

TREES AND TREE-EQUIVALENT GRAPHS

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1. Introduction. As is well known in the theory of graphs a tree is a connected graph without cycles. Many characterizing properties of trees are known (1), for example the cyclomatic number is equal to zero, which is also equal to $p - 1$, where p is the number of connected components of the graph. The graphs with cyclomatic number equal to $p - 1$ are defined here as *tree-equivalent graphs*. A tree is always a tree-equivalent graph but not conversely. The properties of tree-equivalent graphs are studied here. It is shown that by an operation on tree-equivalent graphs one can obtain a tree without disturbing the set of local degrees. The existence of trees with given local degrees follows as a corollary of the existence of tree-equivalent graphs.

2. Definitions.

2.1. An unoriented graph G is defined whenever we have a set X of abstract elements and a set U of edges which are undirected curves joining some pairs of distinct elements of X . It is denoted by (X, U) .

2.2. If the maximum number of edges appearing in a graph which join the same two vertices is S , then the graph is called an S -graph.

2.3. For any vertex a , the subgraph of all vertices and edges that can be arrived at by travelling along the edges of the graph, including a , is called a *connected component* of G .

2.4. The number d_i of edges incident to a vertex x_i of G is called the *local degree* of G at x_i .

2.5. A *cycle* is a sequence of edges (u_1, \dots, u_q) such that

- (1) u_k is attached to u_{k-1} by one of its extremities and to u_{k+1} by the other for $1 < k < q$,
- (2) the initial extremity of u_1 coincides with the terminal extremity of u_q ,
- (3) no edge appears twice.

2.6. The *cyclomatic number* $K(G)$ of an S -graph G is defined to be $m - n + p$ where m is the number of edges, n is the number of vertices, and p is the number of connected components. This number also has the property that $K(G)$ linearly independent cycles exist in the graph.

2.7. Let $\{C_i\}$, $i = 1, 2, \dots, r$, and $\{T_j\}$, $j = 1, 2, \dots, s$, be the p connected

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components of an S -graph G without isolated vertices and let $s > 1$. If each C_i has at least one cycle, if each T_j is a tree, and if

$$K(G) = p - 1 \quad (= r + s - 1),$$

then G is said to be a *tree-equivalent graph*.

3. Properties of tree-equivalent graphs.

3.1. A *tree-equivalent graph does not have isolated vertices* (by definition).

3.2. The local degrees (d_1, d_2, \dots, d_n) of a *tree-equivalent graph satisfy the following*:

- (1) $1 < d_i < n - 1$,
- (2) $\sum_i d_i = 2(n - 1)$.

Proof. Since G is tree-equivalent, we have

$$m = k(G) + n - p = (p - 1) + n - p = n - 1;$$

but $\sum_i d_i = 2m$, since each edge contributes one count each to two of the d_i . Hence $\sum_i d_i = 2(n - 1)$.

By 3.1, each $d_i > 1$. There are exactly $n - 1$ edges in G , since $m = n - 1$ and almost $n - 1$ edges can be incident with any vertex. Therefore $d_i < n - 1$.

3.3. A *graph for which the above property 3.2 holds is a tree-equivalent graph*.

We need only prove that $K(G) = m - n + p = p - 1$, which is obviously true, and that there is at least one connected component which is a tree. But if each connected component contained a cycle, then each component would contain at least as many edges as vertices, whence it would follow that $m > n$, contradicting the hypothesis that $m = \frac{1}{2} \sum d_i = n - 1$. Therefore G has at least one component which is a tree, i.e. $s > 1$.

4. Existence of tree-equivalent graphs.

THEOREM 4.1. *A graph with n vertices with prescribed non-zero local degrees which add up to $2(n - 1)$ always exists.*

By 3.3, the existence of such a graph is equivalent to the existence of a tree-equivalent graph. For every tree-equivalent graph let us construct an incidence matrix $A = (a_{ij})$ of $n - 1$ rows and n columns such that

$$a_{ij} = \begin{cases} 1 & \text{if the edge } i \text{ is incident to the vertex } x_j, \\ 0 & \text{otherwise;} \end{cases}$$

here $i = 1, 2, \dots, n - 1$ are the numbered edges of the graph and $\{x_j\}$, $j = 1, 2, \dots, n$, are the vertices of the graph. The existence of a tree-equivalent graph is equivalent to the existence of an incidence matrix A with each row sum equal to two and the j th column sum equal to d_j . Thus we have to show the

existence of a matrix of 0's and 1's with prescribed row and column sums. This is a particular case of a problem solved by Ryser for which we refer to (2). From the majorization conditions given therein, it follows in our case that our matrix always exists, which in turn implies the existence of a tree-equivalent graph with prescribed local degrees.

THEOREM 4.2. *If $p > 1$, a tree-equivalent graph $p - 1$ components on n vertices can be obtained from a tree-equivalent graph of p components and n vertices without altering the local degrees of the graph.*

Proof. Let G be a tree-equivalent graph with $p (> 1)$ components. Then G has a component which contains a cycle. Let u be an edge of this cycle and v be a pendant edge of a component T_i which is a tree. If u joins the vertices a and b , and v joins c and d , then the removal of u and v and the insertion of an edge (a, c) and an edge (b, d) converts G into a new graph with the same local degrees and $p - 1$ components. This graph still remains a tree-equivalent graph by 3.3.

COROLLARY 4.1. *A tree with prescribed non-zero local degrees (d_1, d_2, \dots, d_n) exists if*

$$\sum_i d_i = 2(n - 1).$$

Proof. Given that d_i are non-zero and that $\sum d_i = 2(n - 1)$, a graph with local degrees d_i exists by Theorem 4.1 and it is tree-equivalent by 3.3. Therefore by Theorem 4.2, one can reduce the number of connected components step by step until one obtains a tree-equivalent graph with one component and local degrees d_i , that is, a tree with these local degrees.

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REFERENCES

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2. H. J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Can. J. Math., 9 (1957), 371-377.

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