Shorted Operators and the Identification Problem—The Real Case

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Abstract—**This paper gives necessary and sufficient conditions under which a real symmetric nonnegative definite matrix can be uniquely determined from the knowledge of a finite number of its shorted versions.**

An earlier paper by one of the authors [2] gave a set of necessary and sufficient conditions under which a complex Hermitian nonnegative definite (n.n.d.) matrix is uniquely identified by a finite number of its shorted versions. The result proved in the context of complex matrices does not hold good for real matrices. The aim of this note is to present the corresponding result for real matrices.

Let $\mathscr{R}^n$ and $\mathscr{R}^{m\times n}$ denote respectively the vector spaces of real n-tuples and real matrices of order $m\times n$. Let $\mathscr{R}_n$ denote the cone of real symmetric n.n.d. matrices of order $n\times n$. For matrices $A, B\in\mathscr{R}_n$ we write $A>B$ if $A-B\in\mathscr{R}_n$. For a matrix A, A' denotes its transpose and $\mathscr{M}(A)$ its column span. A^- denotes a generalized inverse of A and A^+ the Moore Penrose inverse [3]. $\underset{\sim}{A}$ the column string of A is the column vector obtained by writing the columns of A, one below the other in the natural order. Let $A\in\mathscr{R}^{m\times n}$ and $B\in\mathscr{R}^{k\times n}$ and let the jth columns of A and B be denoted by a_j and b_j, respectively. We now introduce a new product $A\boxplus B$ of A and B, which is defined as follows:

$$\begin{aligned} A\boxplus B = (a_1\otimes b_1 : a_1\otimes b_2 + a_2\otimes b_1 : \cdots : a_1\otimes b_n \\ + a_n\otimes b_1 : a_2\otimes b_2 : a_2\otimes b_3 + a_3\otimes b_2 : \ldots : a_2\otimes b_n \\ + a_n\otimes b_2 : \cdots : a_n\otimes b_n) \end{aligned} \tag{1}$$

where $\otimes$ denotes the Kronecker product. $A\boxplus B$ is a matrix of order $mk\times(n(n+1)/2)$. For example, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 2 & 3 \end{pmatrix}$$

then $A\boxplus B$ is the matrix of order 6×3, given by

$$A\boxplus B = \left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}\otimes\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} : \begin{pmatrix} 1 \\ 3 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}\otimes\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} : \begin{pmatrix} 2 \\ 4 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 2 \\ 2 & 7 & 6 \\ 6 & 8 & 0 \\ 3 & 7 & 4 \\ 6 & 17 & 12 \end{pmatrix}$$

If further $C\in\mathscr{R}^{p\times m}$ and $D\in\mathscr{R}^{q\times k}$ as a simple consequence of this definition we have

$$CA\boxplus DB = [C\otimes D][A\boxplus B] \tag{2}$$

noting that the jth columns of CA and DB are Ca_j and Db_j, respectively. Further, the equation

$$A'JA = 0 \tag{3}$$

is equivalent to

$$a_i'Ja_j = 0 \Leftrightarrow [a_j'\otimes a_i']\underset{\sim}{J} = \underset{\sim}{0}, \qquad i, j = 1, 2, \cdots, n.$$

Thus for symmetric matrices J

$$(3) \Leftrightarrow (A\boxplus A)'\underset{\sim}{J} = \underset{\sim}{0}. \tag{4}$$

For a matrix $V\in\mathscr{R}_n$ and a subspace $\mathscr{S}$ of $\mathscr{R}^n$, the shorted matrix $\mathscr{S}(V)$ is the unique matrix in R_n which is such that

$$\mathscr{M}[\mathscr{S}(V)] \subset \mathscr{S}$$

$$V > \mathscr{S}(V)$$

and if $C\in\mathscr{R}_n$, $\mathscr{M}(C)\subset\mathscr{S}$ and $V>C$, then

$$\mathscr{S}(V) > C.$$

The existence of $\mathscr{S}(V)$ was established by Anderson and Trapp [1].

We ask the following question. Given subspaces $\mathscr{S}_1, \mathscr{S}_2, \cdots, \mathscr{S}_b$ of R^n when does the knowledge of shorted matrices $\mathscr{S}_1(V), \mathscr{S}_2(V), \cdots, \mathscr{S}_b(V)$ help us to uniquely identify the matrix V in $\mathscr{R}_n$ of rank v so shorted? Theorem 1 gives a set of necessary and sufficient conditions.

Let the columns of a matrix $X_i\in\mathscr{R}^{n\times r_i}$ span $\mathscr{T}_i = \mathscr{M}[\mathscr{S}_i(V)]$ and the columns of X form a basis of $\mathscr{S} = \mathscr{M}(X_1 : X_2 : \cdots : X_b)$.

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If Rank $X = r < v$

$$\mathscr{S}_i(\mathscr{S}(V)) = \mathscr{S}_i \cap \mathscr{S}(V) = \mathscr{T}_i(V) = \mathscr{S}_i(V), \qquad i = 1,2,\cdots,b$$

and further for any matrix $Z \in \mathscr{R}_n$ with rank$(Z) = v - r$ and $\mathscr{M}(Z)$ virtually disjoint with that of $\mathscr{S}$ (i.e., $\mathscr{M}(Z) \cap \mathscr{S} = \{0\}$)

$$\mathscr{S}_i(\mathscr{S}(V) + Z) = \mathscr{S}_i(V), \qquad i = 1,2,\cdots,b.$$

$\mathscr{S}(V) + Z \in \mathscr{R}_n$ and Rank$(\mathscr{S}(V) + Z) = v$. This shows that the matrix is not uniquely determined by its shorted versions if Rank $X < v$. To avoid triviality, therefore, we shall henceforth assume that

$$\text{Rank } X = v \tag{5}$$

Theorem 1: The matrix $V \in \mathscr{R}_n$ of rank v is uniquely determined by $\mathscr{S}_1(V), \mathscr{S}_2(V), \cdots, \mathscr{S}_b(V)$ iff

$$\text{Rank}\left[X_1 \boxtimes X_1 : X_2 \boxtimes X_2 : \cdots : X_b \boxtimes X_b\right] = v(v+1)/2. \tag{6}$$

Proof: Let $T_i = [X_i'(X_iX_i')^- \mathscr{S}_i(V)(X_iX_i')^- X_i]^*$ From the proof of Theorem 1 in Mitra [2] it is clear that V is uniquely determined by $\mathscr{S}_1(V), \mathscr{S}_2(V), \cdots, \mathscr{S}_b(V)$ iff $X'WX$ is unique and independent of W for every matrix W satisfying the conditions

$$\text{(a)} \qquad X_i'WX_i = T_i, \qquad i = 1,2,\cdots,b \tag{7}$$

and (b) $W \in \{U^-\}$ for some $U \in \mathscr{R}_n$ such that

$$\mathscr{S}_i \subset \mathscr{M}(U). \tag{8}$$

We note that condition (b) could be replaced by

(b′) $X'WX$ is positive definite (p.d.).

Clearly for any matrix W satisfying (b) $X'WX$ is p.d. Hence (b) ⇒ (b′). Conversely if W satisfies (b′), $W \in \{U^-\}$ where $U = X(X'WX)^{-1}X'$. Further (8) is true for such a choice of U. Hence (b′) ⇒ (b) and (b) ⇔ (b′).

Let $P_X = X(X'X)^- X'$ denote the orthogonal projector onto $\mathscr{M}(X)$. Since $X'WX = X'(P_X)'WP_XX = X'XJX'X$ for some J and $X_i'WX_i = X_i'XJX'X_i$, the above condition on W is equivalent to requiring that

$$X_i'XJX'X_i = T_i, \qquad i = 1,2,\cdots,b$$

have a unique p.d. solution J. The existence of one p.d. solution is guaranteed by our assumption that $\mathscr{S}_i(V)$ were indeed obtained by shorting a real symmetric n.n.d. matrix V. The uniqueness of the p.d. solution will require that the null matrix be the only symmetric solution J of the homogeneous equations

$$X_i'XJX'X_i = 0, \qquad i = 1,2,\cdots,b$$

which as we have seen in (4) is equivalent to

$$\left(X'X_i \boxtimes X'X_i\right)'\underset{\sim}{J} = \underset{\sim}{0}, \qquad i = 1,2,\cdots,b$$

or to

$$Q'\underset{\sim}{J} = \underset{\sim}{0}$$

where

$$Q = \left[X'X_1 \boxtimes X'X_1 : X'X_2 \boxtimes X'X_2 : \cdots : X'X_b \boxtimes X'X_b\right].$$

Since the columns of Q are column strings of symmetric matrices and real symmetric matrices span a vector space of dimension $v(v+1)/2$ the equations (9) have the null matrix as the only symmetric solution iff

$$\text{Rank } Q = v(v+1)/2.$$

However, on account of (2)

$$Q = [X' \otimes X']K$$

where

$$K = [X_1 \boxtimes X_1 : X_2 \boxtimes X_2 : \cdots : X_b \boxtimes X_b]$$

and

$$K = \left[X(X'X)^- \otimes X(X'X)^-\right]Q.$$

Hence (11) ⇔

$$\text{Rank } K = v(v+1)/2. \tag{6}$$

Q.E.D.

Remarks: Why should the conditions for the real and complex cases differ? To see this let us amplify the proof of Theorem 1 of Mitra [2]. Proceeding as above we note that the complex Hermitian n.n.d. matrix V is uniquely determined by its shorted versions $\mathscr{S}_1(V), \mathscr{S}_2(V), \cdots, \mathscr{S}_b(V)$ iff the null matrix is the only hermitian solution of the homogeneous equations

$$X_i^* XJX^* X_i = 0, \qquad i = 1,2,\cdots,b \tag{12}$$

If J is a nonnull solution of (12) so is J^*. Further, $J + J^*$ is a Hermitian solution of (12) and this is nonnull unless J is skew Hermitian. If J is a nonnull skew Hermitian solution of (12) $\sqrt{-1}\,J$ is a nonnull Hermitian solution. Hence the above conditions are equivalent to requiring that the null matrix be the only solution of (12) for which it is necessary and sufficient that

$$\text{Rank}\left(X^*X_1 \otimes X'\bar{X}_1 : X^*X_2 \otimes X'\bar{X}_2 : \cdots : X^*X_b \otimes X'\bar{X}_b\right) = v^2$$

$$\Leftrightarrow \text{Rank}\left(X_1 \otimes \bar{X}_1 : X_2 \otimes \bar{X}_2 : \cdots : X_b \otimes \bar{X}_b\right) = v^2.$$

References

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