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DISCOUNTED AND POSITIVE STOCHASTIC GAMES

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1. **Introduction.** The main purpose of this note is to announce a few results on stochastic games. A stochastic game is determined by five objects: S, A, B, q and r. S, A and B are nonempty Borel Subsets of Polish spaces and r is a bounded measurable function on $S\times A\times B$. We interpret S as the state space of some system and A, B as the set of actions available to players I and II respectively at each state. When the system is in state s and players I and II choose action a and b respectively, the system moves to a new state according to the distribution $q(\cdot|s, a, b)$ and I receives from II, $r(s, a, b)$ units of money. Then the whole process is repeated from the new state s'. The problem, then, is to maximize player I's expected income as the game proceeds over the infinite future and to minimize player II's expected loss.

A strategy π for player I is a sequence $\pi_1, \pi_2, \cdots$, where π_n specifies the action to be chosen by player I on the nth day by associating (Borel measurably) with each history

$$h = (s_1, a_1, b_1, \cdots, s_{n-1}, a_{n-1}, b_{n-1}, s_n)$$

of the system a probability distribution $\pi_n(\cdot|h)$ on the Borel sets of A. Call π a *stationary strategy* if there is a Borel map f from S to P_A, where P_A is the set of all probability measures on the Borel sets of A, such that $\pi_n=f$ for each $n\geqq 1$ and in this case, π is denoted by $f^{(\infty)}$. Strategies and stationary strategies are defined similarly for II.

Let β be any fixed nonnegative number satisfying $0\leqq\beta<1$. A pair (π, Γ) of strategies for I and II associates with each initial state s, a nth day expected income $r_n(\pi, \Gamma)(s)$ for I and a total expected discounted income

$$I_\beta(\pi, \Gamma)(s) = \sum_{n=1}^{\infty} \beta^{n-1} r_n(\pi, \Gamma)(s).$$

Such stochastic games are called discounted stochastic games. Positive stochastic games are those where $r(s, a, b)\geqq 0$ $\forall s, a, b$ and $\beta=1$.

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Call π^* optimal for I if

$$I_\beta(\pi^*, \Gamma)(s) \geqq \inf_{\Gamma} \sup_{\pi} I_\beta(\pi, \Gamma)(s)$$

for all Γ and s. Call Γ^* optimal for II if

$$I_\beta(\pi, \Gamma^*)(s) \leqq \sup_{\pi} \inf_{\Gamma} I_\beta(\pi, \Gamma)(s)$$

for all π and s. We shall say that the stochastic game has a value if

$$\sup_{\pi} \inf_{\Gamma} I_\beta(\pi, \Gamma)(s) = \inf_{\Gamma} \sup_{\pi} I_\beta(\pi, \Gamma)(s)$$

for all s.

The case where the stochastic game has a value, sup inf $I_\beta(\pi, \Gamma)(s)$, as a function of S is called the value function.

2. **Main results.** Throughout this paper the following assumptions unless otherwise stated will remain operative.

(i) S will be a complete separable metric space; A and B finite sets.

(ii) The following multifunctions are measurable.

$$r_w : S \to P_A \times P_B,$$

$$r_w(s) = \left\{(\mu', \lambda') : \max_{\mu} \left[r(s, \mu, \lambda') + \beta \int w(\cdot) dq(\cdot \mid s, \mu, \lambda') \right] = \min_{\lambda} \left[r(s, \mu', \lambda) + \beta \int w(\cdot) dq(\cdot \mid s, \mu', \lambda) \right] \right\}$$

where $w \in M(s) =$ space of bounded Borel measurable functions on S and $\beta \in [0, 1]$.

Now we are in a position to state our theorems.

THEOREM 1. *Let S be a complete separable metric space and A, B are finite sets. Further suppose the mulifunctions $\{r_w\}$ are measurable. Then the discounted stochastic game has a value and the value function is Borel measurable. Furthermore players* I *and* II *have optimal stationary strategies.*

THEOREM 2. *Let S be a complete separable metric space and A and B are compact metric. Suppose that, whenever $(a_n, b_n) \to (a_0, b_0)$ in $A \times B$, $r(s, a_n, b_n) \to r(s, a_0, b_0)$ and $\int w(\cdot) dq(\cdot \mid s, a_n, b_n) \to \int w(\cdot) dq(\cdot \mid s, a_0, b_0)$ for every s and w. Further assume that the multifunctions $\{r_w\}$ are measurable. Then the discounted stochastic game has a value and the value function is measurable. Also the two players have optimal stationary strategies.*

THEOREM 3. *Let S be a complete separable metric space and A, B are finite sets. Let $r(s, a, b) \geqq 0$ for all s, a and b. Suppose $I(\pi, \Gamma)(s) \leqq R$ for all π, Γ and s where R is a positive real number independent of π, Γ and s. If $\{r_n\}$ are measurable then the positive stochastic game has a value and the value function is measurable.*

REMARK 1. Proof of Theorem 1 (as well as the other two theorems) depends on a selection theorem that was proved recently by C. J. Himmelberg and F. S. Van Vleck (see [2, Theorem 4, p. 396]).

REMARK 2. We will not attempt to prove these theorems for they follow along similar lines to that of Theorem 4.1 in [3], where we assumed S, A, B are compact metric and $r(s, a, b)$, $q(\cdot \mid s, a, b)$ are continuous in $S \times A \times B$.

REMARK 3. In Theorem 3 we can also prove that the minimizing player has an optimal stationary strategy using a result of Blackwell on positive dynamic programming (see [1, Theorem 2, p. 416]). We have not been able to determine, whether or not, under our conditions, player I has an optimal stationary strategy.

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REFERENCES

1. D. Blackwell, *Positive dynamic programming*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965) vol. 1: Statistics, Univ. of California Press, Berkeley, Calif., 1967, pp. 415–418. MR 36 #1193.

2. C. J. Himmelberg and F. S. Van Vleck, *Some selection theorems for measurable functions*, Canad. J. Math. 22 (1969), 394–397. MR 38 #4637.

3. A. Maitra and T. Parthasarathy, *On stochastic games*, J. Optimization Theory Appl. 5 (1970), 289–300.

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