)=2\nu^2\exp(\lambda^2\eta^2)\,\Phi(2\lambda\eta)-\nu_2^2\,,\\ \rho_1\sigma_1\tau_1&=\mathrm{Cov}(x,y|x\leqslant C)=2\mu\nu\exp(\lambda^2\eta)\,\Phi(-\lambda-\lambda\eta)-\mu_1\nu_1\,,\\ \rho_2\sigma_2\tau_2&=\mathrm{Cov}(x,y|x>C)=2\mu\nu\exp(\lambda^2\eta)\,\Phi(\lambda+\lambda\eta)-\mu_2\nu_2\,. \end{aligned}$$

The elements of the matrix (2.10) are obtained from (2.12) and (2.13). We have

$$\Sigma=\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}=\begin{bmatrix}\sigma_1^2+\tfrac12 M_1M_1^0 & \tfrac12 M_1M_2^0\\ \tfrac12 M_1M_2^0 & \sigma_2^2+\tfrac12 M_2M_2^0\end{bmatrix},$$

$$(2.14)\qquad E=\begin{bmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{bmatrix}=\begin{bmatrix}\rho_1\sigma_1\tau_1+\tfrac12 M_1N_1^0 & \tfrac12 M_1N_2^0\\ \tfrac12 M_2N_1^0 & \rho_2\sigma_2\tau_2+\tfrac12 M_2N_2^0\end{bmatrix},$$

$$T=\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}=\begin{bmatrix}\tau_1^2+\tfrac12 N_1N_1^0 & \tfrac12 N_1N_2^0\\ \tfrac12 N_1N_2^0 & \tau_2^2+\tfrac12 N_2N_2^0\end{bmatrix}.$$

Formulae (2.11) to (2.14) are involved in the asymptotic variance of our estimates of the parameters, particularly of η.

The consistency of the estimate (2.4) may be easily proved by observing that it is primarily a function of the observed fractile means which, in large samples, tend to their respective population values:

$$(2.15)\qquad \begin{aligned} q&=\frac{\bar x_1}{\bar x_1+\bar x_2}\sim\frac{\mu_1}{\mu_1+\mu_2}=\frac{2\mu\Phi(-\lambda)}{2\mu}=\Phi(-\lambda)\,,\\ Q&=\frac{\bar y_1}{\bar y_1+\bar y_2}\sim\frac{\nu_1}{\nu_1+\nu_2}=\frac{2\nu\Phi(-\lambda\eta)}{2\nu}=\Phi(-\lambda\eta)\,,\\ \hat\eta&=\frac{t_Q}{t_q}\sim\frac{t_{Q(-\lambda\eta)}}{t_{\Phi(-\lambda)}}=\eta\,. \end{aligned}$$

It should be noted that the denominator involved in our estimate cannot be zero for, by definition, λ is positive. This implies that $\hat\eta$ is a continuous function of the observed means and therefore consistent. By similar argument we may establish the consistency of the estimates of the remaining parameters (2.6).

The asymptotic variance of the estimate of elasticity is found, after some routine mathematical drill, to be

$$(2.16)\quad m\,\mathrm{Var}(\hat{\eta}) \sim \frac{1}{\lambda^2}\left[\frac{\mathrm{Var}(Q)}{Z^2(\lambda\eta)} - 2\eta\,\frac{\mathrm{Cov}(q,Q)}{Z(\lambda)Z(\lambda\eta)} + \frac{\eta^2\,\mathrm{Var}(q)}{Z^2(\lambda)}\right],$$

where

$$Z(t) = Z(-t) = \Phi'(t) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right),$$

$$\mathrm{Var}(q) = \frac{1}{16\mu^2}\left[\left(\frac{\mu_2}{\mu}\right)^2\Sigma_{11} - 2\left(\frac{\mu_2}{\mu}\right)\left(\frac{\mu_1}{\mu}\right)\Sigma_{12} + \left(\frac{\mu_1}{\mu}\right)^2\Sigma_{22}\right],$$

$$(2.17)\quad \mathrm{Var}(Q) = \frac{1}{16\nu^2}\left[\left(\frac{\nu_2}{\nu}\right)^2 T_{11} - 2\left(\frac{\nu_2}{\nu}\right)\left(\frac{\nu_1}{\nu}\right)T_{12} + \left(\frac{\nu_1}{\nu}\right)^2 T_{22}\right],$$

$$\mathrm{Cov}(q,Q) = \frac{1}{16\mu\nu}\left[\left(\frac{\mu_2}{\mu}\right)\left(\frac{\nu_2}{\nu}\right)E_{11} - \left(\frac{\mu_2}{\mu}\right)\left(\frac{\nu_1}{\nu}\right)E_{12} - \right.$$
$$\left. - \left(\frac{\mu_1}{\mu}\right)\left(\frac{\nu_2}{\nu}\right)E_{21} + \left(\frac{\mu_1}{\mu}\right)\left(\frac{\nu_1}{\nu}\right)E_{22}\right].$$

A consistent estimate of Var $(\hat{\eta})$ may be obtained by replacing the population values in (2.16) by their respective estimates. Once this has been done, the usual large sample statistical tests of significance may be devised for the elasticity. We shall not venture to work out variances for all the other parameters found in equations (2.1) and (2.2), except for the inequality parameter λ. This inequality parameter plays an important role in interregional, intertemporal, and international studies on income distribution; it is directly related to the Lorenz measure of inequality L [1] by the equation

$$(2.18)\quad L = 2\Phi\left(\frac{\lambda}{\sqrt{2}}\right) - 1.$$

As noted earlier in (2.6), the inequality parameter is estimated by $\hat{\lambda} = -t_q$, which has the asymptotic variance

$$(2.19)\quad \mathrm{Var}(\hat{\lambda}) = \frac{\mathrm{Var}(q)}{Z^2(\lambda)}.$$

The above expression, which incidentally appears in the formula for the asymptotic variance of $\hat{\eta}$, may also be computed either directly from the group means or from quantities which may be derived therefrom.

It might be appropriate at this stage to indicate certain procedures to test the underlying basic assumptions. A simple graphical test for log-normality of income distribution and constant income elasticity of demand is provided in terms of the following proportions computed for variable levels of income:

p: proportion of households (or individuals) earning a given income (x) or less,

q: proportion of aggregate income earned by the above stratum of households (or individuals), and

Q: proportion of aggregate consumption accruing to the above stratum of households (or individuals).

The test for the log-normality of income distribution is that the standard normal deviates, t_q and t_p, as defined in (2.5), are linearly related by the equation $t_q = t_p - \lambda$, where λ is the inequality parameter. For constant income elasticity besides log-normality, t_Q and t_p are likewise related by the equation $t_Q = t_p - \lambda\eta$, where η represents the income elasticity of demand.[3] These conditions indeed ensure symmetry of the corresponding concentration curves relating q and Q with p [9, p. 884]. The above tests, though necessary, need not be sufficient, just as symmetry of the Lorenz curve does not ensure log-normality.

The tests proposed above are perhaps stronger then the customary linear log-probit test and are easily adaptable to both fixed-interval and fractile forms of grouped data. It may be noted that a log-probit graphical test can be performed with the usual frequency distribution in fixed classes of income. However, while one has size distribution data giving distribution of households as well as total income in income brackets, such as in the case of income tax statistics or family budget data, the log-probit test does not fully utilize the latter information and is therefore weaker than our tests.

Analytical tests such as the use of the frequency chi-square statistic, $\chi^2 = \Sigma(O-E)^2/E$, where O is observed and E is expected frequency, are inappropriate when the sampling scheme is other than simple (stratified, multistage, pps, etc.). If the sampling scheme is better than the simple one in some sense, the χ^2 statistic would on the average be smaller than the actual χ^2 and would therefore underestimate the significance. If there are g classes and $\hat{\pi}_i$ is the sample estimate of the relative frequency in the ith class, an appropriate statistic for examining the goodness of fit can be given in terms of a consistent estimate $((d_{ij}))$ of the dispersion matrix of the $\hat{\pi}_i$'s. However, since the cost of computing $((d_{ij}))$ is often much higher than the cost of computing the $\hat{\pi}_i$'s, the dispersion matrix is seldom computed.

Roy and Dhar [20] have performed some tests based on the concept of distance between two populations. Their approach may be summed up as follows: Let $d(\pi, \pi^*)$ be a distance function between two discrete probability distributions $\pi = \{\pi_i\}$ and $\pi^* = \{\pi_i^*\}$ in g classes. Let Ω be a family of distributions of which π^* is a typical member. The distance between $\bar{\pi}$ and Ω is defined as

$$(2.20) \qquad \Delta = \inf_{\pi^*} d(\pi, \pi^*)$$

If $\hat{\pi} = \{\hat{\pi}_i\}$ be an estimate of π, then $D = \inf d(\hat{\pi}, \pi^*)$ is taken as an estimate of Δ.

[3] These equations readily follow from the definition of p, q, and Q. If the distribution of income (x) is log-normal, and the Engel curve is of the form $\Psi(x) = Ax^\eta$, then $p = \Phi(t)$, $q = \Phi(t - \lambda)$, and $Q = \Phi(t - \lambda\eta)$, where $t = (\log x - \theta)/\lambda$ is $N(0, 1)$.

Under certain conditions, a normalising constant a_n depending on the sample size n can be chosen so that $a_n(D-\Delta)$ is asymptotically distributed as $N(0, 1)$, provided that $\Delta \neq 0$. If two interpenetrating subsamples each of size n are available, as in the National Sample Survey of India, there will be two independent estimates D_1 and D_2 of Δ, and asymptotically $(D_1+D_2-2\Delta)/|D_1-D_2|$ has a t-distribution with one degree of freedom. From this a confidence interval for Δ can be built up.

The measure of consistency used by them is defined by $C=\sum_{i=1}^{g}\sqrt{\pi_i\pi_i^*}$, which is equivalent to using Bhattacharya's distance function [2]

$$(2.21)\qquad d(\pi, \pi^*)=\cos^{-1}\left(\sum_{i=1}^{g}\sqrt{\pi_i\pi_i^*}\right).$$

Using the estimates of log-normal parameters, θ and λ, computed from two subsamples in each round of the National Sample Survey separately for rural and urban India by Bhattacharya and Iyengar [3], the measure of consistency in each case is computed. In most cases C has been found to be of the order of 0.98, confirming thereby the findings of earlier studies [3, 19].

The test, though admittedly inefficient, is a valid procedure, particularly in the context of a complex sampling design in the National Sample Survey, which renders customary statistical tests somewhat inapplicable [14].

3. AN ALTERNATIVE PROCEDURE

In this section we shall briefly indicate another possibility for obtaining Engel elasticities using grouped expenditure data. One may argue that our estimate $\hat{\eta}$ developed in the previous section, though consistent, need not be the best in the sense of minimum variance, especially when one has more than two fractile classes. The National Sample Survey of India, for instance, provides consumer expenditure data for certain commodities by ten or twenty fractile groups [16, 17]. Our method, if applied to such data, possibly ignores much of the intergroup variation by combining the given classes into two median classes and probably yields inefficient estimates. Under these circumstances, a somewhat different but intuitively satisfactory procedure was proposed by the author in an earlier paper [10]. We shall presently show that the alternative estimate, say $\hat{\eta}_g$, which makes use of all the g pairs of group means $(\bar{y}_i, \bar{x}_i)$, is asymptotically unbiased for large values of g. The alternative procedure consists of the following steps:

First, let us compute the cumulative proportions $(\hat{q}_i, \hat{Q}_i)$ of total income and specific expenditure corresponding to the ith fractile class:

$$(3.1)\qquad \begin{aligned} \hat{q}_i &= \frac{\bar{x}_1+\ldots+\bar{x}_i}{\bar{x}_1+\ldots+\bar{x}_g} \qquad (i=1, 2, \ldots, g), \\ \hat{Q}_i &= \frac{\bar{y}_1+\ldots+\bar{y}_i}{\bar{y}_1+\ldots+\bar{y}_g} \qquad (i=1, 2, \ldots, g). \end{aligned}$$

Second, we calculate the Lorenz ratio $\hat{L}_0$ and the specific concentration ratio $\hat{L}_s$ (for definition of these concepts, see [9, p. 884]) using the cumulative proportions (3.1):

$$\hat{L}_0 = 1 - \frac{1}{g}\sum_{i=1}^{g}(\hat{q}_i + \hat{q}_{i-1}), \tag{3.2}$$

$$\hat{L}_s = 1 - \frac{1}{g}\sum_{i=1}^{g}(\hat{Q}_i + \hat{Q}_{i-1}),$$

where $(\hat{q}_0, \hat{Q}_0) = (0, 0)$, and $(\hat{q}_g, \hat{Q}_g) = (1, 1)$.

Last, we compute the Engel elasticity by using the formula:

$$\hat{\eta}_g = \frac{t_{\frac{1}{2}(1+\hat{L}_s)}}{t_{\frac{1}{2}(1+\hat{L}_0)}}, \tag{3.3}$$

where t is the standard normal deviate defined in (2.5). It will be noted that the proportions $\frac{1}{2}(1+L_0)$ and $\frac{1}{2}(1+L_s)$ are respectively nothing but the areas above the Lorenz curve and the specific concentration curve contained in the unit square.

This procedure is applicable also for grouped data classified according to fixed class intervals, in which case, of course, the definitions (3.1) and (3.2) will have to be slightly generalised (see [10, p. 385]).

We shall next examine whether (3.3) is consistent for η. In order to do this, let g be fixed. With a given g (>2) it is easy to verify the following statement, in view of our assumptions (2.1) and (2.2):

$$\hat{L}_0 \sim 1 - \frac{1}{g}\sum_{i=1}^{g}[\Phi(t_i - \lambda) + \Phi(t_{i-1} - \lambda)], \tag{3.4}$$

$$\hat{L}_s \sim 1 - \frac{1}{g}\sum_{i=1}^{g}[\Phi(t_i - \lambda\eta) + \Phi(t_{i-1} - \lambda\eta)].$$

Thus for fixed g the estimate (3.3) is *not* unbiased, but is negatively biased for items for which η exceeds unity. The magnitude of the bias is not small enough to ignore when g is small.

We shall now show that as $g \to \infty$, this estimate approaches the true elasticity. For this purpose, let us consider the sequence $\{a_g\}$ where

$$a_g = \frac{1}{g}\sum_{i=1}^{g}\Phi(t_i - \lambda\eta). \tag{3.5}$$

Applying the law of large numbers to the above sequence, which is permissible under our assumptions, we see that as $g \to \infty$,

$$a_g \to \mathscr{E}_t\, \Phi(t - \lambda\eta) \tag{3.6}$$

where $\mathscr{E}$ denotes the expected value. It is easy to show that the right-hand side of (3.6) is equal to the probability that any two independent standard normal variates do not exceed each other by $-\lambda\eta$ [11]. This probability is simply $\Phi(-\lambda\eta/\sqrt{2})$. Therefore, as $g \to \infty$,

$$(3.7) \qquad \hat{L}_s \sim 1 - 2\Phi\left(-\frac{\lambda\eta}{\sqrt{2}}\right).$$

Similarly, $\hat{L}_0$, being a special case of (3.7) in which η is set equal to unity, tends to $1-2\Phi(-\lambda/\sqrt{-2})$. It follows at once that as $g \to \infty$,

$$(3.8) \qquad \begin{aligned} \tfrac{1}{2}(1+\hat{L}_s) &\sim \Phi\left(\frac{\lambda\eta}{\sqrt{2}}\right), \\ \tfrac{1}{2}(1+\hat{L}_0) &\sim \Phi\left(\frac{\lambda}{\sqrt{2}}\right), \end{aligned}$$

so that

$$(3.9) \qquad \hat{\eta}_g \sim \frac{t_{\Phi(\lambda\eta/\sqrt{2})}}{t_{\Phi(\lambda/\sqrt{2})}} = \eta .$$

This establishes the asymptotic unbiasedness of our estimate (3.3).

Perhaps it will be possible to derive an expression for the asymptotic variance of $\hat{\eta}_g$. In fact, when g is fixed, this can be worked out using the elements of the generalised matrix (2.10); this and other related problems are being investigated. It should be emphasised that the alternative procedure does not yield consistent estimates for moderately low values of g, and it is conceivable that it may not be superior to the conventional method of least squares. We shall return to this problem in Section 4.

Without sacrificing any information, we may use the method discussed in Section 2 for obtaining a consistent estimate of the Engel elasticity. Let $g=2k$ where k is a positiver integer. Now we extend the definition of q and Q:

$$(3.10) \qquad q = \frac{\bar{x}_1 + \ldots + \bar{x}_k}{\bar{x}_1 + \ldots + \bar{x}_g}, \qquad Q = \frac{\bar{y}_1 + \ldots + \bar{y}_k}{\bar{y}_1 + \ldots + \bar{y}_g}.$$

It is easy to verify that, for a given g, as $m \to \infty$, q and Q tend respectively to $\Phi(-\lambda)$ and $\Phi(-\lambda\eta)$. This is because $\bar{x}_i \sim g\mu\{\Phi(t_i-\lambda)-\Phi(t_{i-1}-\lambda)\}$ and $\bar{y}_i \sim g\nu\{\Phi(t_i-\lambda\eta)-\Phi(t_{i-1}-\lambda\eta)\}$ with $t_0=-\infty$, $t_k=0$, and $t_g=+\infty$.

The estimate of the elasticity is, as before, given by $\hat{\eta}=t_Q/t_q$, where the q and Q are defined in (3.10). The asymptotic variance of this estimate takes the form

$$(3.11) \qquad m \operatorname{Var}(\hat{\eta}) \sim \frac{1}{\lambda^2}[R_{11}\eta^2 - 2R_{12}\eta + R_{22}],$$

where

$$R_{11} = \frac{1}{16\mu^2 Z^2(\lambda)} \left\{ \left(\frac{\mu_2}{\mu}\right)^2 S_{uu} - 2\left(\frac{\mu_2}{\mu}\right)\left(\frac{\mu_1}{\mu}\right) S_{uu^*} + \left(\frac{\mu_1}{\mu}\right)^2 S_{u^*u^*} \right\},$$

$$(3.12) \qquad R_{12} = \frac{1}{16\mu\nu Z(\lambda) Z(\lambda\eta)} \left\{ \left(\frac{\mu_2}{\mu}\right)\left(\frac{\nu_2}{\nu}\right) S_{uv} - \left(\frac{\mu_2}{\mu}\right)\left(\frac{\nu_1}{\nu}\right) S_{uv^*} - \left(\frac{\mu_1}{\mu}\right)\left(\frac{\nu_2}{\nu}\right) S_{u^*v} + \left(\frac{\mu_1}{\mu}\right)\left(\frac{\nu_1}{\nu}\right) S_{u^*v^*} \right\},$$

$$R_{22} = \frac{1}{16\nu^2 Z^2(\lambda\eta)} \left\{ \left(\frac{\nu_2}{\nu}\right)^2 S_{vv} - 2\left(\frac{\nu_2}{\nu}\right)\left(\frac{\nu_1}{\nu}\right) S_{vv^*} + \left(\frac{\nu_1}{\nu}\right)^2 S_{v^*v^*} \right\}.$$

The S's in (3.12) may be obtained from the generalised matrix (2.10):

$$S_{uu} = \sum_{i=1}^{k} \sum_{j=1}^{k} \Sigma_{ij},$$

$$(3.13) \qquad S_{u^*u} = S_{uu^*} = 2 \sum_{i=1}^{k} \sum_{j=k+1}^{g} \Sigma_{ij} = 2 \sum_{i=k+1}^{g} \sum_{j=1}^{k} \Sigma_{ij},$$

$$S_{u^*u^*} = \sum_{i=k+1}^{g} \sum_{j=k+1}^{g} \Sigma_{ij}.$$

We may note in passing that the S's in (3.12) are simply sums of elements of the matrix Σ equally partitioned into four quadrants. Similarly, expressions which appear in R_{12} and R_{22} of (3.11) can be derived in terms of the elements of the partitioned matrices of E and T respectively.

4. THE LEAST SQUARES ESTIMATE

In this section we return the main question—whether or not the above method apparently so simple and probably inexpensive, is better than the commonly used method of least squares, under the given assumptions. In what follows it will be shown that the method of least squares yields asymptotically biased estimates. We shall mainly focus our attention on the estimate of elasticity.

Suppose with the same data $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$ we compute the regression coefficient b_2 assuming the double-log form for the Engel curve. Then

$$(4.1) \qquad b_2 = \frac{\sum_{i=1}^{2} (\log \bar{x}_i - \overline{\log \bar{x}_i})(\log \bar{y}_i - \overline{\log \bar{y}_i})}{\sum_{i=1}^{2} (\log \bar{x}_i - \overline{\log \bar{x}_i})^2},$$

which simplifies to

$$(4.2)\qquad \frac{\log \bar{y}_2 - \log \bar{y}_1}{\log \bar{x}_2 - \log \bar{x}_1}\,.$$

This estimate obtained from the linear double-log hypothesis is the same as Wald's, which was derived to estimate the slope of the linear regression in two variables when both the dependent and independent variables are subject to errors of measurement [23].

It is easy to see that b_2 is not consistent for η, for we observe that b_2 is consistent for

$$(4.3)\qquad \beta(\eta) = \frac{\log \Phi(\lambda\eta) - \log \Phi(-\lambda\eta)}{\log \Phi(\lambda) - \log \Phi(-\lambda)}\,,$$

which is in general not equal to η except for the values -1, 0, and 1. In fact, the asymptotic value β is a monotonic function of η with the asymptotic bias increasing, constant, or decreasing according to whether

$$(4.4)\qquad \frac{Z(\lambda\eta)}{\Phi(\lambda\eta)\Phi(-\lambda\eta)} \gtreqless \frac{1}{\lambda}\{\log \Phi(\lambda) - \log \Phi(-\lambda)\}\,.$$

By means of numerical examples we may show that, for a given value of λ, the left-hand side in (4.4) is smaller than the right-hand side for some values of η, while for other values of η the opposite holds. For illustrating this, we assume $\lambda = 0.6$, in which case the right-hand side becomes 1.6215. Table I gives the values of the left-hand side for $\lambda = 0.6$ and $\eta = -1.0$, -0.5, 0.5, 1.0, 1.5, and 2.0, as well as the percentage asymptotic bias.

TABLE I

η	-1.0	-0.5	0	0.5	1.0	1.5	2.0
LHS	-1.6739	-1.6154	$2\sqrt{2/\pi}$	1.6154	1.6739	1.7715	1.9067
$\beta(\eta)$	-1.0	-0.4940	0	0.6940	1.0	1.5303	2.1201
bias (%)	0	-1.2	0	1.2	0	2.02	6.00

From Table I it is clear that the asymptotic bias in percentage terms is small within the range of elasticities we have considered. The bias, however, increases fast enough as we move along the η-axis. The asymptotic bias regarded as a function of η resembles the letter w intersecting the elasticity axis at $\eta = -1.0$ and 1 (Figure 1). We should now examine whether the bias in the least squares estimate is accompanied by an increase or decrease in its variance.

The asymptotic variance of the regression estimate may be calculated from the variance-covariance matrix (2.10):

$$(4.5)\qquad m \operatorname{Var}(b_2) \sim \frac{1}{k^2}(S_{11}\beta^2 - 2S_{12}\beta + S_{22})\,,$$

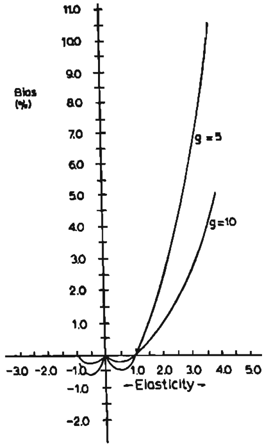

FIGURE 1.—Asymptotic bias of least squares estimate.

where

$$k(\lambda)=\log \Phi(\lambda)-\log \Phi(-\lambda)\,, \tag{4.6}$$

$$S_{11}=\frac{\Sigma_{11}}{\mu_1^2}-2\frac{\Sigma_{12}}{\mu_1\mu_2}+\frac{\Sigma_{22}}{\mu_2^2}\,,$$

$$S_{12}=\frac{E_{11}}{\mu_1\nu_1}-\frac{E_{12}}{\mu_1\nu_2}-\frac{E_{21}}{\mu_2\nu_1}+\frac{E_{22}}{\mu_2\nu_2}\,, \tag{4.7}$$

$$S_{22}=\frac{T_{11}}{\nu_1^2}-2\frac{T_{12}}{\nu_1\nu_2}+\frac{T_{22}}{\nu_2^2}\,.$$

The comparison of the variances (2.16) and (4.5) is not quite straightforward; it involves three sets of comparisons of the coefficients of like terms. The direction of inequality of those coefficients depends on the magnitude and sign of the elements of the variance-covariance matrix.

The expression (2.16) for the variance of $\hat{\eta}$ may be rewritten in the form

$$m\,\mathrm{Var}\,(\hat{\eta})\sim\frac{1}{\lambda^2}\{R_{11}\eta^2-2R_{12}\eta+R_{22}\}\,, \tag{4.8}$$

in which

$$R_{11} = \frac{1}{16\mu^2 Z^2(\lambda)} \left\{ \left(\frac{\mu_2}{\mu}\right)^2 \Sigma_{11} - 2\left(\frac{\mu_2}{\mu}\right)\left(\frac{\mu_1}{\mu}\right)\Sigma_{12} + \left(\frac{\mu_1}{\mu}\right)^2 \Sigma_{22} \right\},$$

$$(4.9) \quad R_{12} = \frac{1}{16\mu\nu Z(\lambda) Z(\lambda\eta)} \left\{ \left(\frac{\mu_2}{\mu}\right)\left(\frac{\nu_2}{\nu}\right) E_{11} - \left(\frac{\mu_2}{\mu}\right)\left(\frac{\nu_1}{\nu}\right) E_{12} - \left(\frac{\mu_1}{\mu}\right)\left(\frac{\nu_2}{\nu}\right) E_{21} + \left(\frac{\mu_1}{\mu}\right)\left(\frac{\nu_1}{\nu}\right) E_{22}, \right.$$

$$R_{22} = \frac{1}{16\nu^2 Z^2(\lambda\eta)} \left\{ \left(\frac{\nu_2}{\nu}\right)^2 T_{11} - 2\left(\frac{\nu_2}{\nu}\right)\left(\frac{\nu_1}{\nu}\right) T_{12} + \left(\frac{\nu_1}{\nu}\right)^2 T_{22} \right\}.$$

Expression (4.8) is a special case of (3.11) in which there are only two fractile groups. In order, therefore, to show that Var $(\hat{\eta}) <$ Var (b_2) we have to examine whether the folowing inequalities are simultaneously satisfied:

$$(4.10) \quad \frac{R_{11}}{S_{11}} < \frac{\lambda^2}{k^2}, \quad \frac{R_{12}}{S_{12}} > \frac{\lambda^2}{k^2}, \quad \frac{R_{22}}{S_{22}} < \frac{\lambda^2}{k^2}.$$

These inequalities may be numerically verified by choosing arbitrary values for the parameters involved, but this is beyond the scope of the present paper. A detailed investigation in this direction is still underway. We shall, however, consider a special case and show that our estimate has a smaller asymptotic variance.

Let us write the variance difference in the form

$$(4.11) \quad m\{\text{Var}(\hat{\eta}) - \text{Var}(b_2)\} = a_{11}\,\text{Var}(q) - 2a_{12}\sqrt{\text{Var}(q)\,\text{Var}(Q)}\,\rho_{qQ} + a_{22}\,\text{Var}(Q),$$

where the a's are given by

$$a_{11} = \frac{\eta^2}{\lambda^2 Z^2(\lambda)} - \frac{\beta^2}{k^2 \Phi^2(\lambda)\Phi^2(-\lambda)},$$

$$(4.12) \quad a_{12} = \frac{\eta}{\lambda^2 Z(\lambda) Z(\lambda\eta)} - \frac{\beta}{k^2 \Phi(\lambda)\Phi(-\lambda)\Phi(\lambda\eta)\Phi(-\lambda\eta)},$$

$$a_{22} = \frac{1}{\lambda^2 Z^2(\lambda\eta)} - \frac{1}{k^2 \Phi^2(-\lambda\eta)\Phi^2(\lambda\eta)}.$$

We observe that the a's are all negative for $\eta \geq 1$[4] by virtue of the following inequality.

THEOREM 1: *For all nonnegative λ and $\eta \geq 1$,*

$$(4.13) \quad H(\lambda\eta) = \frac{\Phi(\lambda\eta)\Phi(-\lambda\eta)}{Z(\lambda\eta)} \leq \frac{\lambda}{k},$$

where k, as defined earlier in (4.6), is equal to $\log \Phi(\lambda) - \log \Phi(-\lambda)$, the equality holding when $\lambda \to 0$.

[4] Computationally it has been verified that this proposition is true for $\eta > 0.7$.

The proof of the theorem consists in showing that $H(\lambda)$ is a monotonic decreasing function of λ and is less than λ/k for all $\lambda > 0$. Differentiating k with respect to λ, we have

$$(4.14) \qquad k' = \frac{Z(\lambda)}{\Phi(\lambda)\Phi(-\lambda)},$$

which is simply the reciprocal of $H(\lambda)$. Thus it is enough to show that k' is monotonically increasing, i.e., $k'' > 0$, and that the elasticity of k with respect to λ is larger than unity, i.e., $\lambda k' \geqslant k$. Both these propositions can be established if we prove a basic lemma concerning the normal distribution.

LEMMA: *If $\Phi(t)$ is the standard normal distribution function, $\Phi(t)=\int_{-\infty}^{t} Z(t)\,dt$ when $Z(t)=1/\sqrt{2\pi}\, e^{-t^2/2}$, and $k(t)=\log\Phi(t)-\log\Phi(-t)$, then $k(t)$ is strictly convex in t for $t>0$ and strictly concave in t for $t<0$.*

PROOF: Noting that $\Phi''(t)=-tZ(t)$, the derivatives of $k(t)$ are

$$(4.15) \qquad k'(t) = \frac{Z(t)}{\Phi(t)\Phi(-t)},$$

and

$$(4.16) \qquad k''(t) = \frac{Z(t)}{\Phi^2(t)\Phi^2(-t)} A(t),$$

where

$$(4.17) \qquad A(t)=[\Phi(t)-\Phi(-t)]Z(t)-t\Phi(t)\Phi(-t).$$

Since $\lim_{t\to\infty} t^n\Phi(-t)=0$, we have $0=A(-\infty)=A(0)=A(\infty)$. We shall now prove by contradiction that $A(t)>0$ for $t>0$. Suppose this were not the case. Then there exists at least one point t^* in $(0,\infty)$ such that $A(t^*)\leqslant 0$, and, since $A(0)=A(\infty)=0$, $A'(t^*)=0$, i.e.,

$$(4.18) \qquad A'(t^*)=2Z^2(t^*)-\Phi(t^*)\Phi(-t^*)=0,$$

so that, say,

$$(4.19) \qquad A(t^*)=[\Phi(t^*)-\Phi(-t^*)]Z(t^*)-2t^*Z^2(t^*)=Z(t^*)B(t^*),$$

where

$$(4.20) \qquad B(t)=[\Phi(t)-\Phi(-t)]-2tZ(t).$$

But $B(0)=0$, $B(\infty)=1$, and $B'(t)=2t^2Z(t)>0$. Hence, $B(t^*)>0$, a contradiction of our assumption. Therefore $A(t)>0$, and consequently, for $t>0$, $k''(t)>0$, that is, $k(t)$ is strictly convex.

Since $\Phi(t)+\Phi(-t)=1$, we have $k(-t)=-k(t)$, so that $k''(-t)=-k''(t)<0$, showing that for $t<0$, $k(t)$ is strictly concave.

COROLLARY: *The function $w(t)=|k(t)|$ is strictly convex and $|\alpha k(t)|>|k(\alpha t)|$ for* *all* $0<\alpha<1$. *Equivalently, for a given t,* $|k(\eta t)/\eta|$ *is a strictly increasing function of* η *for all* $\eta>0$.[5]

The proof of Theorem 1 follows directly from the above lemma. However, for purposes of comparing the magnitudes involved in either side of (4.13), tables and charts have been provided in the Appendix.

Now, since $a_{11}<0$, the necessary and sufficient condition that Var $(\hat{\eta})\leqslant$ Var (b_2) is that the discriminant of the quadratic form (4.11)

$$\begin{vmatrix} a_{11} & a_{12}\rho_{qQ} \\ a_{12}\rho_{qQ} & a_{22} \end{vmatrix} \geqslant 0\,;$$

or, in other words,

$$\rho_{qQ}^2 \leqslant \frac{a_{11}a_{22}}{a_{12}^2}\,. \tag{4.21}$$

As an illustration of this, let $\lambda=0.6$, as before, and $\eta=2.0$. From the available tables for the normal distribution we compute the a's and see that the condition (4.21) takes the form $\rho_{qQ}^2\leqslant 0.9251$. As long as ρ_{qQ}, the correlation between shares of total income and consumption possessed by the lower income class, does not exceed 0.96, our estimate will have a smaller asymptotic variance than Wald's estimate in the double-log case. The a's are, of course, functions of λ and η, and the condition (4.21) varies numerically from one situation to another.

Theoretically it is possible to show that the right-hand side of (4.21) is less than one and is exactly equal to unity for some values of λ and η, in which case

$$H(\lambda) = \frac{\beta}{\eta}\,H(\lambda\eta)\,. \tag{4.22}$$

For a given level of income inequality there do exist some commodities whose income elasticity satisfies equation (4.22). This is because H is a monotonic decreasing function while the asymptotic regression β is larger than the true elasticity for luxuries and smaller for necessities. In such situations, equation (4.21) is automatically satisfied since, by definition, the correlation cannot exceed unity, implying that our estimate has greater asymptotic efficiency than Wald's. In the Appendix, the values of $H(\lambda)$ are tabulated, in order that one may find the values of λ and η that satisfy (4.22).

Regressions based on only two pairs of observations $\{\bar{x}_i, \bar{y}_i\}(i=1,2)$ do not seem realistic in actual practice. We shall, therefore, consider a situation in which we have g pairs of observed means $(\bar{x}_i, \bar{y}_i)$ corresponding to g given fractile groups.

[5] The author gratefully acknowledges the help received from Dr. McFadden and Dr. J. Tsukui in this proof.

The regression coefficient in the double-log model is computed by using the earlier expression (4.1) in which the summation of squares and products of deviations is carried over g classes:

$$b_g = \frac{\sum_{i=1}^{g} (\log \bar{x}_i - \overline{\log \bar{x}_i})(\log \bar{y}_i - \overline{\log \bar{y}_i})}{\sum_{i=1}^{g} (\log \bar{x}_i - \overline{\log \bar{x}_i})^2}. \tag{4.23}$$

By a generalisation of our argument, we may show that (4.23), for a given g, is asymptotically biased, the bias increasing for larger values of the elasticity. However, if in (4.23) the arithmetic means are replaced by the corresponding geometric means, the regression estimate preserves the desirable small-sample properties such as unbiasedness and efficiency. But geometric means are not usually computed in practice because of "zero" observations as well as because of computational inconvenience.

Let $C_0=0, C_1, C_2, \ldots, C_{g-1}, C_g=\infty$ be the $0, 1/g, 2/g, \ldots, (g-1)/g$, and the last fractiles of the distribution of x, assumed to be log-normally distributed with parameters (θ, λ); let t_i be the standard normal deviate corresponding to $C_i (t_0 = -\infty, t_g=\infty)$, i.e.,

$$C_i = \exp(\theta + \lambda t_i). \tag{4.24}$$

We have the following general expressions for the truncated means:

$$\mu_i = g\mu\{\Phi(t_i - \lambda) - \Phi(t_{i-1} - \lambda)\},$$
$$\nu_i = g\nu\{\Phi(t_i - \lambda\eta) - \Phi(t_{i-1} - \lambda\eta\}, \tag{4.25}$$

where μ, ν, and Φ are already defined. In a somewhat similar manner it may be proved [22] that the joint distribution of $\{u_1, \ldots, u_g; v_1, \ldots, v_g\}$, where $u_i = \sqrt{m}(\bar{x}_i - \mu_i)$ and $v_i = \sqrt{m}(\bar{y}_i - \nu_i)$, is asymptotically multivariate normal with zero mean and the generalised variance-covariance matrix (2.10). The elements of the generalised matrix are shown in the Appendix. Applying the foregoing remark to b_g we observe that the latter is consistent for the expression (4.23), in which the observed means are replaced by their corresponding population means, but not for η except for certain values of the elasticity. To evaluate the magnitude of the asymptotic bias in general terms is difficult, but the numerical course is open to us. We assume plausible values for λ and η and compute the asymptotic regression coefficient β, taking $g=5$ and $g=10$.

Let us write

$$q_i = \Phi(t_i - \lambda) - \Phi(t_{i-1} - \lambda),$$
$$Q_i = \Phi(t_i - \lambda\eta) - \Phi(t_{i-1} - \lambda\eta). \tag{4.26}$$

The asymptotic regression coefficients β is given by

$$(4.27)\qquad \beta = \frac{\sum_{i=1}^{g} (\log q_i - \overline{\log q_i})(\log Q_i - \overline{\log Q_i})}{\sum_{i=1}^{g} (\log q_i - \overline{\log q_i})^2}$$

We shall assume the following values for λ and η: $\lambda=0.6$; $\eta=-1.0$, -0.5, 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, and 4.0. The computational results are summarised in Table II:

TABLE II

η	$\beta(\eta)$		bias (%)	
	$g=5$	$g=10$	$g=5$	$g=10$
-1.0	-0.9975	-0.9988	-0.2499	-0.1212
-0.5	-0.4966	-0.4986	-0.6871	-0.2828
0	0	0	0	0
0.5	0.4972	0.4988	0.5632	0.2360
1.0	1	1	0	0
1.5	1.5137	1.5060	0.9145	0.4063
2.0	2.0440	2.0197	2.2005	0.9838
2.5	2.5970	2.5434	3.8797	1.7373
3.0	3.1791	3.0808	5.9698	2.6919
3.5	3.7967	3.6352	8.4772	3.8622
4.0	4.4555	4.2097	11.3884	5.2434

In the computations a sufficient number of decimal places were kept in view of the anticipated magnitude of bias. It is seen that the percentage bias does not appear to be very serious, though for large elasticity values it cannot be ignored. The bias tends to diminish with an increase in the number of fractile groups. Perhaps with $g=20$ the bias may almost tend to be negligible. This fact alone is not a sufficient justification for choosing the regression method unless we have also explored the relative speed of convergence of the estimate to its population value as the number of groups increases. The asymptotic bias is plotted in Figure 1 for chosen values of g, namely, $g=5$ and $g=10$. The graph shows that as the bias is monotonic in the elasticity, no uniform correction for the regression estimate can be suggested.

An expression for the asymptotic variance of the least squares estimate (4.23) can also be worked out, although this involves some tedious algebra. Let us write for each i ($i=1, 2, \ldots, g$):

$$l_t = \frac{1}{2\mu d}\left[\left\{\frac{\log Q_t - \overline{\log Q_t}}{q_t} - \overline{\left(\frac{\log Q_t - \overline{\log Q_t}}{q_t}\right)}\right\} - \right.$$

(4.28)
$$\left. 2\left\{\frac{\log q_t - \overline{\log q_t}}{q_t} - \overline{\left(\frac{\log q_t - \overline{\log q_t}}{q_t}\right)}\right\}\right],$$

$$m_t = \frac{1}{2\nu d}\left[\left\{\frac{\log q_t - \overline{\log q_t}}{Q_t} - \overline{\left(\frac{\log q_t - \overline{\log q_t}}{Q_t}\right)}\right\}\right],$$

$$d = \sum_{t=1}^{g} (\log q_t - \overline{\log q_t})^2 ,$$

where the q's and Q's are as defined in (4.26). The asymptotic variance is given by

(4.29) $$m \text{ Var}(b_g) \sim l\Sigma l' + lEm' + mTm' ,$$

in which $l = (l_1, l_2, \ldots, l_g)$ and $m = (m_1, m_2, \ldots, m_g)$ are g-dimensional vectors.

The variance comparison between the least squares estimate and our estimate is quite difficult as has already been noted even in the simple case of $g = 2$. Numerical-graphical devices, however, are powerful aids in these circumstances, though admittedly they lack in mathematical rigour.

5. ESTIMATION IN THE LOG-LOGISTIC CASE

In this section we relax some of the assumptions of previous sections and assume alternative forms for the distribution of income and the demand relationships. First, we shall consider some plausible hypotheses for the income distribution, retaining the constant elasticity assumption, and then proceed to consider an important case of "variable" elasticity.

As before, let x denote household income, and let us assume that the variable x has a log-logistic distribution [5] which is characterised by the equation

(5.1) $$\log \frac{F(x)}{1-F(x)} = a + b \log x ,$$

where $F(x)$ is the cumulative distribution function. Further, let $\mathscr{E}(y|x) = \Psi(x) = Ax^\eta$, as in (2.1). The problem then is to investigate whether we could still use the methods of Sections 2 and 3 for estimating the distributional parameters of (5.1) and those of the Engel curve. This examination would help us to see whether our methods can be applied to situations where the log-normal hypothesis is replaced by some other plausible alternative, such as (5.1), which may in practice be verified from given size distribution data. We shall be concerned in the main with the estimation of Engel elasticity, though other parameters are by themselves important.

For the log-logistic distribution, the Lorenz curve is given by the parametric equations

(5.2) $$p_\xi = \frac{\xi}{1+\xi},$$

(5.3) $$q_\xi = \frac{1}{B(l,m)}\int_0^\xi \frac{\tau^{l-1}}{(1+\tau)^{l+m}}\,d\tau,$$

where $\xi = e^a x^b$; $l = 1+1/b$, $m = 1-1/b$, so that $l+m=2$. Elimination of ξ between (5.2) and (5.3) yields the Lorenz curve in the form

(5.4) $$q = \frac{1}{B(l,m)}\int_0^p \tau^{l-1}(1-\tau)^{m-1}d\tau.$$

Similarly, the specific concentration curve for the given commodity takes the form

(5.5) $$Q = \frac{1}{B(l^*,m^*)}\int_0^p \tau^{l^*-1}(1-\tau)^{m^*-1}d\tau,$$

where $l^* = 1+\eta/b$, and $m^* = 1-\eta/b$, so that $l^*+m^*=2$. It should be pointed out that the distribution of income assumed above should have $b>1$ and $\eta<b$, so that (5.4) and (5.5) are defined for all $0<p<1$.

It is a simple exercise to show that for the log-logistic distribution (5.1) the Lorenz measure of inequality is given by $1/b$, whereas the specific concentration ratio is η/b.[6] That is, $L_s = \eta L_0$, in the notation of (3.2). Intuitively it appears that the ratio of the specific concentration ratio to the Lorenz ratio gives in this case a consistent estimate of the Engel elasticity. Also, the other distributional parameter a can be computed by

(5.5') $$\hat{a} = \frac{1}{L_0}\log\left(\frac{\Gamma(1+L_0)\Gamma(1-L_0)}{\bar{x}}\right).$$

Let us compute the proportions q and Q corresponding to $p=\frac{1}{2}$, i.e., $q=I_{0.5}$

[6] Since

$$Q = I_p(l^*,m^*) = \frac{1}{B(l^*,m^*)}\sum_{r=0}^{\infty}(-1)^r\binom{m-1}{r}\frac{p^{l^*+r}}{l^*+r},\ 0<p<1,$$

we have for the area under the specific concentration curve

$$A_s = \int_0^1 Q\,dp = 1 - \frac{B(l^*+1,m^*)}{B(l^*,m^*)} = \frac{m^*}{l^*+m^*}.$$

The specific concentration ratio L_s is, by definition, $1-2A_s$, so that

$$L_s = \frac{l^*-m^*}{l^*+m^*}.$$

Also, since $l^* = 1+\eta/b$, $m^* = 1-\eta/b$, the result follows, i.e., $L_s = \eta/b$. Similarly, the Lorenz ratio L_0 is obtained as a special case by putting $\eta = 1$ in the above.

(l, m) and $Q=I_{0.5}(l^*, m^*)$ where $I_p(l, m)$ is the incomplete B-function. The interesting problem then is to apply the formula (2.4) and determine the nature and magnitude of bias, if any. To do this, let $1/b=0.4$ (which is roughly the position in the case of consumer expenditure distributions in urban India). That is, $l=1.4$ and $m=0.6$. For these values of (l, m), the incomplete B-function corresponding to $p=0.5$ gives $q=0.2453{,}521$. Similarly, for the specific commodities with assumed elasticity values such proportions may be computed. For example, if $\eta=-1$, then $l^*=0.6$ and $m^*=1.4$, so that $Q=0.7546{,}479$. In this manner, the proportions Q and their standard normal deviates are computed for $\eta=-1.0, -0.5, 0.5, 1.0, 1.5$, and 2.0, and the main results are summarised in Table III. Column 4 gives estimates of elasticity computed by using the formula (2.4), and the bias is shown in the last column.

TABLE III

η	Q	t_Q	$\hat{\eta}$	% bias
−1.0	0.7546,479	0.6891,733	−1.0000	0
−0.5	0.6273,240	0.3247,748	−0.4712	−5.18
0.5	0.3726,760	−0.3247,748	+0.4712	−5.18
1.0	0.2453,521	−0.6891,733	1.0000	0
1.5	0.1346,750	−1.1045,950	1.6028	6.8
2.0	0.0311,000	−1.8648,900	2.7060	35.3

The bias appears to be considerable and, again, increases with the "true" elasticity. It is thus clear that our method does not necessarily possess all the desired properties of good estimators if the basic assumption of log-normality is changed. However, for the log-logistic case, the ratio of the specific concentration ratio to the Lorenz ratio appears to be more logical.

At this stage it is worth considering the unconventional approach of Section 3 versus the least squares estimate in the log-logistic case, that is, to see the nature of the bias that may arise due to our using the group "arithmetic" means instead of geometric means. We shall briefly outline the procedure and leave out the computations for the present.

For the log-logistic distribution (5.1), the mean is found to be.

(5.6) $\mathscr{E}(x)=e^{-a/b}B(l, m)\,.$

If there are g fixed fractile classes (C_i, C_{i+1}), $i=0, \ldots, g-1$, then the truncated means are given by

(5.7) $\mu_i=\mathscr{E}(x|C_i\leqslant x\leqslant C_{i+1})=ge^{-a/b}[I_{z_{i+1}}(l, m)-I_{z_i}(l, m)]\,,$

where $z=\xi/(1+\xi)$, and $I_z(l, m)$ is the incomplete B-function of the first type ($0<z<1$); $z_1, \ldots, z_{g-1}$ are the $g-1$ fractiles of the B_1-variate with parameters (l, m).

Now, since $\mathscr{E}(y|x) = Ax^{\eta}$ by assumption, the truncated means of y are given by

$$(5.8) \qquad \nu_i = gAe^{-(a/b)\eta}[I_{z_{i+1}}(l^*, m^*) - L_{z_i}(l^*, m^*)] .$$

An interesting problem will be to assign specific values to the parameters involved in (5.7) and (5.8) and compute the series $\{\mu_i, \nu_i\}$ for chosen values of g, and finally work out the regression of μ_i on ν_i. Computations on these lines are omitted as they are expected to yield results similar to those presented in Figure 1.

In an exactly similar manner, we may work out the consequences of the well-known Pareto hypothesis of income distribution. The Pareto distribution is characterised by the double-log linear relationship

$$(5.9) \qquad \log\{1 - F(x)\} = -\alpha \log \frac{x}{x_0}, \quad x > x_0 ,$$

where $F(x)$ is the cumulative distribution function; x_0 is the lower income limit; and $\alpha > 1$ represents the inequality parameter related to the Lorenz measure by the equation

$$(5.10) \qquad L_0 = \frac{1}{2\alpha - 1} .$$

This, however, is omitted from our consideration as a trivial exercise.

Prais and Houthakker [18] in their monumental work *The Analysis of Family Budgets* have made use of five basic forms of Engel curve including the double-log case which gives constant income (expenditure) elasticity. Forms leading to variable elasticities are often found more realistic in economics.[7] The semi-log case, for example, falls in this category.[8] Stated in symbols, the semi-log hypothesis takes the form (5.11) with an implicit additive error term distributed as $N(0, \sigma_0)$:

$$(5.11) \qquad \mathscr{E}(y|x) = \gamma + \delta \log x .$$

Also implicit in this hypothesis is the assumption that the marginal propensity to consume is, on the average, inversely proportional to income. This hypothesis has found some empirical support especially for necessities such as food articles [18, p. 96]. The "variable" elasticity is given by

$$(5.12) \qquad \eta(x) = \frac{\delta}{\gamma + \delta \log x} .$$

For purposes of projection, the elasticity is usually computed at the mean income by the principle of least squares.

[7] These are indicated by the asymmetry of the specific concentration curve.

[8] This form obviously restricts the values of x to the range $0 < \exp(-\gamma/\delta) < x < \infty$. Hence, in the derivation of the Lorenz curve, as well as of the specific concentration curve, the integration will have to be performed over the income range $x > \exp(-\gamma/\delta)$. But, since the proportion of incomes below the "threshold" level is usually small and concentration curves for most necessities seem to rise above the horizontal axis right from the origin, the effect of ignoring the truncation may not be serious.

We shall show below that the method of concentration curves can be used also in the semi-log case, thus relaxing the constant elasticity stipulation of (2.1). The assumption of log-normality will, however, be retained for reasons of simplicity.

Under the log-normal hypothesis the Engel elasticity, computed at the median income C, is given by

$$(5.13) \qquad \eta(C) = \frac{\delta}{\gamma+\delta\theta}.$$

A consistent procedure for estimating the parameters (γ, δ) again involves the use of concentration curves. As pointed out earlier, the Lorenz curve of income distribution is given by $t_q = t_p - \lambda$ while for the semi-log Engel curve (5.11), the specific concentration curve has the equation

$$(5.14) \qquad Q = p - \eta(C)\lambda Z(t_p).$$

The specific concentration ratio is given by $\lambda\eta(C)/\sqrt{\pi}$, so that the semi-log form becomes realistic as long as $\eta(C) < \sqrt{\pi}/\lambda$.

At the median income, $p=0.5$, $t_p=0$, and $\lambda = -t_{\hat{q}_{0.5}}$, so that the "median" elasticity is estimated by

$$(5.15) \qquad \hat{\eta}(C) = \frac{0.5-\hat{Q}_{0.5}}{\hat{\lambda}Z(0)},$$

where $Z(0) = 1/\sqrt{2\pi}$; $\hat{Q}_{0.5}$ and $\hat{q}_{0.5}$ are obtained directly from fractile data, or computed from concentration curves in the case of fixed-interval data. Now, since the denominator in (5.12) is estimated by $\bar{y}$, the overall mean of specific expenditure, δ has the estimate $\hat{\eta}(C)\bar{y}$. If the estimates of δ and θ are substituted again in (5.13), an estimate of γ can be obtained in terms of $\hat{\eta}(C)$, $\hat{\delta}$, and $\hat{\theta}$. Finally, elasticities for the various fractile groups (income classes) can be obtained by substituting group means $\bar{x}_i$'s in (5.12), if desired.

On the other hand, the elasticity $\eta(\mu)$ computed at the mean is obtained by

$$(5.16) \qquad \eta(\mu) = \frac{\delta}{\gamma+\delta\log\mu},$$

which is related to $\eta(C)$ by the relation

$$(5.17) \qquad \eta(\mu) = \eta(C)\left\{\frac{1}{1+\mu(C)\frac{\lambda^2}{2}}\right\}$$

so that

$$(5.18) \qquad \eta(\mu) < \eta(C),$$

provided that the commodity in question is not "inferior"; for inferior goods, the expression (5.15) becomes negative, since in that case the specific concentration curve lies above the Egalitarian line. Also, it will be noted that the "constant" elasticity η is larger than $\eta(C)$. This at once leads to the inequality

$$\eta(\mu) < \eta(C) < \eta \,. \tag{5.19}$$

This inequality is empirically confirmed by Prais and Houthakker [18, p. 94] for six food commodities in their analyses of British family budgets.

The standard errors of the above estimates are difficult to compute, though not impossible, at least in large samples. Some of the empirical studies along the lines suggested in the foregoing sections will be reported in a subsequent note.

6. SOME CONCLUDING REMARKS

The commonly used method of least squares has its general applications in estimating linear regressions in which the equations are subject to error. Among other restrictive assumptions is that the residuals are serially uncorrelated and are also uncorrelated with the explaining variables; the latter are assumed to be free from observational errors. Also, data on individual units, be they households or individuals, are required for obtaining best statistical results. But in situations where we are required to estimate the Engel elasticity from grouped survey materials which are available in the form of grouped arithmetic means in size classes of income (or total expenditure), these assumptions are less likely to hold. Also, from grouped size distribution data with coarse and unequal class intervals, it is not possible to obtain very satisfactory estimates of, for example, the income inequality, though this is often attempted in empirical work [13].

Our method does not require that data on individual units be available. While estimating the Engel elasticity, it explicitly makes use of the knowledge of the distribution of income. The basic assumptions underlying our estimation procedure are easily testable. In the regression, however, the assumption of normality of the residual terms, taken additively or multiplicatively, is often taken for granted.

The fixed class interval data with unequal frequencies has certain disadvantages such as heteroscedasticity from the estimational point of view. The method of fractile analysis seems to be a better method of analysing economic data, particularly in the context of our method of estimation, since it readily provides the basic raw material for our study.

The regression estimate in the present case is shown to be biased; the bias, which arises due to aggregation, increases with the true value of the elasticity and does not tend to vanish even in large samples. The method of fractiles, on the other hand, provides consistent estimates of the Engel curve; the problem of "zero" entries does not seriously arise in it [6, p. 26]. Asymptotic variances which can be estimated in large samples as well as subjected to the usual tests of significance are provided for our estimates. Approximate tests of significance may be readily devised, given two interpenetrating subsamples, by using the fractile error [15].

It is not suggested, however, that our method is a substitute for the general method of least squares, which can be used in a variety of situations involving several variables. But where we have a priori knowledge about the distribution of income and the nature of demand relationships, it may be appropriate to devise special methods which give consistent results, such as those we have proposed. It is generally agreed that the choice of a particular method is dictated by the type of data that are easily and readily available.

An immediate generalisation of our method to cases involving more than two variables seems possible. In that case we may be able to apply this method for estimating, for instance, the well-known Cobb-Douglas production function or the household demand relationships involving family income and family size [7, 4]. Such possible generalisations are still under investigation. It seems also possible to apply our method to estimate the Engel curve in the additive logarithmic forms [8] as well as extend it to other well-known forms [24].

A few important and difficult statistical problems remain. The estimates of standard errors, etc. are all based on random sampling assumptions. It is therefore necessary to build up a satisfactory theory of estimation of the demand curve, at least in large samples, when the sampling has been done by a multistage design. Also, in some family budget surveys the households are classified according to per capita total expenditure, and not according to household income; this must be pointed out. Moreover, in the framework of general equilibrium, the additivity of the Engel curves and the simultaneous character of the system are also important econometric problems that require some consideration, but these complications are not considered in this paper.

Harvard Economic Research Project and
Indian Statistical Institute

APPENDIX

1. Let t_i be the i/g-th quantile of the standard normal distribution, and C_i the corresponding quantiles of the distribution of x, which is assumed to be log-normal with parameters (θ, λ). Then we have

$$C_i = \exp(\theta + \lambda t_i) \qquad (i = 1, 2, \ldots, g-1)$$

with $t_0 = -\infty$ and $t_g = +\infty$.

2. Let $\xi_i = \mathcal{E}(y|x = C_i) = AC_i^{\eta}$,

$$\mu_i = \mathcal{E}(x|C_{i-1} \leqslant x \leqslant C_i) = g\mu[\Phi(t_i - \lambda) - \Phi(t_{i-1} - \lambda)],$$

where $\mu = \xi(x) = \exp(\theta + \frac{1}{2}\lambda^2)$. Similarly, let

$$\nu_i = \mathcal{E}(y|C_{i-1} \leqslant x \leqslant C_i) = g\nu[\Phi(t_i - \lambda\eta) - \Phi(t_{i-1} - \lambda\eta)],$$

where $\nu = A\exp(\eta\theta + \frac{1}{2}\lambda^2\eta^2)$.

TABLE A1[1]

(SEE ALSO FIGURE ON NEXT PAGE)

t	$Z(t)$	$\Phi(t)$	$k(t)$	$H(t)$	t/k
(1)	(2)	(3)	(4)	(5)	(6)
0	0.3989	0.5000	0	0.6267	0.6267
0.10	0.3970	0.5398	0.159569	0.6257	0.6267
0.20	0.3910	0.5793	0.319897	0.6233	0.6252
0.30	0.3814	0.6179	0.480571	0.6190	0.6242
0.40	0.3683	0.6554	0.642903	0.6131	0.6222
0.50	0.3521	0.6915	0.806916	0.6258	0.6196
0.60	0.3332	0.7257	0.973047	0.5975	0.6166
0.70	0.3123	0.7580	1.141665	0.5872	0.6131
0.80	0.2897	0.7881	1.313461	0.5764	0.6091
0.90	0.2661	0.8159	1.488848	0.5644	0.6045
1.00	0.2420	0.8413	1.667897	0.5516	0.5996
1.10	0.2179	0.8643	1.849870	0.5383	0.5946
1.20	0.1942	0.8849	2.039648	0.5242	0.5883
1.30	0.1714	0.9032	2.233341	0.5099	0.5821
1.40	0.1497	0.9192	2.431501	0.4963	0.5758
1.50	0.1295	0.9332	2.636915	0.4811	0.5688
1.60	0.1109	0.9452	2.847808	0.4671	0.5618
1.70	0.0940	0.9554	3.041294	0.4532	0.5590
1.80	0.0790	0.9641	3.290640	0.4380	0.5470
1.90	0.0656	0.9713	3.521635	0.4253	0.5395
2.00	0.0540	0.9772	3.757926	0.4130	0.5322
2.10	0.0440	0.9821	4.004946	0.4000	0.5244
2.20	0.0355	0.9861	4.261822	0.3859	0.5162
2.30	0.0283	0.9893	4.526757	0.3746	0.5081
2.40	0.0224	0.9918	4.794962	0.3616	0.5005
2.50	0.0175	0.9938	5.077028	0.3543	0.4924
2.60	0.0136	0.9953	5.355640	0.3456	0.4855
2.70	0.0104	0.9965	5.651429	0.3365	0.4778
2.80	0.0079	0.9974	5.988942	0.3291	0.4675
2.90	0.0060	0.9981	6.263962	0.3167	0.4630
3.00	0.0044	0.9987	6.644026	0.2954	0.4515

[1] $Z(t) = \exp(-t^2/2)$; $\Phi(t) = \int_{-\infty}^{t} Z(t)\,dt$; $k(t) = \log \Phi(t) - \log \Phi(-t)$; $H(t) = \Phi(t)\Phi(-t)/Z(t)$.

3. Let

$$\sigma_i^2 = \mathrm{Var}(x \mid C_{i-1} \leqslant x \leqslant C_i)$$
$$= \mu^2[ge^{\lambda^2}\{\Phi(t_i - 2\lambda) - \Phi(t_{i-1} - 2\lambda)\}] - \mu_i^2 ;$$
$$\tau_i^2 = \mathrm{Var}(y \mid C_{i-1} \leqslant x \leqslant C_i)$$
$$= \nu^2[ge^{\lambda^2\eta^2}\{\Phi(t_i - 2\lambda\eta) - \Phi(t_{i-1} - 2\lambda\eta)\}] - \nu_i^2 ;$$
$$\varrho_i\sigma_i\tau_i = \mathrm{Cov}(x, y \mid C_{i-1} \leqslant x \leqslant C_i)$$
$$= \mu\nu[ge^{\eta\lambda^2}\{\Phi(t_i - \overline{\lambda 1 + \eta}) - \Phi(t_{i-1} - \overline{\lambda 1 + \eta})\}] - \mu_i\nu_i .$$

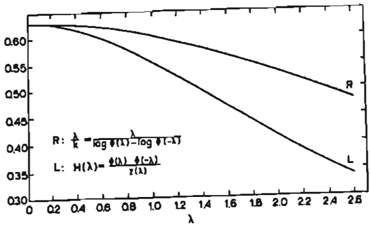

FIGURE A1.—Functions $H(\lambda)$ and λ/k compared.

4. Let us define the following:

$$M_i = i(C_i - \mu_i) - (i-1)(C_{i-1} - \mu_i) \quad (i = 2, \ldots, g-1);$$
$$M_1 = C_1 - \mu_1; \quad M_g = -(g-1)(C_{g-1} - \mu_g);$$
$$M_i^0 = (g-i)(C_i - \mu_i) - (g-i+1)(C_{i-1} - \mu_i) \quad (i = 2, \ldots, g-1);$$
$$M_1^0 = (g-1)(C_1 - \mu_1); \quad M_g^0 = -(C_{g-1} - \mu_g);$$
$$N_i = i(\xi_i - \nu_i) - (i-1)(\xi_{i-1} - \nu_i) \quad (i = 2, \ldots, g-1);$$
$$N_1 = (\xi_1 - \nu_1); \quad N_g = -(g-1)(\xi_{g-1} - \nu_g);$$
$$N_i^0 = (g-i)(\xi_i - \nu_i) - (g-i+1)(\xi_{i-1} - \nu_i) \quad (i = 2, \ldots, g-1);$$
$$N_1^0 = (g-1)(\xi_1 - \nu_1); \quad N_g^0 = -(\xi_{g-1} - \nu_g).$$

5. We shall next define the variance-covariance matrices Σ, T, and E. The elements of Σ are given by:

$$\Sigma_{ij} = \frac{1}{g} M_i M_j^0, \quad j > i,$$
$$= \frac{1}{g} M_j M_i^0, \quad j < i,$$
$$= \sigma_i^2 + \frac{1}{g} M_i M_i^0 + (C_i - \mu_i)(C_{i-1} - \mu_i), \quad i = j \neq 1, g,$$
$$= \sigma_1^2 + \frac{1}{g} M_1 M_1^0, \quad i = j = 1,$$
$$= \sigma_g^2 + \frac{1}{g} M_g M_g^0, \quad i = j = g.$$

Similarly, the elements of T are defined, replacing the σ's by τ's and the M's by N's. Lastly, the elements of the E matrix are given by:

$$E_{ij} = \frac{1}{g} M_i N_j^0, \quad j > i,$$
$$= \frac{1}{g} N_i M_j^0, \quad j < i,$$
$$= \varrho_i \sigma_i \tau_i + \frac{1}{g} M_i N_i^0 + (C_i - \mu_i)(\xi_{i-1} - \nu_i), \quad i = j \neq 1, g,$$
$$= \varrho_1 \sigma_1 \tau_1 + \frac{1}{g} M_1 N_1^0, \quad i = j = 1,$$
$$= \varrho_g \sigma_g \tau_g + \frac{1}{g} M_g N_g^0, \quad i = j = g.$$

6. Then, if $u_i = \sqrt{m}(\bar{x}_i - \mu_i)$ and $v_i = \sqrt{m}(\bar{y}_i - \nu_i)$, the theorem states that the distribution of $w = (u_1, \ldots, u_g;\ v_1, \ldots, v_g)$ is asymptotically normal with mean zero and variance-covariance matrix given by

$$\left(\begin{array}{c:c} \Sigma & E \\ \hdashline & T \end{array}\right) .$$

For proof of this theorem, see Sethuraman [21, 22].

REFERENCES

[1] AITCHISON, J., AND J. A. C. BROWN: *The Log-Normal Distribution with Special Reference to Its Uses in Economics*, Cambridge University Press, 1957.

[2] BHATTACHARYA, A.: "On a Measure of Divergence of Two Multinomial Populations," *Sankhya*, 7, 1946, p. 401.

[3] BHATTACHARYA, N., AND N. S. IYENGAR: "Changes in the Pattern of Distribution of Persons by Monthly Per Capita Total Consumer Expenditure." (Mimeographed.) Working Paper No. 218, Indian Statistical Institute, 1961, 38 pp. (To appear in the printed series, *Studies On Consumer Behaviour*, Vol. 2.)

[4] CROCKETT, JEAN: "Demand Relationships for Food," *Proceedings of the Conference on Consumption and Saving*, eds. Irwin Friend and Robert Jones, University of Pennsylvania, 1960, p. 296.

[5] FISK, P. R.: "The Graduation of Income Distribution," *Econometrica*, 29, 1961, pp. 171–185.

[6] HOUTHAKKER, H. S.: "The Econometrics of Family Budgets," *Journal of the Royal Statistical Society*, A, 115, 1952, pp. 1–20.

[7] ———: "An International Comparison of Household Expenditure Patterns, Commemorating the Centenary of Engel's Law," *Econometrica*, 25, 1957, pp. 532–551.

[8] ———: "Additive Preferences," *Econometrica*, 28, 1960, pp. 244–257.

[9] IYENGAR, N. S.: "On a Method of Computing Engel Elasticities from Concentration Curves," *Econometrica*, 28, 1960, pp. 882–891.

[10] ———: "On a Problem of Estimating Increase in Consumer Demand," *Sankhyā*, 22, 1960, pp. 379–390.

[11] ———: "On the Standard Error of Lorenz Concentration Ratio," *Sankhyā*, 22, 1960, pp. 371–378.

[12] ———: "Estimation of Quality Elasticities for Certain Commodities from National Sample Survey Data," *Sankhyā*, 25, 1963, pp. 15–22.

[13] KRAVIS, I. B.: *The Structure of Income*, University of Pennsylvania, 1962, p. 238.

[14] MAHALANOBIS, P. C.: "Recent Experiments in Statistical Sampling in the Indian Statistical Institute," *Journal of the Royal Statistical Society*, 109, 1946, pp. 325–378.

[15] ———: "A Method of Fractile Graphical Analysis," *Econometrica*, 28, 1960, pp. 325–351.

[16] ———: "Next Steps in Planning," *Sankhya*, 22, 1960, pp. 143–172.

[17] ———: "A Preliminary Note on Consumption of Cereals in India." Paper presented at the thirty-third session of the International Statistical Institute, Paris, 1961.

[18] PRAIS, S. J., AND H. S. HOUTHAKKER: *The Analysis of Family Budgets*, Cambridge University Press, 1955.

[19] ROY, J., AND S. K. DHAR: "A Study on the Pattern of Consumer Expenditure in Rural and Urban India," *Studies Relating to Planning for National Development*, No. 2, Asia Publishing House, 1960, pp. 56–72.

[20] ———: "A Study of the Pattern of Consumer Expenditures in Rural and Urban India." (Part II.) Paper presented at the second Indian Econometric Conference, Waltair, 1961.

[21] SETHURAMAN, J.: "Some Limit Distributions Connected With Fractile Graphical Analysis," *Sankhya*, 23, 1961, pp. 79–90.

[22] ———: "Fixed Interval Analysis and Fractile Analysis." *Contributions to Statistics* (ed. by C. R. Rao and others), Oxford, Pergamon Press and Calcutta, Statistical Publishing Society, 1964, pp. 449–470.

[23] WALD, A.: "The Fitting of Straight Lines If Both Variables Are Subject to Error," *Annals of Mathematical Statistics*, XI, 1940, pp. 284–300.

[24] WOLD, H.: *Demand Analysis*, New York, John Wiley and Sons, Inc., 1953.