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CATENARY TURNPIKES AND ROLLING PLANS: SYNTHESIS AND EXTENSIONS

By
DIPANKAR DASGUPTA
(Indian Statistical Institute, New Delhi)

1. Introduction

This paper has three objectives.

1. First, in the context of finite horizon plans, we compare the Cass-Samuelson [2, 4] results concerning the catenary turnpike theorem for one sector neoclassical optimal growth models with Goldman's [3] results on rolling or continual revision of plans and show that the former set of results imply the latter. We do not pursue the reverse implication in this paper, although a little reflection will show that that is true also.

2. Secondly, we extend Goldman's results to a class of utility functions substantially broader than what he was concerned with and show that most of this results remain valid. (see page 8).

3. Lastly, we carry out the entire analysis in discrete time and by invoking Bellman's principle of optimality we are able to provide particularly simple proofs of all the propositions involved.

In the next section we spell out the model and characterize the optimal policy functions. Many of the results of this section are well-known. However, for the sake of completeness we collect the proofs of these results in section 4. Section 3 connects up the catenary turnpike theorem with the results concerning rolling plans. While otherwise selfcomplete, this section makes use of the results stated in section 2.

2 The Model and Some Preliminary Results

The growth model considered is of the simplest neoclassical variety. Output is a function of capital and labour, where the production function is linear and homogeneous. Formally stated.

$$Y_t = C_t + (K_{t+1} - K_t) = F(K_t, L_t) \qquad (2\cdot1)$$

where,

Y_t is the level of output in period t;

C_t is the level of consumption in period t;

K_t is the capital stock at the beginning of period t;

$(K_{t+1} - K_t)$ is the investment in period t;

L_t is the labour force in existence in period t.

Labour is assumed to grow at a constant exogeneously given rate n. Thus,

$$\frac{L_{t+1} - L_t}{L_t} = n. \tag{2·2}$$

From the assumption of linear homogeneity of F (K, L), we have

$$\frac{Y_t}{L_t} = \frac{C_t}{L_t} + \frac{K_{t+1}}{L_{t+1}} \cdot \frac{L_{t+1}}{L_t} - \frac{K_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right)$$

or, using (2·2),

$$y_t = c_t + (1 + n)k_{t+1} - k_t = f(k_t)$$

or

$$c_t = f(k_t) + k_t - (1 + n)k_{t+1} \tag{2·3}$$

where,

$$y_t = \frac{Y_t}{L_t},\ c_t = \frac{C_t}{L_t},\ k_t = \frac{K_t}{L_t},\quad f(k_t) = F\left(\frac{K_t}{L_t}, 1\right).$$

The following assumptions are made on f (k) :

A (i) $f'(k) > 0,\ f''(k) < 0.$

A (ii) $f'(k) \to +\infty$ as $k \to 0$; $f'(k) \to 0$ as $k \to +\infty$.

A (iii) $f(0) = 0.$

The welfare function of the society is assumed to take the form

$$\sum_{t=1}^{T} \alpha^{t-1} u(c_t) \tag{2·4}$$

where,

u (c) is the stationary per period utility function, representing utility as a function of per capita consumption;

α is a constant, representing discounting of future utilities;

T is the length of the horizon.

The assumptions made with regard to preferences are

B (i) $u'(c) > 0,\ u''(c) < 0$

B (ii) $u(c) \to -\infty$ as $c \to 0$, $u(c) \to +\infty$ as $c \to +\infty$

B (iii) $1 > \alpha > 0$

B (iv) $1 < T < +\infty$.

The planner's task is to choose $\{k_2, k_3, \ldots, k_T\}$ or equivalently $\{c_1, c_2, \ldots, c_T\}$ such that (2·4) is maximised subject to (2·3) for given values of the initial and terminal capital labour rations $k_1 = \underline{k}$ and $k_{T+1} = \bar{k}$ respectively. In what follows, we shall always assume that $\bar{k}$ is attainable from $\underline{k}$. When $\bar{k} = 0$, the assumption will be unnecessary.

It is clear that the optimum value of (2·4) will be a function of $\underline{k}$ and $\bar{k}$. For the T—period problem, let $w_T(\underline{k}; \bar{k})$ be this optimum value. Also, let the corresponding optimum paths of per capita consumption and capital-labour ratio be $\{c_t^T\}_{t=1}^T$, $\{k_t^T\}_{t=1}^T$, where $k_1^T = \underline{k}$. Then, from Bellman's principle of optimality, we have

$$w_T(\underline{k}; \bar{k}) = \max_{k_2} \left\{ u\left(f(\underline{k}) + \underline{k} - (1+n)k_2\right) + \alpha\, w_{T-1}(k_2; \bar{k}) \right\} \quad (2{\cdot}5)$$

The value of k_2 solving (2·5) is k_2^T by definition.

The following two results are of fundamental importance.

Lemma 1 : $w_T(\underline{k}; \bar{k})$ is a strictly concave function of $\underline{k}$, for all $T > 1$.

Lemma 2 : The value of k_2 maximizing $u(f(\underline{k}) + \underline{k} - (1+n)k_2) + \alpha\, w_{T-1}(k_2; \bar{k})$ in (2·5) is unique.

From Lemma 2 it follows that there exists a unique, single-valued optimal policy function $g_1^T(\underline{k}; \bar{k})$ (say) such that $k_2^T = g_1^T(\underline{k}; \bar{k})$, which for any specified value of $\bar{k}$, gives k_2^T as a function of $\underline{k}$. Similarly, there exist optimal policy functions $g_t^T(k_t^T; \bar{k})$, $t = 2, \ldots, T-1$ which give k_{t+1}^T as a function of k_t^T for given values of $\underline{k}$ and $\bar{k}$. The principle of optimality implies that whereas $g_1^T(\underline{k}; \bar{k})$ is the first period optimum policy function for (2·5), $g_2^T(k_2^T; \bar{k})$ is the first period optimum policy function for the welfare function $\sum_{t=2}^{T} \alpha^{t-2} u(c_t)$, with starting and ending stocks given by k_2^T and $\bar{k}$. Similar observations hold for $g_3^T(k_3^T; \bar{k})$ and so on. It is obvious that the process of capital accumulation takes place through a sequential application of these functions.

Using (2·3), we can now define the related policy functions for consumption. Thus, the existence of $g_t^T(k_t^T; \bar{k})$, $t = 1, 2, \ldots, T-1$

implies the existence of unique, single-valued, optimal policy function $h_t^T(k_t^T;\bar{k})$, $t = 1,\dots,T$ such that $c_t^T = h_t^T(k_t^T;\bar{k})$.

The following sequence of results may be easily established.

Lemma 3 : If the sequences $\{k_t^T\}_{t=1}^T$ and $\{c_t^T\}_{t=1}^T$ are optimal, then for $1 \leqslant t \leqslant T-1$

$$\frac{(1+n)\,u'(c_t^T)}{\alpha\, u'(c_{t+1}^T)} = 1 + f'(k_{t+1}^T) \tag{2·6}$$

or equivalently,

$$\frac{(1+n)\,u'(h_t^T(k_t^T;\bar{k}))}{\alpha\, u'(h_{t+1}^T[g_t^T(k_t^T;\bar{k});\bar{k}])} = 1+f'(g_t^T(k_t^T;\bar{k})) \tag{2·6$'$}$$

Conditions (2·6) and (2·6)′ will, henceforth, be referred to as the intertemporal optimality conditions.

Lemma 4 : $g_t^T(k_t^T;\bar{k})$ and $h_t^T(k_t^T;\bar{k})$ are strictly monotone increasing functions of k_t^T

Let $\underline{k}_t$ be the minimum value of k_t from which $\bar{k}$ is attainable if planning starts in period $t < T$. If $\bar{k} = 0$, $\underline{k}_t = 0$ also.

Lemma 5 : $g_t(k_t^T;\bar{k})$ and $h_t^T(k_t^T;\bar{k})$ are continuous functions of of k_t^T for $k_t^T \geqslant \underline{k}_t$, where the inequality becomes strict if $\bar{k} > 0$.

The modified golden rule capital-labour ratio k^*, which occupies a central position in the literature on infinite horizon plans satisfies the equation

$$1 + f'(\dot{k}) = \frac{1+n}{\alpha} \tag{2·7}$$

We do not provide a proof of this result in this paper. It can, however, be established by using the optimality condition developed in Brock and Gale [1].

Finally we note that the neo-classical growth model rules out unbounded accumulation of capital. The result is a consequence of assumption A (i).

Lemma 6 : Given any k_1, let $\{k_t\}_{t=1}^{\infty}$, $\{c_t\}_{t=1}^{\infty}$ satisfy (2·3). Then, there exists a $\hat{k} < +\infty$, such that if $k_1 < \hat{k}$, $k_t \leqslant \hat{k}$, for all t.

3. Turnpike Theorems Versus Rolling Plans

Goldman, in his treatment of rolling plans, recognized that with the passage of time, the planner's interest is naturally extended beyond the year "T". More people come to exist at the beginning of period 2 than were existing at the beginning of period 1. If nothing else, simply to take account of the interests of these people, the planner may wish to extend his horizon beyond the year "T". Goldman considers a special case of this problem by introducing a planner, who at the termination of each year within the horizon, adds another year to the horizon, itself. Thus at the beginning of period 2, the planner's welfare function transforms into

$$\sum_{t=2}^{T+1} \alpha^{t-2}\, u(c_t) \tag{3·1}$$

This process is assumed to be repeated in each period. Therefore, at the beginning of period, θ the planner's welfare function is

$$\sum_{t=\theta}^{T+\theta-1} \alpha^{t-\theta}\, u(c_t) \tag{3·2}$$

Goldman starts with the case where, although the planner keeps changing his welfare function every year, the terminal target he wishes to achieve remains the same. Thus, $k_{T+1} = k_{T+2} = \ldots = k_{T+\theta} = \ldots = \bar{k}$. In what follows, we shall always assume that $\bar{k} > 0$.

At the beginning of every year, the planner is faced with a T–period plan, the solution to which can be obtained by a sequential application of the functions $g_1^T(\cdot), g_2^T(\cdot), \ldots, g_{T-1}^T(\cdot)$. However, it is only the function $g_1^T(.)$ that is used in practice, since, at the beginning of any year the planner is faced with a horizon which is exactly the horizon he faced at the beginning of the previous year. Hence, in order to study the nature of capital accumulation under this scheme, one merely has to characterize the properties of $g_1^T(\cdot)$ in detail. For the sake of notational convenience, the capital-labour ratio at the beginning of any period will be denoted by k_1. Thus, at $t = 1$, $k_1 = \underline{k}$. Similarly, given k_1 and $\bar{k}$, the choice of next period's capital-labour ratio via $g_1^T(.)$ will be denoted k_2. Unless otherwise specified, we shall stick to this notation for the remainder of the paper.

We may now state Goldman's results. First, he proves that if the terminal target ratio lies below the modified golden age k^*, revision of plans leads to a unique stationary state in the limit for any (finite) time horizon. Moreover, the value of the stationary state lies strictly above the terminal target. If, on the other hand, the terminal target is above k^*, revision of plans leads to a stationary state below the terminal target. Secondly, an increase in the time horizon increases the value of the stationary state to which the system converges in the first case and reduces it in the second. Thirdly, if the planner decides to leave a capital-labour ratio at the end of the horizon which is exactly equal to the value of the initial capital-labour ratio at each stage, revision of plans leads the system to k^*. At this stage, it is worth noting that apart from B (i) and D (ii), Goldman assumes

$$(\text{Bv}) \; -\frac{u'(c)}{u''(c)} \text{ is a decreasing function of c.}$$

In what follows, we do not use (Bv). Precisely two results of Goldman fail to hold in the absence of (Bv). First, the uniqueness of the stationary state cannot be established any longer. Secondly, in case $\bar{k} = 0$, a non-zero stationary state may even cease to exist. Since, however, $\bar{k}$ is rarely zero in a planning context, the non-existence of a stationary state in this case may not be a serious problem.

Cass and Samuelson on the other hand, are concerned with the once for all maximization of (2·4). Their results which we have referred to above as the catenary turnpike theorem, may be summarized in two propositions. First, the optimum path of k_t displays a catenary behaviour with respect to the modified golden rule k^*. In other words, whatever may be the initial and terminal restrictions, the optimal path arches towards k^*. Secondly, as T increases, the optimal path gets closer to k^*.

The stage is now set to study interconnections. Consider, to start with, $k_1 = \underline{k} < \bar{k} < k^*$. So long as $k_1 < \bar{k}$, one would usually expect k_2 to be larger than k_1. However, in specifying $k_1 < k^*$, we are appealing to one interesting feature of the turnipike theory. Suppose $k_1 = \bar{k}$. In this case, provided the optimum path has a catenary behaviour, $k_2 > k_1$ $(=\bar{k})$ must hold. Thus, for each T the function $g_1^T(k_1 ; \bar{k})$ may be expected to lie above the 45° – line (See Figure I) in the $k_1 - k_2$ plane for $k_1 \leqslant \bar{k}$. Theorem 1 establishes this result.

Theorem 1 : Let $k_1 \leqslant \bar{k}$, $k_1 < k^*$. Then $k_2 > k_1$.

Proof : The proof proceeds by induction. Consider a 2 period problem, where we maximize.

$$u(f(k_1) + k_1 - (1+n)k_2) + \alpha u(f(k_2) + k_2 - (1+n)\bar{k}).$$

Suppose, $k_2 \leqslant k_1$. The intertemporal optimality condition implies

$$\frac{(1+n)\,u'(f(k_1) + k_1 - (1+n)k_2)}{\alpha\, u'(f(k_2) + k_2 - (1+n)\bar{k})} = 1 + f'(k_2) \qquad (3\cdot3)$$

$k_2 < k_1 < k^*$ implies $1 + f'(k_2) > \frac{1+n}{\alpha}$,

since $1 + f'(k^*) = \frac{1+n}{\alpha}$.

But,

$$f(k_1) + k_1 - (1+n)k_2) > f(k_2) + k_2 - (1+n)\bar{k},$$

since, $k_2 < k_1 < \bar{k}$.

$$\therefore\ u(f(k_1) + k_1 - (1+n)k_2 < u'(f(k_2) + k_2 - (1+n)\bar{k})$$

$$\therefore\ \frac{(1+n)u'(f(k_1) + k_1 - (1+n)k_2)}{\alpha u'(f(k_2) + k_2 - (1+n)\bar{k})} < \frac{1+n}{\alpha}$$

Hence, (3·3) gives a contradiction. Therefore, $k_2 > k_1$.

Assume now that the result holds for a T-period probelm. We shall show that it must hold for a T + 1-period problem also. We have

$$w_{T+1}(k_1; \bar{k}) = u(f(k_1) + k_1 - (1+n)k_2)$$
$$+ \alpha u(f(k_2) + k_2 - (1+n)k_3) + \alpha^2 w_{T-1}(k_3; \bar{k}).$$

Assume, $k_2 < k_1$. From our assumption, then, $k_3 > k_2$, using the principle of optimality. From the intertemporal optimality condition, however,

$$\frac{(1+n)u'(f(k_1) + k_1 - (1+n)k_2)}{\alpha u'(f(k_2) + k_2 - (1+n)k_3)} = 1 + f'(k_2) \qquad (3\cdot 4)$$

It is easily shown that the left hand side of (3·4) is strictly less than $\frac{1+n}{\alpha}$ and the right hand side is strictly greater than $\frac{1+n}{\alpha}$, which is a contradiction.

Therefore, $k_2 > k_1\ \forall\ T$, where T is the time horizon involved.

Q. E. D.

For each T and $\bar{k} < k^*$ the function $g_1^T(k_1; \bar{k})$, when plotted against k_1, will behave as shown in Figure I. Since, $\bar{k} > 0$, and T is strictly finite, $-g_1^T(k_1; \bar{k})$ will not exist at $k_1 = 0$, as in that case. $\bar{k}$ is not attainable from k_1. Whenever $g_1^T(k_1; \bar{k})$ exists and $k_1 < \bar{k}$, $g_1^T(k_1; \bar{k}) > k_1$, which follows from Theorem 1. As shown in Figure I, $g_1^T(k_1; \bar{k})$ lies strictly above the 45°-line for $k_1 < \bar{k}$. Hence, any fixed point of the function $g_1^T(k_1; \bar{k})$, if it exists, must be larger than $\bar{k}$. That such fixed points must exist follows from the fact that $g_1^T(k_1; \bar{k})$ is continuous (Lemma 5) and that it must eventually lie below the 45°-line (Lemma 6). The fixed point, however, may not be unique.

The process of capital accumulation and its limiting properties can now be demonstrated. The arrows in Figure I illustrate the path of accumulation starting from some $k_1 = \underline{k} < k^*$, when $\bar{k} < k^*$ also. The important point to note is that since $g_1^T(k_1; \bar{k})$ starts above the 45°-line and eventually lies below it, the accumulation process must lead to a steady state. Moreover, for $\bar{k} < k^*$, the steady state must lie above $\bar{k}$. This shows that the first part of the catenary turnpike theorem implies the first Goldman result summarized above. Analogous results are easily established for the case where $\bar{k} > k^*$.

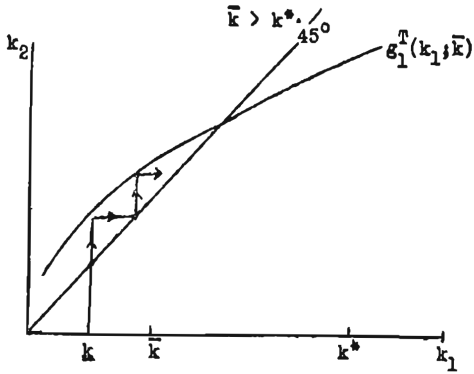

Figure I

Next, we go on to study the effect of an increasing time horizon. In particular, suppose $k_1 < \bar{k} < k^*$. Let $k_2^{T+1} = g_1^{T+1}(k_1; \bar{k})$ and $k_2^T = g_1^T(k_1; \bar{k})$. From Theorem 1 we know that $k_2^{T+1} > k_1$ as well as $k > k_1$ What is the relationship between k_2^{T+1} and k_2^T ? The second part of catenary turnpike theorem tells us that $k^* > k_2^{T+1} > k_2^T$ since with increasing time horizon the optimum path moves closer to k^*. In general, we may prove the following two results.

Theorem 2 : In any T-period plan, along the optimal path,

$k_t \leqslant k^*$ for $t \leqslant T$, if $\bar{k}_1, \bar{k} \leqslant k^*$.

Proof : The intertemporal optimality conditions for any two adjacent periods are

$$\frac{(1+n)\,u'(f(k_t)+k_t-(1+n)k_{t+1})}{\alpha\, u'(f(k_{t+1})+k_{t+1}-(1+n)k_{t+2})} = 1+f'(f_t+1) \quad (3.5)$$

$$\frac{(1+n)\,u'(f(k_{t+1})+k_{t+1}-(1+n)k_{t+2})}{\alpha\, u'(f(k_{t+2})+k_{t+2}-(1+n)k_{t+3})} = 1+f'(k_{t+2}) \quad (3\cdot 6)$$

Suppose, the theorem is false. Then, let k_{t+1} be the first element to exceed k^*. Then, from (3.5), since $1+f'(k_{t+1}) < \frac{1+n}{\alpha}$:

$$u'(f(k_t)+k_t-(1+n)k_{t+1}) < u' f(k_{t+1}) + k_{t+1}-(1+n)k_{t+2})$$

$$\therefore f(k_t)+k_t-(1+n)k_{t+1} > f(k_{t+1})+k_{t+1}-(1+n)k_{t+2} \quad (3.7)$$

But, $k_t < k_{t+1}$, by assumption.

Thus, $k_{t+1} < k_{t+2}$, from (3.7).

Repeating the same argument, from (3.6),

$k_{t+2} < k_{t+3}$ and so on...

Consider now,

$$\frac{(1+n)u'(f(k_{T-1})+k_{T-1}-(1+n)k_T)}{\alpha\, u'(f(k_T)+k_T-(1+n)\bar{k})} = 1+f'(k_T) \quad (3.8)$$

which is the optimality condition for the last period.

We have,

$$f(k_{T-1})+k_{T-1} < f(k_T)+k_T$$

and $k_T > \bar{k}$, since $k_T > k^*$ and $\bar{k} \leqslant k^*$.

$$\therefore f(k_{T-1})+k_{T-1}-(1+n)k_T < f(k_T)+k_T-(1+n)\bar{k}$$

$$\therefore \frac{(1+n)\,u'(f(k_{T-1})+k_{T-1}-(1+n)k_T)}{\alpha\, u'(f(k_T)+k_T-(1+n)\bar{k}} > \frac{1+n}{\alpha}$$

But, $1+f'(k_T) < \frac{1+n}{\alpha}$, since $k_T > k^*$. Therefore, (3.8) is contradicted.

Q. E. D.

Theorem 3 : Let $\bar{k} < k^*$. Then, there exists a unique $\bar{k}_1 > 0$, such that $k_2^{T+1} < k_2^T$ if $k_1 < \bar{k}_1$, and $k_2^{T+1} > k_2^T$ if $k_1 > \bar{k}_1$, where $k_2^{T+1} = g_1^{T+1}(k_1; \bar{k})$, $k_2^T = g_1^T(k_1; \bar{k})$.

Proof : Note first of all that given any $\bar{k}$, there exists a minimum value of k_1 from which $\bar{k}$ is attainable by pure accumulation of capital in T periods. Let $\underline{k}_1$ be this value. If the initial stock is $\underline{k}_1$ $k_2^T = \frac{f(\underline{k}_1) + \underline{k}_1}{1+n}$, from (2·3) by putting $c_1^T = 0$. Clearly, for any k_1, $g_1^T(k_1; \bar{k})$ cannot lie below $\frac{f(\underline{k}_1) + \underline{k}_1}{1+n}$, or else, $\bar{k}$ will not be attainable from k_2^T. On the other hand, for the T + 1 – period problem, $\bar{k}$ is attainable from $\underline{k}_1$, allowing positive consumption in every period, since we have one additonal period to work with in this case. Thus, the optimal T + 1 – period programme exists for $k_1 = \underline{k}_1$. Moreover $g_1^{T+1}(\underline{k}_1; \bar{k}) < \frac{f(\underline{k}_1) + \underline{k}_1}{1+n}$ or else, $c_1^{T+1} = 0$, which is ruled out by Assumption B (ii). Appealing to the continuity of $g_1^{T+1}(k_1; \bar{k})$ for k_1 close enough to $\underline{k}_1$ (from the right), $g_1^{T+1}(k_1; \bar{k})$ must still be less than $\frac{f(\underline{k}_1) + \underline{k}_1}{1+n}$. On the other hand, for k_1 very close to $\underline{k}_1$ (from the right), $g_1^T(k_1; \bar{k})$ must lie above $\frac{f(\underline{k}_1) + \underline{k}_1}{1+n}$.

Hence, for k_1 close to $\underline{k}_1$, $g_1^{T+1}(k_1; \bar{k}) < g_1^T(k_1; \bar{k})$.

Consider now the zone for which $k_1 > \bar{k}$. In this zone it can be easily shown that $k_2^{T+1} > k_2^T$ Assume the contrary. Consider the first period intertemporal optimality conditions for the T and T + 1 – period problems respectively. We have

$$\frac{(1+n)u'(f(k_1) + k_1 - (1+n)k_2^T)}{\alpha u'(f(k_2^T) + k_2^T - (1+n)k_3^T} = 1 + f'(k_2^T) \quad (3·9)$$

and

$$\frac{(1+n)u'(f(k_1) + k_1 - (1+n)k_2^{T+1})}{\alpha u'(f(k_2^{T+1}) + k_2^{T+1} - (1+n)k_3^{T+1})} = 1 + f'(k_2^{T+1}) \quad (3·10)$$

By assumption,

$$1 + f'(k_2^{T+1}) > 1 + f'(k_2^T)$$

Hence,

$$\frac{u'(f(k_1) + k_1 - (1+n)k_2^{T+1})}{u'(f(k_2^{T+1}) + k_2^{T+1} - (1+n)k_3^{T+1})} > \frac{u'(f(k_1) + k_1 - (1+n)k_2^T}{u'(f(k_2^T) + k_2^T - (1+n)k_3^T} \quad (3.11)$$

But, $f(k_1) + k_1 - (1+n)k_2^{T+1} > f(k_1) + k_1 - (1+n)\, k_2^T$.

Therefore, $u'(f(k_1) + k_1 - (1+n)k_2^{T+1} \leq u'(f(k_1) + k_1 - (1+n)k_2^T)$

Hence, $u'(f(k_2^{T+1}) + k_2^{T+1}(1+n)k_3^{T+1}) \leq u'(f(k_2^T) + k_2^T - (1+n)\, k_3^T)$.

or, $f(k_2^{T+1}) + k_2^{T+1} - (1+n)\, k_3^{T+1} > f(k_2^T) + k_2^T - (1+n)\, k_3^T$

or, $k_3^{T+1} \leq k_3^T$.

Carrving on in this fashion, it can be easily shown that $k_{T+1}^{T+1} \leq \bar{k}$.

However, it follows from Theorem 1 that $g_t^T(\bar{k};\bar{k}) > \bar{k}$ for all T.

Thus, the monotonicity of $g_t^T(k_t^T;\bar{k})$ from Lemma 4 implies $g_t^T(k_t^T;\bar{k})\bar{k})$ $\bar{k}$ for all T when $k_1 > \bar{k}$. Thus, $k_{T+1}^{T+1} \leq \bar{k}$ is a contradiction.

Thus, $g_1^{T+1}(k_1;\bar{k})$ starts below and eventually lies above $g_1^T(k_1;\bar{k})$ when $\bar{k} < k^*$. Appealing to the continuity of the functions established in Lemma 5, it follows, therefore, that there exists some

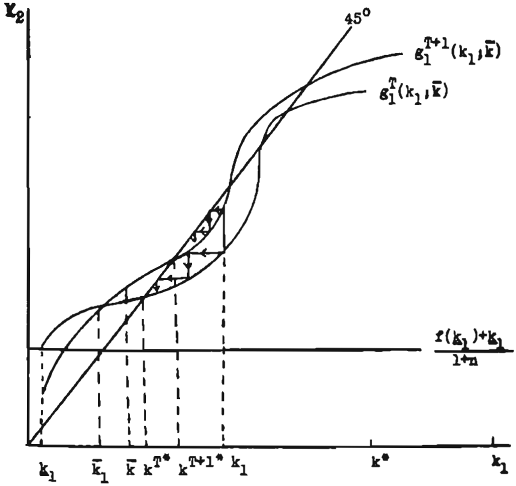

Figure 2

$\bar{k}_1 > \underline{k}_1$, such that $g_1^{T+1}(\bar{k}_1 ; \bar{k}) = g_1^T(\bar{k}_1 ; \bar{k})$. The last step in the proof of the theorem will be to show that $\bar{k}_1$ is unique.

Suppose, there exists $\bar{k}_1' > \bar{k}_1$ such that $g_1^{T+1}(\bar{k}_1' ; \bar{k}) = g_1^T(\bar{k}_1' ; \bar{k})$. Let the optimum $T+1$ – period path starting from $\bar{k}_1'$ be denoted by $\{k_t^{T+1'}\}_{t=1}^{T+1}$. Using repeatedly the fact that the right hand sides, and hence the left hand sides of (3·9) and (3·10) are equal, we have $k_{T+1}^{T+1'} = \bar{k}$. On the other hand, $\{k_t^{T+1}\}_{t=1}^{T+1}$, the optimum $T+1$ – period path starting from $\bar{k}_1$ also satisfies the condition $k_{T+1}^{T+1} = \bar{k}$. However, the $g_t^{T+1}(\cdot)$ functions being strictly monotone increasing, it follows then, $k_T^{T+1} = k_T^{T+1'}$ and $k_{T-1}^{T+1} = k_{T-1}^{T+1'}$, ..., $k_2^{T+1} = k_2^{T+1'}$. But $k_2^{T+1'} = g_1^{T+1'}(\bar{k}_1' ; \bar{k})$ and $k_2^{T+1} = g_1^{T+1}(\bar{k}_1 ; \bar{k})$. By assumption, however, $\bar{k}_1' > \bar{k}$. Hence, $k_2^{T+1'} > k_2^{T+1}$, which is a contradiction.

Q. E. D.

The behaviour of $g_1^{T+1}(k_1; \bar{k})$ vis-a-vis $g_1^T(k_1; \bar{k})$ is illustrated in Figure II. As the arrows indicated, the $T+1$ – period revised plan leads to a higher value of the stationary state as compared to the T-period stationary state. A rigorous proof of this statement is contained in Theorem 4. Starting from a common initial capital-labour ratio k_1, let $\{k_n^{T+1}\}$ and $\{k_n^T\}$, $n = 1,2....$ be the paths arising out of continual revision of a $T+1$ – period and T-period plan respectively. Suppose $k_n^{T+1} \longrightarrow k^{T+1*}$ and $k_n^T \longrightarrow k^{T*}$ as $n \longrightarrow \infty$. Then we have

Theorem 4 : Suppose the conditions of Theorem 3 are satisfied. Moreover, let $\{k_n^{T+1}\}$, $\{k_n^T\}$, k^{T+1*} and k^{T*} be as defined above. Then $k^{T+1*} > k^{T*}$.

Proof : Case (i). Let $k_1 > \bar{k}_1$, where $\bar{k}_1$ is as defined in Theorem 3. Then, by Theorem 3 and Lemma 4, $k_n^{T+1} > k_n^T \; \forall n$. Hence, $k^{T+1*} > k^{T*} > \bar{k}_1$. Applying Theorem 3 once again, the case where $k^{T+1*} = k^{T*}$ is ruled out.

Case (ii). Let $k_1 < \bar{k}_1$. In this case, using Theorem 1, there exists N such that $k_n^{T+1} > \bar{k}_1$, $\forall n > N$. The analysis of case (i) now applies.

Q. E. D.

Analogous results can be shown to hold for the case $\bar{k} > k^*$.

The connection between the results concerning the effect of increasing time horizon on continual revision of plans and that on catenary turnpike theorem is, therefore, again established.

Finally, Goldman discusses the assymptotic properties of revision of plans when $\bar{k} = k_1$ at every stage of the plan. That is to say, if at each point of time the planner intends to bequeath to the future exactly the capital labour ratio he started out with, then it is shown that the capital-labour ratio converges to k^* in the limit. This result may again be seen to be an implication of the catenary turnpike result. Take, for example, the case $\bar{k} = k_1 < k^*$. In this case, the catenary turnpike theorem tells us that $k_2 > k_1$. On the other hand, Theorem 2 implies $k_2 < k^*$. Repeating the same argument, we expect the above mentioned result of Goldman to be true. Theorem 5 proves the result.

Theorem 5 : Let $k_1 < k^*$. If $\bar{k} = k_1$ at each stage of the plan then the capital-labour ratio converges to the modified golden rule k^*.

Proof : Let the path of accumulation be denoted by the sequence $\{k_n\}$, i. e., $k_{n+1} = g_1^T(k_n ; k_n) \forall n$. From Theorem 1 and Theorem 2, $\{k_n\}$ is a monotone increasing sequence of real numbers which is uniformly bounded above by k^*. Hence, there exists k^{**} such that $k_n \longrightarrow k^{**}$ as $n \longrightarrow \infty$.

Suppose $k^{**} < k^*$. By Theorem 1 and lemma 5, there exists $\theta > 0$ such tath $| k_n - k^{**} | < \theta \Rightarrow k_{n+1} = g_1^T(k_n; k_n) > k^{**}$ which is a contradiction.

Q. E. D.

Once again, analogous result can be proved when $k_1 > k^*$.

4. Proofs of Lemmas in Section 2

Proof of Lemma 1 : Let $\underline{k}$ and $\underline{k}^*$ $(\underline{k} \neq \underline{k}^*)$ be two initial stocks giving rise to optimum per capita consumption streams $\{c_t\}_{t=1}^T$, $\{c_t^*\}_{t=1}^T$ and corresponding streams $\{k_t\}_{t=1}^T$, $\{k_t^*\}_{t=1}^T$ for the optimum capital-labour ratio. Choose $1 > \eta > 0$. We verify, first of all, that $\eta c_t + (1 - \eta) c_t^*$ is feasible. From strict concavity of $f(k)$,

$$f(\eta k_t + (1-\eta) k_t^*) > \eta f(k_t) + (1-\eta) f(k_t^*).$$

Thus,

$$\begin{aligned} & f(\eta k_t + (1-\eta)k_t^* + (\eta k_t + (1-\eta)k_t^*) \\ & \qquad -(1+n)(\eta k_{t+1} + (1-\eta)k_{t+1}^*) \\ > \; & \eta(f(k_t) + k_t - (1+n)k_{t+1}) + (1-\eta)(f(k_t^*) + k_t^* \\ & \qquad -(1+n)k_{t+1}^*) \\ = \; & \eta c_t + (1-\eta)c_t^*, \end{aligned} \tag{4.1}$$

since, $\{c_t\}_{t=1}^{T}$ and $\{c_t^*\}_{t=1}^{T}$ are individually feasible.

But (4·1) shows $\{\eta c_t + (1-\eta)c_t^*\}_{t=1}^{T}$ is feasible.

From the definition of $w_T(\underline{k};\bar{k})$,

$$w_T(\eta\underline{k} + (1-\eta)\underline{k}^*, \bar{k}) = \sum_{t=1}^{T} \alpha^{t-1}[u(\eta c_t + (1-\eta)c_t^*)].$$

But, $u(\cdot)$ is strictly concave. Hence,

$$w_T(\eta\underline{k} + (1-\eta)\underline{k}^*; \bar{k}) > \eta\sum_{t=1}^{T}\alpha^{t-1}u(c_t) + (1-\eta)\sum_{t=1}^{T}\alpha^{t-1}u(c_t^*)$$

$$= \eta w_T(\underline{k};\bar{k}) + (1-\eta)w_T(\underline{k}^*;\bar{k}).$$

Therefore, $w(\underline{k};\bar{k})$ is a strictly concave function of k.

Q. E. D.

Proof of Lemma 2 : The result is true if

$$u(f(\underline{k}) + \underline{k} - (1+n)k_2) + \alpha w_{T-1}(k_2;\bar{k})$$

is a strictly concave function of k_2. From Lemma 1, $w_{T-1}(k_2;\bar{k})$ is a strictly concave function of k_2. Also, differentiating $u(f(\underline{k}) + \underline{k} - (1+n)k_2)$ twice with respect to k_2, we have

$$(1+n)^2 u''(f(\underline{k}) + \underline{k} - (1+n)k_2) < 0.$$

Hence, $u(f(\underline{k}) + \underline{k} - (1+n)k_2) + \alpha w_{T-1}(k_2;\bar{k})$ is a strictly concave function of k_2.

Q. E. D.

Proof of Lemma 3 : If $\{k_t^T\}_{t=1}^{T}$ is optimal, then it must maximise $u(c_t^T) + \alpha u(c_{t+1}^T)$ among all programmes $\{k_t^{T'}\}_{t=1}^{T}$ such that $k_t^{T'} = k_t^T$ and $k_{t+2}^{T'} = k_{t+2}^T$. Thus, k_{t+1}^T maximises

$$u(f(k_t^T) + k_t^T - (1+n)k_{t+1}^T) + \alpha u(f(k_{t+1}^T) + k_{t+1}^T - (1+n)k_{t+2}^T)$$

The maximum must be an interior one, or else, either $c_t^T = 0$, or, $c_{t+1}^T = 0$, which is ruled out since $u(c) \longrightarrow -\infty$, as $c \longrightarrow 0$. Therefore,

$$-(1+n)u'(f(k_t^T)+k_t^T-(1+n)k_{t+1}^T)$$
$$+\alpha u'(f(k_{t+1}^T)+k_{t+1}^T-(1+n)k_{t+2}^T)(1+f'(k_{t+1}^T))=0$$
$$\text{or}\quad \frac{(1+n)u'(c_t^T)}{\alpha u'(c_{t+1}^T)}=1+f'(k_{t+1}^T)$$

Now, put $c_t^T=h_t^T(k_t^T;\bar{k})$, $k_{t+1}^T=g_t^T(k_t^T;\bar{k})$.

Q. E. D.

Proof of Lemma 4 : It is sufficient to prove the lemma for $g_1^T(\cdot)$ and $h_1^T(\cdot)$. We first show $g_1^T(k_1^T;\bar{k})$ is strictly monotone increasing in k_1^T. Assume to start with, $h_1^T(k_1^T;\bar{k})$ is strictly monotone increasing in k_1^T. We know that the optimum solution must satisfy

$$(1+n)\ u'\ (h_1^T(k_1^T;\bar{k})=\alpha.\ \frac{dw_{T-1}(g_1^T((k_1^T;\bar{k});\bar{k})}{dk_1^T} \tag{4.2}$$

For an increase in k_1^T, by assumption $h_1^T(k_1^T;\bar{k})$ rises, and hence, the left hand side of (4.2) falls. If $g_1^T(k_1^T;\bar{k})$ is non-increasing in k^T, the right hand side of (4.2) will be non-decreasing in k_1^T, by concavity of $V_{T-1}(\cdot)$. Hence, $g_1^T(k_1^T;\bar{k})$ must be strictly increasing in k_1^T if $h_1^T(k_1^T;\bar{k})$ is strictly increasing in k_1^T also. Assume, therefore $h_1^T(k_1^T;\bar{k})$ is non-increasing in k_1^T. From (2.3),

$$h_1^T(k_1^T;\bar{k})+(1+n)g_1^T(k_1^T;\bar{k})=f(k_1^T)+k_1^T \tag{4.3}$$

With an increase in k_1^T, the right hand side of (4.3) goes up and thus, if $h_1^T(k_1^T;\bar{k})$ is non-increasing, $g_1^T(k_1^T;\bar{k})$ must be strictly monotone increasing in k_1^T.

We now show, $h_1^T(k_1^T;\bar{k})$ cannot have any non-increasing portion. Assume the contrary, and let $h_1^T(k_1^T;\bar{k})$ be non-increasing for $\underline{k}_1<k_1^T<\bar{k}_1$. Consider $k_1^{T'}$ such that $\underline{k}_1<k_1^T<k_1^{T'}<\bar{k}_1$. and the corresponding optimal paths $\{c_t^{T'}\}_{t=1}^T$ and $\{k_t^{T'}\}_{t=1}^T$. By assumption $c_1^{T'}<c_1^T$. However, $w_1(k_1^T;\bar{k})>w_1(k_1^T;\bar{k})$ and hence, there must exist some $t>1$ such that $c_{t+1}^{T'}>c_{t+1}^T$ and $c_t^{T'}<c_t^T$. On the other hand, since $g_t^T(k_t^T;\bar{k})$ is strictly monotone increasing in k_t^T, $k_t^{T'}>k_t^T$ for all $1\leq t<T$. Consider now, the intertemporal optimality condition for c_t^T, c_{t+1}^T. We have,

$$(1+n)u'(c_t^T) = \alpha u'(c_{t+1}^T)(1+f'(k_{t+1}^T)) \tag{4.4}$$

Similarly, for $c_t^{T'}, c_{t+}^{T'}$ we have,

$$(1+n)u'(c_t^{T'}) = \alpha u'(c_{t+1}^{T'})(1+f'(k_{t+1}^{T'})) \tag{4.5}$$

Since, $c_t^{T'} < c_t^T$, $(1+n)u'(c_t^{T'}) > (1+n)u'(c_t^T)$.
However, $c_{t+1}^{T'} > c_{t+1}^T$ and $k_{t+1}^{T'} > k_{t+1}^T$ imply

$$\alpha u'(c_{t+1}^{T'})(1+f'(k_{t+1}^{T'})) < \alpha u'(c_{t+1}^T)(1+f'(k_{t+1}^T))$$

Thus, (4·4). and (4.5) cannot hold together. Therefore, $h_1^T(k_1^T;\bar{k})$ is strictly monotone increasing.

Q. E. D.

Proof of Lemma 5 : It is sufficient to prove the lemma for $t = 1$. Let $\underline{k}_1$ be the minimum k_1 from which $\bar{k}$ is attainable in T−period. If one starts from $\underline{k}_1$, $c_t^T = 0$ for all t, i. e., all output has to be reinvested. Hence, for $k_1 = \underline{k}_1$, the optimum path will not exist. However, it clearly exists for $k_1 > \underline{k}_1$. Assume now, that $g_1^T(k_1;\bar{k})$ is discontinuous at $k_1 = \bar{k}_1 > \underline{k}_1$. Consider two sequences $\{k_1^n\}$ and $\{k_1^{n'}\}$ going to $\bar{k}_1$ from below and above respectively. Let $g_1^T(k_1^n;\bar{k}) \longrightarrow g_1^T(\bar{k}_1;\bar{k})^+$ and $g_1^T(k_1^{n'};\bar{k}) \longrightarrow g_1^T(\bar{k}_1;\bar{k})^-$. Both limits must exist, since $g_1^T(\cdot)$ is monotone increasing and uniformly bounded above and below (see Lemma 6). For similar reasons, let $h_1^T(k_1^n;\bar{k}) \longrightarrow h_1^T(\bar{k}_1;\bar{k})^+$ and $h_1^T(k_1^{n'};\bar{k}) \longrightarrow h_1^T(\bar{k}_1;\bar{k})^-$. However, $\{f(k_1^n)+k_1\}$ and $\{f(k_1^{n'})+k_1^{n'}\}$ must both converge to $f(\bar{k}_1)+\bar{k}_1$, since $f(k)$ is continuous. Thus

$$h_1^T(\bar{k}_1;\bar{k})^+ + (1+n)g_1^T(\bar{k}_1;\bar{k})^+ = f(\bar{k}_1)+\bar{k}_1$$

and

$$h_1^T(\bar{k}_1;\bar{k})^- + (1+n)\,g_1^T(\bar{k}_1;\bar{k})^- = f(\bar{k}_1)+\bar{k}_1 .$$

By subtraction,

$$[h_1^T(\bar{k}_1;\bar{k}) - h_1^T(\bar{k}_1;\bar{k})^-] + (1+n)[g_1^T(\bar{k}_1;\bar{k})^+ - g_1^T(\bar{k}_1;\bar{k})^-] = 0 \tag{4·6}$$

By assumption and monotonicity of $g_1^T(\cdot)$
$[g_1^T(\bar{k}_1;\bar{k})^+ - g_1^T(\bar{k}_1;\bar{k})^-] > 0$ and monotonicity of $h_1^T(\cdot)$ implies $[h_1^T(\bar{k}_1;\bar{k})^+ - h_1^T(\bar{k}_1;\bar{k})^-] \geqslant 0$. Hence, (4·6) is violated. Therefore, $g_1^T(k_1;\bar{k})$ is continuous for all $k_1 > \underline{k}_1$.

If $\bar{k} = 0$ is allowed, them $\underline{k}_1 = 0$ and continuity of $g_1^T(k_1; 0)$ at $k_1 = 0$ is easily shown by noting that $g_1^T\ (k_1; 0) = 0$ for $k_1 = 0$ and that

$$g_1^T\ (k_1; 0) = \frac{f(k_1) + k_1 - h(k_1)}{1 + n}$$

goes to zero as $k_1 \longrightarrow 0$.

Similar methods show that $h_1^T\ (\cdot)$ is continuous also.

Q. E. D.

Proof of Lemma 6 : Define the path of pure capital accumulation $\bar{k}_t$ by letting $c_t = 0$ in (2·3). Then,

$$(1 + n)\,\bar{k}_{t+1} - \bar{k}_t = f(\bar{k}_t) \tag{4·7}$$

The sequence $\{\bar{k}_t\}$ obviously bounds from above any feasible accumulation path $\{k_t\}$. Assume, $\bar{k}_{t+1} > \bar{k}_t\ \forall\ t$ is possible. In particular, let us say, $\bar{k}_{t+1} = \bar{k}_t + \epsilon_t$, $\epsilon_t > 0\ \forall t$, and $\epsilon_t \neq 0$. Then, from (4·7),

$$(1 + n)\ (\bar{k}_t + \epsilon_t) - \bar{k}_t = f(\bar{k}_t)$$

or, $$n\bar{k}_t + (1 + n)\epsilon_t = f\ (\bar{k}_t) \tag{4·8}$$

But, by assumption, $f\ (k)$ is a strictly concave function of k and $f'(k) \longrightarrow 0$, as $k \longrightarrow \infty$. Hence, $\exists\ k' \ni \bar{k}_t > k'$ implies

$$n\bar{k}_t + (1 + n)\ \epsilon_t > f(\bar{k}_t)$$

But, by assumption, $\bar{k}_t > k'$ from some $t = t'$ onwards. Thus, the path of pure accumulation will eventually violate (4·8). Hence, $\exists\ \hat{k} \ni k_t < \hat{k}\ \forall\ t$.

Q. E. D.