CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of Czechoslovak Academy of Sciences
V. 24 (99), PRAHA 15. 11. 1974, No 4

DECOMPOSITIONS OF THE STATE SPACE, HOMOMORPHISMS AND PRODUCTS OF SEMIGROUP ACTS

KRIPASINDHU SIKDAR, Calcutta

(Received November 7, 1972)

1. INTRODUCTION

Let S be a (topological) semigroup and X a nonvoid T_2-space. Then an *act* [cf 4, 5, 6], denoted by the pair (X, S), is a continuous function $f : X \times S \to X$ such that $f(x, s_1 s_2) = f(f(x, s_1), s_2)$ for all $x \in X$ and all $s_1, s_2 \in S$. Throughout this paper, X and S, which are often termed as the *state space* and the *input semigroup*, respectively, will refer to an act (X, S) and $f(x, s)$ will be simply denoted by xs.

For $\emptyset \neq A \subseteq X$ and $\emptyset \neq T \subseteq S$, let $AT = \{xs : x \in A \text{ and } s \in T\}$ and $AT^{(-1)} = \{x : x \in X \text{ and } xT \cap A \neq \emptyset\}$. An *orbit* (*a point-inverse set*) is a set of the form xS ($xS^{(-1)}$) for some $x \in X$. An orbit is *maximal* if it is not properly contained in an orbit. A *minimal* orbit and a *maximal* (*minimal*) *point-inverse set* are analogously defined. An act (X, S) is *compact* if both X and S are so, and is *unitary* if $x \in xS$ for each $x \in X$. An act whose orbits, or maximal orbits (point-inverse sets) form a partition of the state space will be called a *quasi-transitive*, or *disjoint* (i-disjoint) act, respectively. For all other unexplained concepts concerning acts reference is made to DAY [4].

If S is a group and $XS = X$ the orbits partition X but if S is merely a semigroup various kinds of overlapping of orbits are possible. This paper results from an attempt to study semigroup acts from the above consideration and results concerning disjoint (and i-disjoint) acts, quasi transitive acts, how a homomorphism maps a maximal (minimal) orbit (point-inverse set), or a disjoint (i-disjoint) act onto a similar object, and how a product of acts inherit similar properties from the component acts, are presented in Sections 2, 3, 4 and 5, respectively. Some of these results were reported in [10].

2. DISJOINT (I-DISJOINT) ACTS

To start with let us state the following remarks without proof.

Remark 2.1. Let (X, S) be a compact act.

(a) Every orbit is contained in a maximal orbit [cf. 10, 1] and every orbit contains a minimal orbit. If $XS = X$, then the family F of maximal orbits form a minimal cover of X (i.e., $UF = X$ and no sub-family of F has this property).

(b) If the act is also unitary, then xS is a maximal (minimal) orbit iff $xS^{(-1)}$ is a minimal (maximal) point-inverse set. Consequently, statements similar to those in (a) hold good for maximal point-inverse sets.

Though, in general, an act need not be disjoint, the following is true.

Proposition 2.2. [cf. 10, 1]. *Let (X, S) be a compact act. Then there exists a disjoint act (X^*, S) whose homomorphic image is (X, S). Further, if the set $Y = \{x : xS$ is maximal orbit of $(X, S)\}$ is closed, then X^* is compact.*

The following gives several characterizations of disjoint acts.

Proposition 2.3. *Let (X, S) be a compact unitary act. Then the following statements are equivalent.*

(1) *The maximal orbits form a decomposition of X.*
(2) *For any distinct pair $x, y \in X$, $xS \cap yS \neq \emptyset$ implies that $xS^{(-1)} \cap yS^{(-1)} \neq \emptyset$.*
(3) *For any $\emptyset \neq A \neq B \subseteq X$, $AS \cap BS \neq \emptyset$ implies that $AS^{(-1)} \cap BS^{(-1)} \neq \emptyset$.*
(4) *Each point-inverse set contains a unique minimal point-inverse set.*
(5) *Each orbit is contained in a unique maximal orbit.*
(6) *Each maximal orbit is a union of maximal point-inverse sets.*
(7) *Each maximal orbit is a union of point-inverse sets.*
(8) *There exists a (unique) equivalence relation on X with closed graph such that each equivalence class is an orbit.*

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5). Easy.

(5) ⇒ (6). Suppose xS is a maximal orbit and $\{x_\alpha S\}$ are all the minimal orbits contained in xS. We claim that $\bigcup x_\alpha S^{(-1)} = xS$. If $y \in x_\alpha S^{(-1)}$, then $x_\alpha S \subseteq yS \subseteq xS$. So $y \in xS$. Conversely, if $y \in xS$, then $yS \subseteq xS$ and if $y'S$ is a minimal orbit contained in $yS \subseteq xS$, then $y \in yS^{(-1)} \subseteq y'S^{(-1)} \subseteq x_\alpha S^{(-1)}$

(6) ⇒ (7). Trivial.

(7) ⇒ (6). Let xS be a maximal orbit which is a union of point-inverse sets $\{x_\alpha S^{(-1)}\}$. Let $\{x^\alpha S^{(-1)}\}$ be all the maximal point-inverse sets in which one or more of $x_\alpha S^{(-1)}$ are contained. We claim that $xS = \bigcup x^\alpha S^{(-1)}$. Clearly, $xS \subseteq x^\alpha S^{(-1)}$ which contains some $x_\alpha S^{(-1)}$, as $x_\alpha S^{(-1)} \subseteq xS$, $xS^{(-1)} \subseteq x_\alpha S^{(-1)} \subseteq x^\alpha S^{(-1)}$. Hence,

for some $s \in S$, $xs = x^\alpha$. Thus $x^\alpha \in xS$ and for some x_β such that $x_\beta S^{(-1)} \subseteq xS$, $x^\alpha \in x_\beta S^{(-1)}$ so that $x^\alpha S^{(-1)} = x_\beta S^{(-1)}$ as $x^\alpha S^{(-1)}$ is maximal. Hence, $x^\alpha S^{(-1)} \subseteq xS$.

(6) $\Rightarrow$ (1). We first make two observations.

(a) For any two minimal orbits xS and yS, $xS \cap yS^{(-1)} \neq \emptyset$ iff $xS = yS$. For, if $xS \cap yS^{(-1)} \neq \emptyset$, then there exist $s, t \in S$ such that $xst = y$ and, hence, $xS = yS$.

(b) If a maximal orbit xS is a union of maximal point-inverse sets $\{x_\alpha S^{(-1)}\}$, then $\{x_\alpha S\}$ are indeed all the minimal orbits contained in xS. For, (a) implies that if yS is any minimal orbit contained in xS, then $ys = x_\alpha S^{(-1)} \neq \emptyset$ for some α, and so $yS = x_\alpha S$.

Now suppose x_1S and x_2S are any two maximal orbits which intersect. Then there exists a minimal orbit $yS \subseteq x_1S \cap x_2S$ and, hence, by (b), $yS^{(-1)} \subseteq x_1S \cap x_2S$. Let $zS^{(-1)}$ be a minimal point-inverse set contained in $yS^{(-1)}$. Then zS, a maximal orbit, is contained in $x_1S \cap x_2S$ as $z \in zS^{(-1)}$ and, therefore, $zS = x_1S = x_2S$.

(1) $\Rightarrow$ (8). Define $\varrho \subseteq X \times X$ as $x \varrho y$ if x and y are contained in the same maximal orbit. Then ϱ has the required properties.

(8) $\Rightarrow$ (1). If ϱ is such an equivalence relation then note that each equivalence class is a maximal orbit. For, let $[x]$ be an equivalence class containing x and suppose $[x] = xS$. If $xS \subseteq yS \subseteq [y]$, the equivalence class containing y, for some $y \in X$, then $x \in yS \subseteq [y]$ implies that $xS = [x] \subseteq [y]$. Then it follows that $[y] = [x]$ and, hence, $xS = yS$. Further, as ϱ has closed graph each equivalence class is closed and, hence, compact.

Remark 2.4. There exist analogous characterizations for i-disjoint acts.

A characterization of acts which are both disjoint and i-disjoint is the following

Proposition 2.5. *Let (X, S) be a compact unitary act. Then the following two statements are equivalent.*

(a) *(X, S) is both disjoint and i-disjoint.*

(b) *Each maximal orbit is a maximal point-inverse set and vice-versa.*

Proof. (a) $\Rightarrow$ (b). Let xS be a maximal orbit. Then, as (X, S) is i-disjoint, by virtue of Remark 2.4, if yS is the unique minimal orbit contained in xS, we claim that $xS = yS^{(-1)}$. If $z \in xS$, then $zS \subseteq xS$ and zS contains a unique minimal orbit which must be yS and, hence, $z \in yS^{(-1)}$. Conversely, if $z \in yS^{(-1)}$, then $yS \subseteq zS$, and, hence, as (X, S) is disjoint, by virtue of Proposition 2.3 (5), the unique maximal orbit in which zS is contained in must be xS. Therefore, $z \in xS$.

To prove that each maximal point-inverse set is a maximal orbit we can apply similar arguments.

(b) $\Rightarrow$ (a). Suppose two maximal orbits x_1S and x_2S intersect and suppose $y_1S^{(-1)}$ and $y_2S^{(-1)}$ are two maximal point-inverse sets which equal x_1S and x_2S, respectively.

There exists a minimal orbit $zS \subseteq x_1S \cap x_2S$ so that $zS \subseteq y_1S^{(-1)} \cap y_2S^{(-1)}$ which implies that both y_1 and y_2 are in zS and, therefore, equivalently, $y_1S = y_2S = zS$ as zS is minimal. Therefore, $x_1S = x_2S$ and, hence, (X, S) is disjoint.

Similarly, it can be shown that (X, S) is i-disjoint.

3. QUASI-TRANSITIVE ACTS

In this section acts for which any two distinct orbits are disjoint are studied. A semigroup S acts on X *point-transitively* if $xS = X$ for some $x \in X$, *quasi-transitively* if $XS = X$ and for any $x, y \in X$, $y \in xS$ implies that $x \in yS$ and *transitively* if $xS = X$ for all $x \in X$.

First, note that an act (X, S) is quasi-transitive iff it is unitary and each orbit is minimal as well as maximal, and is transitive iff it is point-transitive and quasi-transitive. Then some characterizations for quasi-transitive compact acts are stated below.

In what follows let K, E and R stand for the minimal ideal, the set of idempotents and any minimal right ideal of S respectively and $H(e)$ stand for the maximal subgroup of S containing $e \in E$.

Proposition 3.1. *Let (X, S) be a compact act. Then the following statements are equivalent.*

(1) *S acts quasi-transitively on X.*
(2) *The orbits form a decomposition of X (i.e., the orbits partition X and each orbit is closed).*
(3) *R acts unitarily on X.*
(4) *For each $e \in K \cap E$, $(Xe, H(e))$ is a topological transformation group and $\bigcup\{Xe : e \in R \cap E\} = X$.*
(5) *For each $x \in X$, there exists $e \in R \cap E$ such that $x = xe$.*
(6) *For each $x \in X$, there exists $e \in K \cap E$ such that $x = xe$.*
(7) *K acts unitarily on X.*

Proof.

$(1) \Rightarrow (2)$. Trivial.

$(2) \Rightarrow (3)$. For any $x \in X$, $x \in xS$ and xS is a minimal orbit. Now xR is a minimal orbit and $xR \subseteq xS$. So $x \in xR = xS$.

$(3) \Rightarrow (4)$. For any $e \in K \cap E$, $Xe\,H(e) = Xe\,eSe = Xe$. $Se = XRe = Xe$. Also note that $XH(e) = Xe$. Now $X = XR = X(\bigcup\{H(e) : e \in R \cap E\}) = \bigcup\{XH(e) : e \in R \cap E\} = \bigcup\{Xe : e \in R \cap E\}$.

$(4) \Rightarrow (5)$. Since for any $x \in X$, $x \in Xe$ for some $e \in R \cap E$, we then have $x = xe$.

$(5) \Rightarrow (6) \Rightarrow (7)$. Trivial.

(7) ⇒ (1). Since $K = \bigcup R$, for any $x \in X$, $x \in xK$ implies that $x \in xR$ for some R and xR is a minimal ideal and, hence, $xR = xS$ because $x \in xR$ implies that $xS \subseteq \subseteq xRs \subseteq xR$. Thus, each orbit xS is minimal and S acts unitarily on X. Hence, (1) follows.

It is worth noting that R (or K) acts unitarily on X iff $XR = X$ (or $XK = X$) [cf. 2].

With further restrictions on X or S or both we have a few more results regarding quasi-transitive acts. Some parts of Propositions 3.2 and 3.3 are similar to results of Stadlander [7, 12].

Proposition 3.2. *Let (X, S) be a compact act. If either, S is left simple or $S^2 = S$ and S is normal, or S acts commutatively on X, then the following statements are equivalent.*

(1) *S acts quasi-transitively on X.*

(2) *For each $e \in K \cap E$, $(xS, H(e))$ is a topological transformation group for any $x \in X$ and $XS = X$.*

(3) *For each $e \in K \cap E$, $(X, H(e))$ is a topological transformation group.*

Proof. (1) ⇒ (2). Note that $H(e)$ is a compact topological group for each $e \in K \cap \cap E$ and, by virtue of our assumptions, for each $x \in X$, $xS\,H(e) = x\,Se\,Se = xS^2e = = xSe = XeS = xR = xS$. Also $XS = X$.

(2) ⇒ (3). For any $e \in K \cap E$, note that, $X\,H(e) = (\bigcup\{xS : x \in X\})\,H(e) = = \bigcup\{xS : x \in X\} = X$.

(3) ⇒ (1). Since for any $e \in K \cap E$, $H(e)$ acts on X unitarily and so far any $x \in X$, $xS\,H(e) \subseteq xS$ implies that $xS = xS\,H(e) = xS\,x\,Se = x\,e\,S = xR =$ a minimal ideal. Note that S acts unitarily on X and so (1) follows.

Proposition 3.3. *Let (X, S) be a compact act. If either, S is left-simple or S is normal, or S acts commutatively on X, then the following statements are equivalent.*

(1) *S acts quasi-transitively on X.*

(2) *$\varrho_s : X \to X$, $\varrho_s(x) = xs$, is a homeomorphism for all $s \in K$.*

(3) *For any $e \in K \cap E$, $x = xe$ for all $x \in X$.*

(4) *ϱ_s, as defined in (2), is a homeomorphism for some $s \in K$.*

(5) *For some $e \in K \cap E$, $x = xe$ for all $x \in X$.*

Proof. (1) ⇒ (2). For each $e \in K \cap E$, $Xe = Xe\,H(e) = X\,e\,S\,e = X\,e\,S = XR = X$.

Since $(Xe, H(e))$ is a topological transformation group for each $e \in E \cap K$, it follows that ϱ_s is a homeomorphism for each $s \in H(e)$ and, hence, for each $s \in K = \bigcup\{H(e) : : e \in K \cap E\}$.

(2) ⇒ (3) ⇒ (1). Let $e \in K \cap E$ and $s \in H(e)$. Then ϱ_s is a homeomorphism and hence $XH(e) = X$. Therefore, $x = xe$ for all $x \in X$ and so, by Proposition 3.4, S acts quasi-transitively on X.

(2) ⇒ (4) ⇒ (5) ⇒ (1). Trivial.

In the above proposition we established equivalence of the statements (2) and (4) under the assumption of normality of S or the commutativity of the act. This is, however, not necessary as we have the following simple result.

Proposition 3.4. *Let (X, S) be a compact act such that $\varrho_s : X \to X$, $\varrho_s(x) = xs$, is a homeomorphism for some $s \in K$. Then ϱ_s is a homeomorphism for all $s \in K$.*

Proof. Note that $K = \bigcup\{H(e) : e \in K \cap E\}$ and so, if, for $s \in H(e)$, ϱ_s is a homeomorphism, then $(X, H(e))$ is a topological transformation group. Then, via isomorphisms of $H(e)$ and $H(f)$, $e, f \in E \cap K$ [cf. Theorem 1.2.6,9], it follows that $(X, H(f))$ is also a topological transformation group for all $f \in K \cap E$. Hence, the assertion follows.

In Proposition 3.3 we have proved equivalence of quasitransitive acts and acts where each transition map $\varrho_s : X \to X$, $\varrho_s(x) = xs$, for $s \in K$ is a homeomorphism under certain hypotheses. The implication from the homeomorphism of ϱ_s's to quasi-transitivity of the acts does not demand all these hypotheses. However, the assumption of ϱ_s's to be homeomorphisms is sufficiently strong and has some implication towards the algebraic structure of the input semigroup as seen in the following result.

Proposition 3.5. *Let a compact semigroup S act effectively on X (i.e., for $s, t \in S$, $s \neq t$ implies that for some $x \in X$, $xs \neq xt$). Then for each $s \in S$, the transition map $\varrho_s : X \to X$, $\varrho_s(x) = xs$, is $1-1$ iff (X, S) is a topological transformation group.*

Proof. It is sufficient to verify that under the hypothesis if each ϱ_s is $1-1$, then S is a topological group and $x1 = x$ for all $x \in X$ where 1 is the identity in S.

To prove that S is a topological group, by theorem 1.1.15 [9], we need only to show that S is cancellative. Now if for $s_1, s_2, t \in S$, $s_1 t = s_2 t$, then $xs_1 t = xs_2 t$ for all $x \in X$, and, as ϱ_t is $1-1$, $xs_1 = xs_2$ for all $x \in X$. Again, S acts effectively on X, and hence, $s_1 = s_2$. Similarly, $ts_1 = ts_2$ implies that $s_1 = s_2$ since $xt = x$. Thus, S is cancellative. Further, if 1 is the identity in S, then, as ϱ_1 is $1-1$, it follows that $xs = x$ for all $x \in X$. Hence, the result follows.

Note that in the above proposition the assumption of effective action can be dropped if we demand that $(X, S/\varrho)$ should be a topological transformation group where $s\varrho t$ if $xs = xt$ for all $x \in X$.

There is an analogous result in Day [4] which states that if (X, S) is effective and compact such that $Xs = X$ for all $s \in S$, then S must be a group.

There exist somewhat similar results concerning transitive acts [10]. Further, via a result on point-transitive acts [7, 8, 12] and a result on transitive acts [5, 8] it is easy to give characterizations of decompositions of a nonvoid T_2-space X induced by a disjoint or quasi-transitive action of a semigroup on it [10]. All these are simple and, hence, omitted.

4. ON HOMOMORPHISMS OF ACTS

Throughout this section we let h to be a homomorphism from a compact act (X, S) onto a compact act (Y, S), that is, h is a map from X onto Y, which need not be continuous, such that $h(xs) = h(x)s$ for all $x \in X$ and all $s \in S$. We investigate how h maps each maximal (minimal) orbit (point inverse set) or a disjoint (i-disjoint) acts onto a maximal (minimal) orbit (point inverse set) or a disjoint (i-disjoint) act respectively. This section is mainly algebraic.

Clearly, h maps an orbit onto an orbit and every maximal orbit yS of (Y, S) is h-image of some maximal orbit xS of (X, S).

But h-image of a maximal orbit need not be a maximal orbit. However, we have:

Proposition 4.1. *h maps each maximal orbit onto a maximal orbit if, for any $x_1, x_2 \in X$, (1) $h(x_1 S \cap x_2 S) = h(x_1 S) \cap h(x_2 S)$, and (2) $x_3 \notin x_1 S \cap x_2 S$ implies that $h(x_3) \notin h(x_1 S \cap x_2 S)$.*

Proof. Easy.

Proposition 4.2. *h maps each maximal orbit onto a maximal orbit if for any $x_1, x_2 \in X$, $C = h(x_1) S \cap h(x_2) S \neq \emptyset$ implies that if $C \subsetneq h(x_1) S$, then $C \subsetneq h(x_3) S$ for some $x_3 \in X$ such that $x_2 S \subseteq x_3 S$.*

Proof. Easy.

Corollary 4.3. *If h is $1-1$, then h maps maximal orbit onto a maximal orbit.*

Regarding disjoint acts, we have the following two results.

Proposition 4.4. *h maps a disjoint act (X, S) onto a disjoint act (Y, S) if, for any $y \in Y$, $h^{-1}(y) = xA$ for some $x \in X$ and $\emptyset \neq A \subseteq S$.*

Proof. Let, if possible, two maximal orbits $y_1 S$ and $y_2 S$ of (Y, S) intersect. Then suppose $x_1 S$ and $x_2 S$ are two maximal orbits of (X, S) such that $h(x_i) S = y_i S$, $i = 1, 2$. Now, for $y \in y_1 S \cap y_2 S \neq \emptyset$, $h^{-1}(y) \cap x_i S \neq \emptyset$, $i = 1, 2$. Now note that as (X, S) is disjoint, $h^{-1}(y) = xA$ for some $x \in X$ and $\emptyset \neq A \subseteq S$ iff $h^{-1}(y)$ is contained in a unique maximal orbit; and, furthermore, $h^{-1}(y) \subseteq x_1 S \cap x_2 S$ which implies that $x_1 S = x_2 S$. Hence, $y_1 S = y_2 S$.

Proposition 4.5. *The following two statements are equivalent.*

(a) *(Y, S) is disjoint and h maps each maximal orbit onto a maximal orbit.*

(b) *For any two maximal orbits x_iS, $i = 1, 2$, of (X, S) $\bigcap h(x_i) S \neq \emptyset$ implies $h(x_1S) = h(x_2S)$.*

Proof. Easy.

Concerning minimal orbits, note that h maps each minimal orbit onto a minimal orbit, and each minimal orbit of (Y, S) is h-image of some minimal orbit of (X, S). Therefore, a homomorphic image of a quasi-transitive (transitive) act is quasi-transitive (transitive).

We next consider maximal point-inverse (mpi) sets and homomorphisms.

Proposition 4.6. *Every mpi set $yS^{(-1)}$ of (Y, S) is h-image of a union of mpi sets $\{x_\alpha S^{(-1)}\}$ of (X, S) such that $h(x_\alpha) S = yS$.*

Proof. Notice that $yS^{(-1)}$ is an mpi set iff yS is a minimal orbit and there exists a minimal orbit in (X, S) whose h-image is yS. So, suppose $\{x_\alpha S\}$ are all the minimal orbits of (X, S) such that $h(x_\alpha) S = yS$. We claim that $yS^{(-1)} = \bigcup h(x_\alpha S^{(-1)})$. Note that $h(xS^{(-1)}) \subseteq h(x) S^{(-1)}$ for any $x \in X$ and $h(x_\alpha S) = yS$ if $h(x_\alpha) S^{(-1)} = yS^{(-1)}$. Therefore, $h(x_\alpha S^{(-1)}) \subseteq yS^{(-1)}$ and, hence, $\bigcup h(x_\alpha S^{(-1)}) \subseteq yS^{(-1)}$. Conversely, let $z \in yS^{(-1)}$. Then, for some $x \in X$, $h(x) = z$ and there is $s \in S$ such that $h(x) s = y$ and $h(x) s S = yS$. There exists a minimal orbit $x'S \subseteq x s S \subseteq xS$ so that $h(x'S) \subseteq \subseteq h(x) s S = yS$. Now $x' = xst$ for some $t \in S$ and so $x \in x'S^{(-1)}$. So $h(x) = z \in \in h(x'S^{(-1)}) \subseteq \bigcup h(x_\alpha S^{(-1)})$.

Proposition 4.7. (Y, S) *is i-disjoint iff for any two mpi sets $x_iS^{(-1)}$, $i = 1, 2$, of (X, S), $\bigcap h(x_iS^{(-1)}) \neq \emptyset$ implies that $h(x_1) S = h(x_2) S$.*

Proof. 'Only if' part follows from Proposition 4.6.

Conversely, let for any two mpi sets $x_iS^{(-1)}$, $i = 1, 2$, of (X, S), $\bigcap h(x_iS^{(-1)}) \neq 0$ Then $\bigcap h(x_i) S^{(-1)} \neq \emptyset$ as $h(x, S^{(-1)}) \subseteq h(x) S^{(-1)}$ for any $x \subseteq X$. Then, as $(Y, S$ is i-disjoint, it follows that $\bigcap h(x_i) S \neq \emptyset$ and, hence, $h(x_1) S = h(x_2) S$.

In general, $h(xS^{(-1)}) \subseteq h(x) S^{(-1)}$ for any $x \in X$ and $h(xS^{(-1)}) = h(x) S^{(-1}$ iff for any $a \in h(x) S^{(-1)}$, $h^{-1}(a) \cap xS^{(-1)} \neq \emptyset$. The following gives a sufficient condition for the latter to happen in case of mpi sets.

Proposition 4.8. *h maps each mpi set of (X, S) onto an mpi set of (Y, S) if for any two mpi sets $x_iS^{(-1)}$, $i = 1, 2$ of (X, S), $\bigcap h(x_iS^{(-1)}) \neq \emptyset$ implies that $h(x_1S^{(-1)}) = = h(x_2S^{(-1)})$.*

Proof. Let $xS^{(-1)}$ be an mpi set of (X, S). Let $h(xS^{(-1)}) \subseteq yS^{(-1)}$, an mpi set in (Y, S) such that $h(x) S = yS$. Now $yS^{(-1)} = \bigcup\{h(x_\alpha S^{(-1)}) : h(x_\alpha S) = yS\}$ and for any α, β such that $h(x_\alpha S) = h(x_\beta S) = yS$, since $\emptyset \neq yS \subseteq h(x_\alpha S^{(-1)}) \cap$

$\cap h(x_\beta S^{(-1)})$, it follows that $h(x_\alpha S^{(-1)}) = h(x_\beta S^{(-1)})$. So, $h(xS^{(-1)}) = yS^{(-1)} = h(x)\,S^{(-1)}$.

Proposition 4.9. *Let (X, S) be disjoint. Then (Y, S) is i-disjoint if for any two mpi sets $x_iS^{(-1)}$ of (X, S) that intersect $h(x_1S^{(-1)}) = h(x_2S^{(-1)})$. If h maps each mpi set onto an mpi set then this condition is also necessary.*

Proof. It is sufficient to show that any mpi set $yS^{(-1)}$ of (Y, S) is a union of orbits. By Proposition 4.6, $yS^{(-1)} = \bigcup h(x_\alpha S^{(-1)})$ where $x_\alpha S$ are all the minimal orbits of (X, S) such that $h(x_\alpha)\,S = yS$. Since (X, S) is disjoint, each maximal orbit xS is a union of mpi sets corresponding to the minimal orbits contained in xS, and, by the condition of the Proposition, if $xS = \bigcup x_\beta S^{(-1)}$, then $x \in \bigcap x_\beta S^{(-1)}$ implies that $h(xS) = h(x_\beta S^{(-1)})$ for each β. So, if $yS^{(-1)} = \bigcup h(x_\alpha S^{(-1)})$, from the disjointness of (X, S) and the condition of the Proposition, it follows that there exist maximal orbits $\{x^\alpha S\}$ such that $\bigcup x^\alpha S = \bigcup x_\alpha S^{(-1)}$ and $\bigcup h(x^\alpha S) = \bigcup h(x_\alpha S^{(-1)}) = yS^{(-1)}$ which is a union of orbits.

To prove the other way, suppose (Y, S) is i-disjoint and h maps each mpi set onto an mpi set. Each mpi set of (Y, S) is a union of maximal orbits. Suppose two mpi sets $x_iS^{(-1)}$, $i = 1, 2$, of (X, S) intersect. As (X, S) is disjoint, $\bigcup x_iS^{(-1)} \subseteq xS$, a maximal orbit. So, $\bigcup h(x_iS^{(-1)}) \subseteq h(x)\,S \subseteq yS$, a maximal orbit. As (Y, S) is i-disjoint yS is contained in some mpi set $y'S^{(-1)}$ and since $h(x_iS^{(-1)}) = h(x_i)\,S^{(-1)}$, an mpi set for $i = 1, 2$, it follows that $\bigcup h(x_i)\,S^{(-1)} \subseteq y'S^{(-1)}$ and, hence, $h(x_i)\,S^{(-1)} = y'S^{(-1)}$. Thus, $h(x_1S^{(-1)}) = h(x_2S^{(-1)})$.

5. PRODUCTS OF ACTS

Let (X_i, S) and (X, S) be two families of acts. The product acts $(\Pi X_i, \Pi S_i)$ and $(\Pi X_i, S)$ are defined in a natural way using coordinatewise operations. In this section we make note of how does a product of acts inherit a given property P from the component acts where P may be disjointness (i-disjointness), transitiveness (quasi-transitiveness) of acts, etc. We first note the following

Proposition 5.1. *Let a compact semigroup S act quasi-transitively on X. Then the equivalence relation R on X defined by xRy if $xS = yS$ is open and has a closed graph and, consequently, the quotient space X/R is Hausdorff.*

Proof. Let $A = \bigcup_{x \in A} xS \subseteq X$. Then note that $\bar{A} = \bigcup_{x \in \bar{A}} xS$ where $\bar{A}$ is the closure of A. For, $A \subseteq \bigcup_{x \in \bar{A}} xS$ since the action is unitary. Further, if $y \in \bar{A}$, then $ys \in \bar{A}$ for all $s \in S$ since there exists in A a net $y_\alpha \to y$ implies that, by the continuity of act, for any $s \in S$ in A the net $y_\alpha s \to ys$.

Then, by Proposition 6 in [p. 54, 3], R is open. That R has a closed graph needs a standard net argument.

Therefore, by Proposition 8 in [p. 79, 3], X/R is Hausdorff.

Remark 5.2. *Let $\{S_i\}$ be a family of compact semigroups. Then ΠK_i is the minimal ideal of ΠS_i iff K_i is the minimal ideal of S_i for each i.*

Remark 5.3. *$(\Pi X_i, \Pi S_i)$ is unitary iff (X_i, S_i) is so for each i.*

Proposition 5.4. *Let $\{S_i\}$ be a family of compact semigroups. Then $(\Pi X_i, \Pi S_i)$ is quasi-transitive (transitive) iff (X_i, S_i) is so for each i. Further, in that case, $(\Pi X_i/R, \Pi S_i)$ is isomorphic to $(\Pi(X_i/R_i), \Pi S_i)$ where R and R_i are the equivalences of ΠX_i and X_i induced by the quasi-transitive actions of ΠS_i and S_i, respectively.*

Proof. The first part for quasi-transitive case follows from Proposition 3.1 and Remarks 5.2 and 5.3. The second part follows from Proposition 5.1 and corollary to Proposition 8 in [p. 55, 3].

Proposition 5.5. *Let $\{(X_i, S_i)\}$ be a family of compact acts.*

(1) *For each $(x_i) \in \Pi X_i$, $(x_i)\, \Pi S_i$ is a maximal (minimal) orbit iff each $x_i S_i$ is so.*
(2) *$(\Pi X_i, \Pi S_i)$ is disjoint (i-disjoint) iff each (X_i, S_i) is so.*

Proof. Easy.

While $(\Pi X_i, \Pi S_i)$ inherits most of the properties mentioned in the beginning of this section it is not so for $(\Pi X_i, S)$. In fact, without much restriction on S nothing can be said. In view of Proposition 3.2, we can only state the following

Proposition 5.6. *Suppose a compact semigroup S acts on X_i, $i \in J$. If either (1) S is left-simple or (2) $S^2 = S$ and S is normal or S acts commutatively on X_i, then S acts quasi-transitively on ΠX_i iff S acts quasi-transitively on each X_i.*

References

[1] *J. T. Borrego* and *E. E. De Vun:* Maximal semigroup orbits, to appear.

[2] *J. T. Borrego:* Compact actions, Proc. 2nd. Fl. Symp. on Automata and Semigroups, Part *I* (1971), University of Florida.

[3] *N. Bourbaki:* General Topology, Part I, Addison Wesley Publishing Company, 1966.

[4] *J. M. Day:* Semigroup acts, algebraic and topological. Proc. 2nd. Fl. Symp. on Automata and Semigroups, Part *I* (1971), University of Florida.

[5] *J. M. Day* and *A. D. Wallace:* Semigroups acting on continua, J. Austr. Math. Soc. *VII* (1967), 327—340.

[6] *J. M. Day* and *A. D. Wallace:* Multiplication induced in the state-space of an act, Math. Systems Theory *I* (1967), 305—314.

[7] *C. F. Keleman:* Transitive semigroup actions, Tr. Amer. Math. Soc. *146* (1969), 369–375.
[8] *Y. F. Lin:* On input semigroups of automata, Math. Systems Theory, 4 (1970), 35–39.
[9] *A. B. Paalman-de Miranda:* Topological semigroups, Mathematisch Centrum, Amsterdam, 2nd Ed., 1970.
[10] *K. Sikdar:* Some remarks on semigroups actions and decompositions of the state space, Indian Statistical Institute Tech. Report. Math. Stat. (20) 71, 11 August, 1971. Read at the 37th Annual Conference of the Indian Math. Society.
[11] *D. Stadlander:* Semigroup actions and dimension, Acq. Math. *3* (1969), 1–14.
[12] *D. Stadlander:* Semigroup actions on topological spaces, Ph. D. Thesis, Penn. St. University 1966.

Author's address: 203, Barrackpore Trunk Road, Calcutta-35, India (Indian Statistical Institute).