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CHARACTERIZATION OF SELF-COMPLEMENTARY GRAPHS WITH 2-FACTORS*

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Let G be a self-complementary graph (s.c.) and π its degree sequence. Then G has a 2-factor if and only if $\pi - 2$ is graphic. This is achieved by obtaining a structure theorem regarding s.c. graphs without a 2-factor. Another interesting corollary of the structure theorem is that if G is a s.c. graph of order $p \geq 8$ with minimum degree at least $p/4$, then G has a 2-factor and the result is the best possible.

0. Introduction

Clapham [1] proved that every self-complementary graph (abbreviated s.c. graph) has a hamiltonian chain. It has been shown by Rao [10] that every s.c. graph of order $p \geq 8$ has an l-cycle for every integer l, $3 \leq l \leq p-2$.

A *k-factor* of a graph G is a spanning subgraph of G which is regular of degree k. Clapham's result [1] implies that every s.c. graph of even order has a 1-factor. The aim of this paper is to characterize s.c. graphs having a 2-factor.

Let G be a s.c. graph of order p and σ be a permutation of the vertices which maps G onto its complement $\bar{G}$. Such a permutation is referred to as a *complementing permutation* of G. (For properties of s.c. graphs and complementing permutations, see [1, 2, 3, 10, 11, 13, 14].) Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ be the decomposition of the permutation σ into disjoint cycles. It is known that the length of σ_i is a multiple of 4 for every i except possibly one i_0 (say) and the exceptional one has length 1 (the latter can occur only in the case $p = 4N+1$). Let σ_i have length $p_i = 4n_i$, $1 \leq i \leq k$, $i \neq i_0$ (possibly). Let

$$\sigma_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,p_i}), \qquad i \neq i_0.$$

We may assume that $(a_{i,1}, a_{i,3}) \in E(G)$ (for if not, $(a_{i,2}, a_{i,4}) \in E(G)$ and we can relabel the vertices appropriately), and this implies that $(a_{i,j}, a_{i,j+2}) \in E(G)$ for all odd j. We call the vertices $a_{i,1}, a_{i,3}, \ldots, a_{i,p-1}$ the *odd vertices* of σ_i and denote the set by A_i; the vertices $a_{i,2}, a_{i,4}, \ldots, a_{i,p_i}$ are the *even vertices* of σ_i and we denote the set by B_i, $1 \leq i \leq k$, $i \neq i_0$. The vertices of $A_i \sqcup B_i$ are the *vertices of* σ_i, $1 \leq i \leq k$, $i \neq i_0$.

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We label the vertices such that in each cycle consecutive odd vertices are joined by an edge. Define a directed graph $D(\sigma)$ whose vertex set is the set of all cycles of σ and the cycles σ_i and σ_j $(i \neq j)$ are joined by an arc (σ_i, σ_j), if there is an edge in G from some even vertex of σ_i to some odd vertex of σ_j if $i, j \neq i_0$; if $i = i_0$, then (σ_i, σ_j) is an arc of $D(\sigma)$ if the unique vertex of σ_i is joined to some odd vertex of σ_j; if $j = i_0$, then (σ_i, σ_j) is an arc of $D(\sigma)$ if some even vertex of σ_i is joined to the vertex of σ_j. It is shown in [1] that $D(\sigma)$ is a complete directed graph. Further, if (σ_i, σ_j) with $i, j \neq i_0$ is an arc of $D(\sigma)$ then *every* even vertex of σ_i is joined to some odd vertex of σ_j; and *every* odd vertex of σ_j is joined to some even vertex of σ_i; if $i = i_0$, then σ_i is joined to every odd vertex of σ_j and to no even vertex of σ_j; if $j = i_0$, then σ_j is joined to every even vertex of σ_i and to no odd vertex of σ_i.

We make use of the following lemma repeatedly in our discussion.

Lemma 0.1. (Clapham [1]; compare Rao [10]). *Let $\sigma_{i_1}, \ldots, \sigma_{i_\theta}$ be a path in $D(\sigma)$ where all $i_j \neq i_0$, $1 \leq j \leq \theta$, then G has a chain containing the vertices of all σ_{i_j}, $1 \leq j \leq \theta$, and no vertex outside, in which two consecutive odd vertices of σ_{i_1} appear consecutively and whose end vertices are consecutive even vertices of σ_{i_θ}.*

The *condensation* D^* of a directed graph D has for its vertices the strong components of D, and two vertices α, β of D^* are joined by an arc $\alpha \to \beta$ if for some $a \in V(\alpha)$ and $b \in V(\beta)$, (a, b) is an arc of D.

As always, $K = K_n$ denotes the complete graph of order n, $K^c = K_n^c$ denotes the empty graph; i.e. the graph with no edges. Similarly, $K = K_{m,n}$ denotes the complete bipartite graph with two independent sets having m and n vertices, respectively. $K^c = K_{n,m}^c$ denotes the empty graph on $n + m$ vertices.

If X and Y are sets of vertices of G, $G[X, Y]$ denotes the subgraph of G, generated by X, Y, i.e. the graph with $X \sqcup Y$ as its vertices, which includes exactly those edges of G, having one end vertex in X and the other in Y. We write $G[X]$ for $G[X, X]$.

1. The structure of s.c. graphs without 2-factors: the case $p = 4N$.

Lemma 1.1. *Let G be a s.c. graph of order $p = 4N$ (> 4) and σ, a complementing permutation of G. Suppose the digraph $D(\sigma)$ is strongly connected. Then G has a 2-factor.*

Proof. First suppose $n_i > 1$, for every i, $1 \leq i \leq k$. Then $G[A_i, B_i]$ is a regular graph of regularity n_i, $1 \leq i \leq k$. Hence $G[A_i, B_i]$ has r-factor, for every r, $1 \leq r \leq n_i$, (see Harary [4, p. 85]), in particular, since $n_i > 1$, it has a 2-factor, $1 \leq i \leq k$. Therefore, G has a 2-factor. Thus we may take that some cycle of σ is of length 4. Since $D(\sigma)$ is a strongly connected complete digraph, by Camion's theorem [4, p. 207], it has a hamiltonian circuit, $(\sigma_1, \ldots, \sigma_k)$ (say), where $n_1 = 1$. Now let μ_1 be a hamiltonian chain of G given by Lemma 0.1 in which the vertices

$a_{1,1}, a_{1,3}$ appear consecutively and whose end vertices are $a_{k,j}, a_{k,j+2}$ with j even, where as always the suffixes are to be taken modulo the length of the cycle of σ in which they appear. Since (σ_k, σ_1) is an arc of $D(\sigma)$, there exists an odd i such that $e_1 = (a_{k,j}, a_{1,i}) \in E(G)$. Then $e_2 = (a_{k,j+2}, a_{1,i+2}) \in E(G)$. Note that $i = 1$ or 3 and $n_1 = 1$. Now

$$\mu = \mu_1 - (a_{1,1}, a_{1,3}) + e_1 + e_2$$

is a 2-factor of G. This completes the proof.

Theorem 1.2. *Let G be a s.c. graph of order $p = 4N$ (>4). Then G does not have a 2-factor if and only if $V(G)$ can be partitioned into two sets V_1, V_2 of order $4N_1, 4N_2$ (say) respectively where $N_1 + N_2 = N$ such that the following conditions hold.*

(0) *$H_i = G[V_i]$ is a s.c. graph, $i = 1, 2$.*

(1) *Let L be the set of all vertices of H_2 whose degree in H_2 is at least $2N_2$; and $R = V_2 - L$. Then $G[L] = K$ and $G[R] = K^c$.*

(2) *$G[V_1, L] = K$ and $G[V_1, R] = K^c$.*

(3) *If $N_2 > 1$, then H_1 does not have a 2-factor.*

Proof. First we prove the sufficiency. Since H_2 is a s.c. graph of order $4N_2$, we have $|L| = |R| = 2N_2$. Now if $N_2 = 1$, then by (2), G has a vertex of degree 1 and therefore G does not have a 2-factor. Thus we may take that $N_2 > 1$. If now G has a 2-factor, then by (1) and (2), note that, because of $|L| = |R|$, a 2-factor of G cannot contain any edge connecting H_1 with H_2, it follows that H_1 also has a 2-factor, contradicting (3).

To prove the necessity, let G be a s.c. graph of order $p = 4N$ (>4) without a 2-factor, and σ, a complementing permutation of G. By Lemma 1.1, $D(\sigma)$ is not strongly connected. Since $D(\sigma)$ is a complete digraph, the condensation of $D(\sigma)$ is a nontrivial transitive tournament (Harary et al. [5, p. 298]). Let $C_1, \ldots, C_s$ be the strong components of $D(\sigma)$ arranged in such an order that *every* even vertex of all $\sigma_u \in V(C_i)$ is adjacent in G to *all* odd vertices of *every* σ_v in $V(C_j)$, $1 \leq i < j \leq s$, where $s \geq 2$. Define V_1 to be the set of all vertices of the cycles of σ in $\bigsqcup_{i=1}^{s-1} V(C_i) = W_1$ (say); and V_2 to be the set of all vertices of the cycles of σ in $V(C_s) = W_2$ (say). We show that V_1, V_2 satisfy the conditions (0) through (3) of the statement of the theorem. Clearly $G[V_i] = H_i$ is a s.c. graph of order $4N_i$ (say), $i = 1, 2$; with $N_1 + N_2 = N$. We first prove three assertions (a), (b) and (c) below and then complete the proof.

(a) $(a_{u,i}, a_{v,j}) \notin E(G)$, *whenever $\sigma_u \in W_1$, $\sigma_v \in W_2$ and i, j even.*

Suppose $e_1 = (a_{u,i}, a_{v,j}) \in E(G)$, with u, v, i, j as above. Then $e_2 = (a_{u,i+2}, a_{v,j+2}) \in E(G)$. Let $\sigma_u \in V(C_{l_0})$ where $1 \leq l_0 \leq s - 1$. Note that there is a $\sigma_w - \sigma_u$ path in $D(\sigma)$, containing all the vertices of $\bigsqcup_{i=1}^{l_0} V(C_i)$ and none of $\bigsqcup_{i=l_0+1}^{s} V(C_i)$, where $\sigma_w \in V(C_1)$. Now obtain, by Lemma 0.1, a chain μ_1 in G by combining the cycles in this $\sigma_w - \sigma_u$ path, for which $a_{u,i}, a_{u,i+2}$ are end vertices.

Similarly, obtain a chain μ_2 in G by combining the cycles of σ in $\bigsqcup_{i=i_0+1}^{r} V(C_i)$ in which two consecutive odd vertices of a cycle of σ in $V(C_{i_0+1})$ appear consecutively and whose end vertices are the consecutive even vertices $a_{v,j}, a_{v,j+2}$. Now a hamiltonian cycle μ in G may be obtained by defining

$$\mu = \mu_1 + e_1 + e_2 + \mu_2,$$

and this is a contradiction.

(b) $(a_{u,i}, a_{w,j}) \notin E(G)$, *whenever* $\sigma_v, \sigma_w \in W_2$, $v \neq w$ *and* i, j *even.*

Suppose $e_1 = (a_{v,i}, a_{w,j}) \in E(G)$, where v, w, i, j are as above. Then $e_2 = (a_{v,i+2}, a_{w,j+2}) \in E(G)$. Let $\rho_1, \ldots, \rho_r$ be a hamiltonian circuit (the case $r = 2$ is also included) in C_s with $\rho_1 = \sigma_v$ and $\rho_l = \sigma_w$, $2 \leq l \leq r$. Let μ_1 be a hamiltonian chain in ρ_1 in which two consecutive odd vertices of ρ_1 say $a_{v,\alpha}, a_{v,\alpha+2}$ (α odd) appear consecutively and whose end vertices are $a_{v,i}, a_{v,i+2}$. Obtain a chain μ_2, by combining the cycles $\rho_2, \ldots, \rho_l$, in which two consecutive odd vertices of ρ_2, $b_{2,\beta}, b_{2,\beta+2}$ (say) appear consecutively and whose end vertices are $a_{w,j}, a_{w,j+2}$. Let μ_3 be a hamiltonian chain in H_1 whose end vertices are consecutive even vertices of some cycle σ_u (say) of σ in $V(C_{s-1})$, $a_{u,\theta}, a_{u,\theta+2}$ (say). We now consider two cases.

Case (*i*) $l = r$. Then

$$\mu^* = \mu_3 + (a_{u,\theta}, b_{2,\beta}) + (a_{u,\theta+2}, b_{2,\beta+2}) + \mu_2 - (b_{2,\beta}, b_{2,\beta+2}) + e_1 + e_2 + \mu_1,$$

is a hamiltonian cycle in G. Thus we may take

Case (*ii*) $r \geq l + 1$. Since (ρ_r, ρ_1) is an arc of $D(\sigma)$ and α is odd, $(a_{v,\alpha}, b_{r,t}) \in E(G)$ for some even t. Now let μ_4 be a chain obtained by combining the cycles $\rho_{l+1}, \ldots, \rho_r$ of σ in which two consecutive odd vertices of ρ_{l+1} appear consecutively and whose end vertices are the consecutive even vertices $b_{r,t}, b_{r,t+2}$ of ρ_r. Then μ_4 and μ^* of case (i) may be combined by defining

$$\mu = \mu^* + \mu_4 + (a_{v,\alpha}, b_{r,t}) + (a_{v,\alpha+2}, b_{r,t+2}) - (a_{v,\alpha}, a_{v,\alpha+2}).$$

Now μ is a hamiltonian cycle of G, a contradiction.

(c) $(a_{v,i}, a_{v,j}) \notin E(G)$, *where* $\sigma_v \in W_2$ *and* i, j *even.*

This is clearly true if $n_v = 1$. So we may take that $n_v > 1$. Now it is enough to show that $(a_{v,2}, a_{v,j}) \notin E(G)$, whenever j is even, $4 \leq j \leq 4n_v$. First let $j \neq 2n_v + 2$. Then $G[B_v]$ has a 2-factor μ_0 (say). Let μ_1 be the cycle $(a_{v,1}, a_{v,3}, \ldots, a_{v,4n_v-1})$. Let $(\rho_1, \ldots, \rho_r)$ be a hamiltonian circuit in C_s with $\rho_1 = \sigma_v$. Obtain a hamiltonian chain μ_2 in H_1 whose end vertices are consecutive even vertices of a cycle $\sigma_u \in V(C_{s-1})$. Now if $r = 1$, then μ_2, μ_1, μ_0 can be combined to yield a 2-factor of G. Thus we may take that $r > 1$. Since (ρ_r, σ_v) is an arc of $D(\sigma)$, $(a_{v,1}, b_{r,t}) \in E(G)$ for some even t. Now let μ_3 be a chain obtained by combining the cycles $\rho_2, \ldots, \rho_r$ in which two consecutive odd vertices of ρ_2 appear consecutively and whose end vertices are $b_{r,t}, b_{r,t+2}$. Then μ_3, μ_2, μ_1 and μ_0 can be combined suitably to get a 2-factor of G.

Thus we may take $j = 2n_v + 2$. Then

$$F = \{(a_{v,i}, a_{v,i+1}), (a_{v,2i-1}, a_{v,2i+1}),\ i \text{ odd},\ 1 \leq i \leq 4n_v\}$$
$$+ \{(a_{v,i}, a_{v,i+2n_v}),\ i \text{ even},\ 1 \leq i \leq 4n_v\},$$

is a 2-factor of $G[A_v \cup B_v]$. This F and the chains μ_2, μ_3 described above may be combined to yield a 2-factor of G itself, a contradiction.

Now we are ready to prove the necessity of conditions (1) through (3). Let A^*, B^* be the sets of the odd vertices or even vertices of the cycles of σ in $V(C_s)$, respectively. Since H_1, H_2 are s.c. graphs and $\sigma(A^*) = B^*$, $\sigma(B^*) = A^*$, and C_s being the bottom most strong component of the complete digraph $D(\sigma)$, it follows, by assertion (a), that $G[V_1, A^*] = K$ and $G[V_1, B^*] = K^c$. By assertions (b) and (c), $G[B^*] = K^c$, hence $G[A^*] = K$. Now it is clear that L equals A^* and R equals B^*. Thus by what has been proved above it follows that conditions (0), (1) and (2) are satisfied. If $N_2 > 1$ and H_1 has a 2-factor, then since C_s is a strong component, it follows, by Lemma 1.1, that the s.c. graph H_2 has a 2-factor. This in turn implies that G also has a 2-factor, contradicting the hypothesis. This completes the proof of the theorem.

The following remark and Lemma will be used in Section 2.

Remark 1.3. The 2-factors obtained in the proofs of assertions (b) and (c) have two consecutive odd vertices of a cycle of σ in $V(C_s)$ appearing consecutively in them.

Lemma 1.4. *Let G be a s.c. graph of order $4N$, and σ, a complementing permutation of G. Suppose $D(\sigma)$ is strongly connected. Then G has a 2-factor in which two consecutive odd vertices of a cycle of σ appear consecutively, if and only if $G[A] \neq K$, where A is the set of all odd vertices of σ.*

Proof. The proof is similar to the proof of assertions (b) and (c) of Theorem 1.2.

2. The structure of s.c. graphs without 2-factors: the case $p = 4N + 1$

Lemma 2.1. *Let G be a s.c. graph of order $4N + 1$, and σ a complementing permutation of G. Suppose the unique fixed point σ_0 of σ belongs to the bottom strong component C_s (say) of $D(\sigma)$ (the case $D(\sigma)$ is strong is not excluded). Then G has a hamiltonian cycle.*

Proof. Let $\rho_1, \ldots, \rho_r$ with $\rho_r = \sigma_0$ be a hamiltonian circuit in C_s ($r = 1$ is possible). Note that $\sigma - \sigma_0$ is a complementing permutation of the s.c. graph $G - \sigma_0$ which is of even order. By Lemma 0.1, there exists a hamiltonian chain μ_1 (say) in $G - \sigma_0$ whose end vertices are consecutive even vertices of ρ_{r-1} if $r \geq 2$, or consecutive even vertices of a cycle of σ in $V(C_{s-1})$ if $r = 1$. Since σ_0 is joined to all even vertices of ρ_{r-1} if $r \geq 2$, and also to all even vertices of every cycle of σ in $V(C_{s-1})$, the vertex σ_0

can be incorporated at the ends of the hamiltonian chain μ_t to get a hamiltonian cycle in G. This completes the proof.

Theorem 2.2. *Let G be a s.c. graph of order $4N+1$. Then G does not have a 2-factor if and only if G can be partitioned into two sets V_1, V_2 of order $4N_1+1$, $4N_2$ respectively where $N_1+N_2=N$ and $N_1 \geq 0$, $N_2 \geq 1$, such that the conditions* (0) *through* (3) *of the statement of Theorem* 1.2 *hold.*

Proof. The proof of the sufficiency is exactly similar to the proof of the sufficiency of Theorem 1.2.

To prove the necessity, let G be a s.c. graph of order $4N+1$ without a 2-factor and σ, a complementing permutation of G and σ_0 the unique fixed point of σ. By Lemma 2.1, $\sigma_0 \notin V(C_s)$, the bottom strong component of $D(\sigma)$. Now define H_1, H_2 as in the proof of Theorem 1.2. Since the vertex $\sigma_0 \in V(H_1) = V_1$, it follows that H_1, is a s.c. graph of odd order, $4N_1+1$ say. Let H_2 be of order $4N_2$, then $N_1+N_2=N$. If $N_1=0$, then we assert that $G[A]=K$ where A is the set of all odd vertices of $\sigma-\sigma_0$ which is a complementing permutation of $G-\sigma_0$. Suppose $G[A] \neq K$. Note that $D(\sigma-\sigma_0)$ is strongly connected. Hence by Lemma 1.4, $G-\sigma_0$ has a 2-factor F in which two consecutive odd vertices of some cycle σ_u ($\neq \sigma_0$) of σ appear consecutively. Then σ_0 may be incorporated in between these two odd vertices of F to get a 2-factor of G, contradicting the hypothesis. Thus $G[A]=K$. Since $\sigma(A)=B$, we have $G[B]=K^c$. Further, since $G[\sigma_0, A]=K$, we have $G[\sigma_0, B]=K^c$. Thus G satisfies the properties (0) through (3) with $V_1=\{\sigma_0\}$, $L=A$, $R=B$ and $V_2=A \sqcup B$. Therefore, henceforth we may take that $N_1 \geq 1$.

We now prove the three assertions (a), (b) and (c) of Theorem 1.2. Suppose (a) does not hold with $\sigma_u \in V(C_m)$ and $\sigma_v \in V(C_s)$. Let $\sigma_0 \in V(C_l)$, $1 \leq l, m \leq s-1$. We consider three subcases according as $l<m$, $l=m$, or $l>m$.

Case (i) $l<m$. Then in $D(\sigma-\sigma_0)$, C_{m+1} is the immediate successor of C_m and C_s is the bottom strong component. Then, as in the proof of assertion (a) of Theorem 1.2, we obtain a 2-factor F_0 of $G-\sigma_0$ in which two consecutive odd vertices of some cycle of σ in $V(C_{m+1})$ appear consecutively. Now σ_0 may be incorporated in between these odd vertices of F_0 to get a 2-factor of G, a contradiction.

Case (ii) $l=m$. Let $\rho_1, \ldots, \rho_r$ be a hamiltonian circuit in C_m with $\rho_1=\sigma_u$ and $\rho_t=\sigma_0$, $2 \leq t \leq r$. If $t=2$, then as in case (i) we get a 2-factor of G. Thus we may take that $2<t \leq r$. Now $G_1 = G - \bigsqcup_{\bullet=2}^{t} \rho_\bullet$ is a s.c. graph with $\sigma - \bigsqcup_{\bullet=2}^{t} \rho_\bullet$ as a complementing permutation. As in the proof of (a) of Theorem 1.2 it can be shown that G_1 has a 2-factor F_1 (say). Now $\rho_2, \ldots, \rho_t$ can be combined to get a cycle F_2 (note $\rho_t=\sigma_0$). Then F_1+F_2 is a 2-factor of G, a contradiction.

Case (iii) $l>m$. Now $C_1, \ldots, C_{m-1}, C_{m+1}, \ldots, C_l$ may be combined to get a F_1 in G (note $\sigma_0 \in C_l$). Also $C_m, C_{l+1}, \ldots, C_s$ may be combined, as in the proof of (a) of Theorem 1.2, to get a 2-factor F_2 of the corresponding graph. But then F_1+F_2 is a 2-factor of G, a contradiction.

In case assertions (b) or (c) of Theorem 1.2 are not valid, we get, by Remark 1.3, a 2-factor F_0 of $G - \sigma_0$ in which two consecutive odd vertices of a cycle of σ in the bottom most strong component of $D(\sigma - \sigma_0)$ (which is C_t) appear consecutively. Then σ_0 may be incorporated in between these odd vertices of F_0 to get a 2-factor of G, contradicting the hypothesis that G does not have a 2-factor. Thus the assertions of (a), (b) and (c) in Theorem 1.2 are valid in the case $p = 4N + 1$ also. Now as in the proof of Theorem 1.2, it can be shown that H_1, H_2 satisfy the conditions (0) through (3) of Theorem 1.2. This completes the proof of Theorem 2.2.

3. Characterization of s.c. graphs with 2-factors

In this section we prove the following:

Theorem 3.1. *Let G be a s.c. graph of order p, and $\pi = (d_1, \ldots, d_p)$ be its degree sequence. Then G has a 2-factor if and only if $\pi - 2 = (d_1 - 2, \ldots, d_p - 2)$ is graphic.*

We use the following three theorems

Theorem 3.2. (Kundu [8], Kleitman, Wang [6]). *Let π and $\pi - k$ be both graphic. Then there is a realization of the former which has one of the latter as a subgraph.*

Theorem 3.3. (Koren [7], Compare Rao, Rao [9, p. 187–188]). *Let $\pi = (d_1, \ldots, d_p)$ be a graphic nonincreasing sequence. Let $\delta(j, \pi) = j(j-1) + \sum_{i=j+1}^{p} \min(d_i, j) - \sum_{i=1}^{j} d_i$. Suppose $\delta(j, \pi) = 0$ for some j, $1 \leqslant j < p$. If $d_{j+1} > j$, let $r = r(j)$ be an index such that $d_r \geqslant j \geqslant d_{r+1}$. If $d_{j+1} \leqslant j$, let $r = j$. For any realization $H = H(u_1, \ldots, u_p)$ of π with degree of $u_i = d_i$, $1 \leqslant i \leqslant p$, define*

$$S = \{u_1, \ldots, u_j\}, \quad T = \{u_{r+1}, \ldots, u_p\}, \quad U = \{u_{j+1}, \ldots, u_r\}.$$

Then

(1) $H[S] = K$,

(2) $H(T) = K^c$. *If* $U \neq \emptyset$,

then

(3) $H[S, U] = K$,

and

(4) $H[T, U] = K^c$.

Theorem 3.4. (Koren [7]). *Suppose $H(u_1, \ldots, u_p)$ realizes π, $S = \{u_1, \ldots, u_j\}$, $p > j \geqslant 1$, $T = \{u_{r+1}, \ldots, u_p\}$, $(r \geqslant j)$, $U = \{u_{j+1}, \ldots, u_r\}$ and conditions* (1), (2) *hold for S and T, and if $U \neq \emptyset$, then conditions* (3), (4) *hold as well. Then $\delta(j, \pi) = 0$.*

Proof of Theorem 3.1. The proof is by induction on p. For $p = 4$, the result is vacuously true. Assume the result for all values less than p and let G be a s.c. graph with degree sequence $\pi = (d_1, \ldots, d_p)$ such that $\pi - 2$ is also graphic. Suppose G does not have a 2-factor. Then by Theorems 1.2 and 2.2, $V(G)$ can be partitioned into two sets V_1, V_2 of order $4N+\delta$, $4N_2$ (where $\delta=0$ or 1 according as p is $4N$ or $4N+1$ respectively) such that the conditions (0) through (3) of Theorem 1.2 hold. Put

$$S=\{u_1,\ldots,u_{2N_2}\},$$

$$U=\{u_{2N_2+1},\ldots,u_{\theta-1}\},$$

$$T=\{u_\theta,\ldots,u_p\},$$

where $\theta=2N_2+4N_1+1+\delta$.

Now it is not difficult to check that for $u\in S$, $v\in U$ and $w\in T$, we have degree $u>$ degree $v>$ degree w where the degree is to be taken in the graph G. Further, G satisfies conditions (1) through (4) of Theorem 3.3. Hence, by Theorem 3.4, $\delta(2N_2,\pi)=0$. Now by Theorem 3.2, π has a realization G^* (say) such that G^* has a 2-factor. Since $\delta(2N_2,\pi)=0$, it follows by Theorem 3.3, that G^* satisfies the conditions (1) through (4) of Theorem 3.3 (with H replaced by G^*). Since G^* has 2-factor, it is evident that the graphs $G^*[U]$, $G^*[S\sqcup T]$ have 2-factors. By the structure of G and G^* it is also evident that degree sequence of $G^*[U]=$ degree sequence of $G[U]=$ degree sequence of $H_1=\pi_1$ (say); and also degree sequence of $G^*[S\sqcup T]=$ degree sequence of $G[S\sqcup T]=$ degree sequence of $H_2=\pi_2$ (say).

Since $G^*[U]$, $G^*[S\sqcup T]$ have 2-factors it follows that π_i-2 is graphic, $i=1,2$.

Thus H_i is a s.c. graph with degree sequence π_i such that π_i-2 is graphic, $i=1,2$. Hence by induction hypothesis, H_i has a 2-factor F_i (say), $i=1,2$. But then F_1+F_2 is a 2-factor of G, a contradiction. This completes the proof of the theorem.

Theorem 3.5. *Let G be a s.c. graph of order $p\geq 8$ such that minimum degree of $G\geq p/4$, then G has a 2-factor.*

Proof. Suppose G does not have a 2-factor. Then let V_1, L, R be as in Theorems 1.2 and 2.2. It is clear that $q(H_2[L,R])=2N_2^2$ (where $q=$ number of edges). It follows that for some vertex w of R, $q(H_2[L,\{w\}])\leq N_2$. Since $G[R]$ is the empty graph, we have $q(G[L,\{w\}])\leq N_2$. Thus minimum degree in $G\leq N_2$. Since $p=4N_1+4N_2+\delta$, it can be easily seen that $N_2<p/4$, a contradiction to the hypothesis.

To show that the result is the best possible, we consider two cases:

Case (i) $p=4N$. A required graph G whose vertex set is $V=\{u_1,\ldots,u_p\}$ may be constructed as follows: Define $V_1=\{u_1,u_2,u_3,u_4\}$, $V_2=V-V_1$,

$G[V_1]$ is the s.c. graph of order 4,

$L=\{u_5,\ldots,u_{2N+2}\}$, $R=V_2-L$;

$G[L]=K$, $G[R]=K^c$, $G[V_1,L]=K$,

$G[V_1,R]=K^c$ and

$G[L,R]$ is the disconnected graph having exactly two components each of which is regular of degree $N-1$. Clearly, G is a s.c. graph of order $4N$ in which minimum degree is $N-1$. Further, G does not have a 2-factor.

Case (ii) $p = 4N + 1$. A required graph G whose vertex set is $V = \{u_0, \ldots, u_{4N}\}$ may be constructed as follows:

$V_1 = \{u_0\}$, $V_2 = V - V_1$;

$L = \{u_1, \ldots, u_{2N}\}$, $R = V_2 - L$;

$G[L] = K$, $G[R] = K^c$, $G[V_1, L] = K$, $G[V_1, R] = K^c$, and

$G[L, R]$ is the disconnected graph having exactly two components each of which is regular of degree N. G is a s.c. graph in which minimum degree is N and G does not have a 2-factor.

4. Epilog

The problem of characterizing s.c. graphs with k-factors seems to be much deeper. In this connection we take the risk of conjecturing the following:

Conjecture. *Let G be a s.c. graph of order p, π its degree sequence. Then G has a k-factor if and only if $\pi - k$ is graphic.*

In a forthcoming paper Rao [12] we characterize, by using the techniques developed in the present paper, hamiltonian s.c. graphs. For a characterization of the degree sequences of self-complementary graphs, see Clapham and Kleitman [2].

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