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ALTERNATING EULERIAN TRAILS WITH PRESCRIBED DEGREES IN TWO EDGE-COLORED COMPLETE GRAPHS

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Let K_n be the complete graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and let $g = (g_1, \ldots, g_n)$ be a sequence of positive integers. Color each edge of this K_n red or blue. In this paper necessary and sufficient conditions are given which guarantee the existence of a connected spanning subgraph F in K_n (as colored) with both red degree and blue degree in F at vertex v_i equal to g_i. When each $g_i = 1$ this answers a question of Erdös proved in this special case in [1].

1. Introduction and definitions

All graphs considered in this paper are finite, whose edges are bicolored with colors red r and blue b and are referred to as *colored graphs* only and the color of an edge uv will be denoted by $c(uv)$. In Bánkfalvi and Bánkfalvi [1] the following problem of Erdös was solved. Characterize the colored complete graphs in which there exists an alternating red–blue hamiltonian cycle. In this paper we solve the following generalization of the above problem and deduce the results of [1]: Given a sequence $(f_1, \ldots, f_n)$ of *length* n of even positive integers, characterize the colored complete graphs K_n of order n with vertex set $\{v_1, \ldots, v_n\}$ in which there exists a connected spanning subgraph F such that for every i, $1 \leq i \leq n$, the number of red edges of F incident at v_i is equal to the number of blue edges of F incident at v_i and each is equal to $\frac{1}{2}f_i$. The problem of Erdös mentioned above is a particular case of this problem with $f_i = 2$ for every i, $1 \leq i \leq n$. The method of proof uses the theory of alternating chains and is a generalization of the proof technique in [1] and Rao and Rao [7]. For results regarding the existence of hamiltonian cycles having adjacent edges with different colors in k-colorations, satisfying some degree constraints, of the edges of the complete graphs, the reader is referred to the recent papers of Daykin [5], Chen and Daykin [3]. For some related problems and results we refer to Chen, Daykin and Erdös [4] and also to the unsolved problem 2 of the book mentioned in [4].

Pertinent definitions are given below; for definitions not given and notation not explained here the reader is referred to Bondy and Murty [2].

A colored graph G is said to be *equitably colored* if for every vertex $v \in V(G)$, the vertex set of G, the number of red edges in G incident at v, called the *red*

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degree of v in G and denoted by $r_G(v)$, is equal to the blue degree of v in G denoted by $b_G(v)$. An *f-factor* F of a given colored K_n, where $f=(f_1,\ldots,f_n)$ is a sequence of positive integers, is a spanning subgraph of K_n such that $r_F(v_i)+b_F(v_i)=f_i$ for every i, $1\leq i\leq n$; and the f-factor F is said to be an *equitably colored f-factor* if F is equitably colored; which then implies that f_i is an even positive integer and $r_F(v_i)=b_F(v_i)=\frac{1}{2}f_i$ for every i, $1\leq i\leq n$.

An eulerian closed trail $E_1=x_1x_2\cdots x_{2\mu}x_1$ (where $x_1,\ldots,x_{2\mu}$ are the vertices occurring in the trail) in a colored graph G is said to be an *alternating eulerian trail* if the edges $x_1x_2, x_2x_3,\ldots,x_{2\mu}x_1$ in E_1 in this ordering are alternately red and blue; further, if $V(G)=\{v_1,\ldots,v_n\}$, the sequence $(f_1,\ldots,f_n)$, where $f_i=r_G(v_i)+b_G(v_i)$, $1\leq i\leq n$, is called *the degree sequence* of G and also of the alternating eulerian trail E_1. The *classes* of E_1, by definition, are the vertex sets $A_1=\{x_1,x_3,\ldots,x_{2\mu}\}$ and $B_1=\{x_2,x_4,\ldots,x_{2\mu}\}$; note that A_1, B_1 need not be disjoint. For arbitrary integer j we mean by x_j the vertex x_i of E_1 where $1\leq i\leq 2\mu$ and $i\equiv j \pmod{2\mu}$.

The edges of K_n with one end vertex in A and the other end vertex in B are referred to as AB-edges, where A, B are subsets of $V(K_n)$. If E_1 is an alternating eulerian trail of a subgraph C_1 of a colored K_n and C_2 is another subgraph of this K_n with $V(C_1)\cap V(C_2)=\emptyset$, then we define the symbol $C_1\to C_2$ if all A_1 $V(C_2)$-edges are of the same color and all B_1 $V(C_2)$-edges are of the other color where A_1, B_1 are the classes of E_1; in particular then $A_1\cap B_1=\emptyset$ and C_1 is a bipartite graph with the bipartition A_1, B_1. $\sim(C_1\to C_2)$ denotes the negation of $(C_1\to C_2)$.

A *red* (respectively, *blue*) *exchangeable trail* T with respect to a subgraph F of a colored K_n is a closed trail T of even length in K_n whose edges are all red (respectively, blue) and which alternately belong to $E(F)$ and $E(K_n)-E(F)$. A coloration of K_n is said to be a *self-complementary coloration* if the red subgraph of K_n, with vertex set $V(K_n)$ is isomorphic to the blue subgraph of K_n; in that case $n\equiv 0$ or $1 \pmod 4$.

If $E_1=x_1x_2\cdots x_s$ and $E_2=y_1y_2\cdots y_t$ are two chains in a graph G with $x_s=y_1$, then the *concatenation* denoted by E_1+E_2 denotes the chain $E_1+E_2=x_1\cdots x_sy_2\cdots y_t$; and the chain $x_sx_{s-1}\cdots x_1$, obtained by reversing E_1, will be denoted by E_1^{-1}.

2. Characterization and corollaries

We start this section with the following lemma (proved earlier in Kotzig [6]) which characterizes colored graphs having alternating eulerian trails and is a basic tool in the proof of the main theorem of this paper. The proof is omitted.

Lemma 1. *If G is a colored graph, then G has an alternating eulerian trail if and only if G is connected and is equitably colored.*

Now we state and prove, using the theory of alternating chains, the main theorem of this paper which characterizes the colorations of K_n having connected equitable colored f-factors where $f=(f_1,\dots,f_n)$ is a given sequence of positive integers.

Main Theorem. *Let K_n be the complete graph with vertex set $V(K_n)=\{v_1,\dots,v_n\}$ and each of its edges colored red or blue, and let $f=(f_1,f_2,\dots,f_n)$ be a fixed sequence of positive integers. Let $r_i(b_i)$ denote the red (blue) degree of vertex v_i. Assume that $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_n$. This graph as colored contains a connected equitable colored f-factor if and only if*

(a) *each f_i is even and K_n has both red and blue $\frac{1}{2}f=(\frac{1}{2}f_1,\frac{1}{2}f_2,\dots,\frac{1}{2}f_n)$ factor, and*

(b) *if $n \geqslant 7$, then for each pair of positive integers k_1, k_2 with $k_1+k_2 \leqslant n-4$, $r_{n-k_2} > r_{n+1-k_2}$, $b_{k_1} < b_{k_1+1}$, and $\sum_{i=1}^{k_1} f_i = \sum_{j=1}^{k_2} f_{n+1-j}$, the following inequality holds:*

$$\sum_{i=1}^{k_1} b_i + \sum_{j=1}^{k_2} r_{n+1-j} > k_1 k_2. \tag{2.1}$$

Proof. To prove the necessity, let F be a connected equitably colored f-factor of the given colored K_n. Then by Lemma 1, F has an alternating eulerian trail $T=x_1x_2\cdots x_{2\mu}x_1$. Since F is eulerian each f_i is even. The red (respectively, blue) subgraph of F is a red (respectively, blue) $\frac{1}{2}f$-factor of K_n. Thus (a) holds.

We now show that (2.1) holds for all positive integers k_1, k_2 with $k_1+k_2 \leqslant n-4$. Suppose there exist positive integers k_1, k_2 with $k_1+k_2 \leqslant n-4$ for which (2.1) does not hold. Then let

$$X=\{v_1,\dots,v_{k_1}\}, \qquad Y=\{v_{n+1-k_2},\dots,v_n\}$$

and

$$Z=\{v_{k_1+1},\dots,v_{n-k_2}\}=V(K_n)-(X\cup Y).$$

Since $k_1+k_2 \leqslant n-4$, we have that X, Y, Z are nonempty pairwise disjoint sets. Clearly,

$$\begin{aligned}\sum_{i=1}^{k_1} b_i + \sum_{j=1}^{k_2} r_{n+1-j} &\geqslant \text{the number of } XY\text{-edges of } K_n \\ &= k_1 k_2.\end{aligned} \tag{2.2}$$

Since (2.1) does not hold, we have that equality holds in (2.2). This implies that any blue (respectively, red) edge with one end vertex in X (respectively, Y) has the other end vertex in Y (respectively, X). This further implies that all XZ-edges are red and all YZ-edges are blue. We may suppose that x_1 is in Z. Since T is eulerian there exists an integer i, $1 \leqslant i \leqslant 2\mu$, such that $x_i \in Z$ and $x_{i+1} \in X \cup Y$. Without loss of generality assume that $x_{i+1} \in X$. Since T is alternating, $x_{i+1}x_{i+2}$ is blue and $x_{i+2}x_{i+3}$ is red; and therefore $x_{i+2} \in Y$ and $x_{i+3} \in X$; and a simple induction argument implies that the edge $x_{2\mu}x_1$ has one end vertex in X and other end in Y contradicting the fact that $x_1 \in Z$. This implies that (b) holds.

To prove the sufficiency we proceed as follows: Out of all red $\frac{1}{2}f$-factors F_R and blue $\frac{1}{2}f$-factors F_B of the given colored K_n (one such pair exists by condition (a)) choose one pair for which the number d of the connected components in the graph $F_R \cup F_B$ is the minimum possible. Let $C_1, \ldots, C_d$ be the connected components in such an $F_R \cup F_B = F_0$ (say). Note that F_0 is an equitably colored f-factor of K_n and each C_i is an equitably colored graph and by Lemma 1 each C_i has an alternating eulerian trail. We shall prove that $d=1$. First we prove several assertions regarding color constraints on C_iC_j-edges and then complete the proof.

Assertion 1. *There is no red (respectively, blue) exchangeable trail T with respect to F_0 such that $|E(T) \cap E(C_i)| \leq 1$ for every i, $1 \leq i \leq d$.*

Suppose that the contrary holds. Then $d > 1$. Let $F = F_0 \Delta T$; Δ being the symmetric difference of the edge sets of F_0 and T. Then F is an equitably colored f-factor since T is an exchangeable trail. Since each C_i is equitably colored by Lemma 1, C_i is eulerian and hence a 2-edge connected graph. By hypothesis it then follows that F has fewer components than F_0, contradicting the selection of F_0.

In particular, we have the following assertion.

Assertion 2. *There is no red (respectively, blue) exchangeable trail T of length 4, hitherto referred to as a* quadrangle, *with respect to F_0 with $|E(T) \cap E(C_i)| \leq 1$, $1 \leq i \leq d$.*

Assertion 3. *If $i \neq k$, with $1 \leq i$, $k \leq d$, and $\sim(C_i \to C_k)$ and $\sim(C_k \to C_i)$, then each class of C_i (respectively, C_k) is joined in both colors to at least one class of C_k (respectively, C_i).*

Suppose that the contrary holds. Without loss of generality assume that $i=1$ and $k=2$. Let E_1, E_2 be alternating eulerian trails in C_1, C_2 respectively. We may assume that for some class of C_1, A_1 (say) all A_1A_2-edges are of the same color red (say) and A_1B_2-edges are of the same color; where A_2, B_2 are the classes of E_2. If A_1B_2-edges are red, then $A_1V(C_2)$-edges are red and by Assertion 2 and the fact that E_1, E_2 are alternating eulerian trails, it follows that $B_1V(C_2)$-edges are blue implying that $C_1 \to C_2$. If A_1B_2-edges are blue, then $A_2 \cap B_2 = \emptyset$ and by Assertion 2 we have that B_1A_2-edges are red and B_1B_2-edges are blue and this implies that $C_2 \to C_1$.

Assertion 4. *If $i \neq k$, $1 \leq i$, $k \leq d$, $\sim(C_i \to C_k)$ and $\sim(C_k \to C_i)$ and $E_i = x_1x_2 \cdots x_{2\mu}x_1$, $E_k = y_1y_2 \cdots y_{2\nu}y_1$ are alternating eulerian trails in C_i, C_k with respect to which the classes of C_i, C_k are defined, then there exists an alternating eulerian trail $E_k^* = z_1z_2 \cdots z_{2\nu}z_1$ of C_k with $\{A_k, B_k\} = \{A_k^*, B_k^*\}$ and having the*

same cycles as that E_k such that the following color restrictions hold:
For every integer j, all the edges

$$x_{1+2j}x_{2+2j},\quad z_{1+2j}z_{2+2j},\quad x_{1+2j}z_{1+2j}\quad \textit{and}\quad z_{2j}x_{2+2j}, \tag{2.3}$$

have the same color and all the edges

$$x_{2j}x_{1+2j},\quad z_{2j}z_{1+2j},\quad x_{2j}z_{2j}\quad \textit{and}\quad z_{1+2j}x_{3+2j}, \tag{2.4}$$

have the other color.

Without loss of generality assume that $i=1$, $k=2$ and $c(x_1x_2)$ is red. By Assertion 3, A_1 is joined in both colors to at least one of A_2, B_2. If A_1 is joined in both colors to A_2, then let $y'_s = y_s$ and in the other case let $y'_s = y_{s+1}$; and define $E'_2 = y'_1 y'_2 \cdots y'_{2\nu} y'_1$. Then A_1 is joined in both colors to A'_2 and $\{A'_2, B'_2\} = \{A_2, B_2\}$. Let $y'_{2l+1}x_{2m+1}$ be red. If $y'_{2l+1}y'_{2l+2}$ is red, then let $y''_s = y'_{s+2l}$ and in the other case let $y''_s = y'_{2l+2-s}$; and further define $E''_2 = y''_1 y''_2 \cdots y''_{2\nu} y''_1$. Then $y''_1 x_{1+2m}$ is red and $A''_2 = A'_2$, $B''_2 = B'_2$ and since E''_2 is an alternating eulerian trail of C_2 we have, for every integer j that

$$y''_{1+2j}y''_{2+2j} \text{ is red} \quad \text{and} \quad y''_{2j} + y''_{1+2j} \text{ is blue.} \tag{2.5}$$

Then by Assertion 2 it follows that $y''_2 x_{2+2m}$ is blue, $y''_3 x_{3+2m}$ is red and so on implying, by induction, that for every nonnegative integer r,

$$y''_{1+2r}x_{1+2m+2r} \text{ is red} \quad \text{and} \quad y''_{2r}x_{2m+2r} \text{ is blue.} \tag{2.6}$$

But for each negative integer s there is an even integer k such that $0 \leq r = k\mu \cdot \nu + s$ and this implies by (2.6) that $y''_{1+2r}x_{1+2m+2r} = y''_{1+2s}x_{1+2m+2s}$ is red and $y''_{2s}x_{2m+2s}$ is blue. Therefore for every integer r (2.6) holds. Thus, in particular, each vertex of A''_2 is joined in red to at least one vertex of A_1. As $A''_2 = A'_2$ is joined in both colors to A_1, there exists a positive integer p such that y''_{1+2p} is joined in both colors to A_1. Considering the edges $y''_{1+2p}x_{1+2r}$, $r = 0, 1, 2, \ldots$, we see that there exists an integer t such that $y''_{1+2p}x_{1+2t}$ is red and $y''_{1+2p}x_{3+2t}$ is blue. Then, by Assertion 2, it follows that for every nonnegative integer r, the edges $y''_{1+2p+2r}x_{1+2t+2r}$, $y''_{2p-2r}x_{2+2t-2r}$ are red and $y''_{2p+2r}x_{2t+2r}$, $y''_{1+2p-2r}x_{3+2t-2r}$ are blue; and then as in the proof of (2.6) for every r, it can be proved that for every r, the edges

$$\begin{cases} y''_{1+2p+2r}x_{1+2t+2r},\ y''_{2p+2r}x_{2+2t+2r} & \text{are red} \\ y''_{2p+2r}x_{2t+2r},\ y''_{1+2p+2r}x_{3+2t+2r} & \text{are blue.} \end{cases} \tag{2.7}$$

In particular when $r = -t$ we get that the edge $y''_{1+2p-2t}x_1$ is red and $y''_{1+2p-2t}x_3$ is blue. Define

$$z_s = y''_{s+2p-2t} \quad \text{and} \quad E^*_2 = z_1 z_2 \cdots z_{2\nu} z_1.$$

Then $A^*_2 = A''_2$ and $B^*_2 = B''_2$, E^*_2 and E_2 have the same cycles and further by (2.5), (2.6) for every r and by (2.7) it follows that (2.3) and (2.4) are satisfied, completing the proof of the assertion.

We now state and prove the crucial assertion we need in the proof of the Main Theorem.

Assertion 5. *For every i, j, with $1 \leq i < j \leq d$, either $C_i \to C_j$ or $C_j \to C_i$.*

Suppose that the assertion is false and without loss of generality assume that $i = 1$, $j = 2$. Let E_1, E_2 be alternating eulerian trails in C_1, C_2, respectively with respect to which the classes are defined. We consider two cases.

Case I: *There is an alternating cycle in either E_1 or E_2.*

Without loss of generality let $x_1 x_2 \cdots x_{2l} x_1$ be an alternating cycle with $x_1 x_2$ red and assume that it is in E_1 and let $E_1 = x_1 x_2 \cdots x_{2l} x_1 = x_{2l+1} x_{2l+2} \cdots x_{2\mu} x_1$. Let $E_2 = y_1 y_2 \cdots y_{2\nu} y_1$. We consider two subcases.

Subcase I.1: $\nu \leq l$. By Assertion 4 there exists an alternating eulerian trail $E_2^* = z_1 z_2 \cdots z_{2\nu} z_1$ in C_2 with color restrictions stated therein. Then

$$T_j = z_{1+2j} x_{3+2j} x_{2+2j} z_{2+2j} z_{3+2j}, \quad 0 \leq j \leq \nu - 1,$$

is a blue path with edges alternately from $E(K_n) - E(F_0)$ and $E(F_0)$. Let $T = \sum_{j=0}^{\nu-1} T_j$. Since $\nu \leq l$ and $x_1, \ldots, x_{2\nu}$ are distinct vertices, we have that T is a blue exchangeable trail with respect to F_0, (note that $z_{2\nu+1} = z_1$); and consequently $F = F_0 \Delta T$ is an equitably colored f-factor in which $C_3, \ldots, C_d$ are unaltered (if $d \geq 3$). Further,

$$\left(\sum_{j=0}^{\nu-1} (x_{1+2j} x_{2+2j} z_{2+2j} z_{1+2j} x_{3+2j})\right) + (x_{2\nu+1} x_{2\nu+2} \cdots x_{2\mu} x_1)$$

is a connected subgraph of F which contains all the vertices of C_1 and C_2. This implies that F has fewer components than F_0, contradicting the minimality of d.

Subcase I.2: $l < \nu$. By Assertion 3, A_1 is joined in both colors to at least one of A_2, B_2, we may assume that it is A_2 (by relabeling E_2 if necessary) and $y_{1+2l} y_{2+2l}$ is red (by considering E_2^{-1}, if necessary). Now suppose that $x_{1+2t} y_{1+2m}$ is red. Then as in the proof of (2.6) for every r we have that $x_{1+2t+2r} y_{1+2m+2r}$ is red. Hence each vertex of A_1 is joined in red to at least one vertex of A_2. Therefore, there exists an integer p such that x_{1+2p} is joined in blue also to a vertex of A_2. Considering the edges $x_{1+2p} y_{1+2r}$, $r = 0, 1, 2, \ldots$, it is clear that there exists an integer q such that $x_{1+2p} y_{1+2q}$ is red and $x_{1+2p} y_{3+2q}$ is blue. Then as in the proof of (2.7) we get for every r that $x_{1+2p+2r} y_{1+2q+2r}$ is red and $x_{1+2p+2r} y_{3+2q+2r}$ is blue. This implies, for every integer s, that we have

$$x_{1+2s} y_{1+2(q-p)+2s} \text{ is red} \quad \text{and} \quad x_{1+2s} y_{3+2(q-p)+2s} \text{ is blue.}$$

By Assertion 2, we get for every integer s

$$x_{2s} y_{2+2(q-p)+2s} \text{ is red} \quad \text{and} \quad x_{2s} y_{2(q-p)+2s} \text{ is blue.}$$

Let $r=2(q-p)$. Then

$$T=\sum_{j=0}^{l-1}(x_{1+2j}y_{r+3+2j}y_{r+2+2j}x_{2+2j}x_{3+2j}),$$

is a blue exchangeable trail with respect to F_0, since $l<\nu$ and $x_1,\ldots,x_{2l}$ are all distinct (note that $x_{2l+1}=x_1$). Let $F=F_0\Delta T$. Then F is an equitably colored f-factor in which the components $C_3,\ldots,C_d$ (if $d\geq 3$) are unaltered. Further,

$$\sum_{j=0}^{l-1}(y_{r+2+2j}x_{2+2j}x_{1+2j}y_{r+3+2j}y_{r+4+2j})+(y_{r+2l+2}y_{r+2l+3}\cdots y_{2\nu}y_1y_2\cdots y_{r+1})$$

and $(x_{2l+1}x_{2l+2}\cdots x_{2\mu}x_1)$, (note $x_{2l+1}=x_1$) are connected subgraphs of F having the vertex x_1 in common and together contain all the vertices of C_1 and C_2. This implies that F has fewer components than F_0, a contradiction.

Case II. *Neither E_1 nor E_2 has an alternating cycle.*

Let k be the minimum integer such that there is a cycle C, (say), of length k in either E_1 or E_2. Note that k is well defined and since we are in Case II, $k=2m+1\geq 3$. Without loss of generality assume that $C=x_1x_2\cdots x_{2m+1}x_1$ is in $E_1=x_1x_2\cdots x_{2\mu}x_1$ where $x_{2m+2}=x_1$ and x_1x_2 is red, a consequence of which is that x_1x_{2m+1} is also red. By Assertion 4 there exists an alternating eulerian circuit $E_2^*=z_1z_2\cdots z_{2\nu}z_1$ in C_2 satisfying the color and cycle restrictions stated therein. Define l^* to be the smallest positive integer such that $z_{l^*+1}\in\{z_1,z_2,\ldots,z_{l^*}\}$. By the selection of k, l^* we have that $l^*\geq k$.

We first prove

$$\text{edge } x_2z_{2j} \text{ is blue} \qquad \text{for } 1\leq j\leq[\tfrac{1}{2}l^*]. \tag{2.8}$$

By Assertion 4 we already have that x_2z_2 is blue. Suppose then that for some j with $2\leq j\leq[\frac{1}{2}l^*]$, the edge x_2z_{2j} is red. Let p be the smallest such $j\geq 2$. Let $e_1=x_1x_2$, $e_2=x_{2p-1}x_{2p}$. If $e_1=e_2$, then $x_1=x_{2p-1}$ and $x_2=x_{2p}$ and by (2.3) x_2z_{2p-2} is red contradicting the minimality of p. So $e_1\neq e_2$. Using (2.3), (2.4), the fact that x_2z_{2p-2} is blue and Assertion 2 repeatedly we get that $x_{2+(2\mu\nu-1)}z_{2p-2+(2\mu\nu-1)}=x_1z_{2p-3}$ and $z_{2p-2}x_{2p}$ are red. Therefore

$$T=(x_1z_{2p-3}z_{2p-2}x_{2p}x_{2p-1}z_{2p-1}z_{2p}x_2x_1)$$

is a red exchangeable trail with respect to F_0 as the vertices $z_{2p-3},\ldots,z_{2p}$ are distinct. So $F=F_0\Delta T$ is an equitably colored f-factor in which

$$(x_1x_{2\mu}x_{2\mu-1}\cdots x_{2p}z_{2p-2}z_{2p-1}x_{2p-1}x_{2p-2}\cdots x_2z_{2p}z_{2p+1}\cdots z_{2\nu}z_1z_2\cdots z_{2p-3}x_1)$$

is a connected subgraph of F containing all the vertices of C_1 and C_2, a contradiction. Therefore (2.8) holds.

Then, by Assertion 2, we get successively for $i=3,4,\ldots,l^*$ that whenever $i\leq i+2j\leq l^*$,

$$\text{edge } x_iz_{i+2j} \text{ is } \begin{cases}\text{blue} & \text{if } i \text{ is even,}\\ \text{red} & \text{if } i \text{ is odd.}\end{cases} \tag{2.9}$$

Also (2.8), (2.3), (2.4) and repeated use of Assertion 2 imply that the edge $x_{2+(2\mu\nu-1)}z_{2j+(2\mu\nu-1)} = x_1 z_{2j-1}$ is red whenever $1 \leq j \leq [\frac{1}{2}l^*]$, i.e.

$$\text{edge } x_1 z_{1+2j} \text{ is red} \qquad \text{whenever } 1 \leq 1+2j \leq l^*-1. \tag{2.10}$$

Now we consider two subcases.

Subcase II.1: *l^* is odd*. Let $l^* = 2r+1$. Then by the minimality of k, $m \leq r$. Also as there are no alternating cycles, equivalently even cycles, in E_2 and hence in E_2^*, we have that $z_{l^*+1} = z_{2p+1}$ for some integer $p \geq 0$. This means that the edge $z_{2r+1}z_{2r+2} = z_{2r+1}z_{2p+1}$ is red. Therefore by (2.9) and (2.10), $T = (x_1 x_{2m+1} z_{2r+1} z_{2p+1} x_1)$ is a red quadrangle with respect to F_0, contradicting Assertion 2.

Subcase II.2: *l^* is even*. Let $l^* = 2r$. By the minimality of k, $2m+1 \leq 2r$. Then $z_{2r+1} = z_{2p}$ for some integer p, $1 \leq p \leq r-1$; and as there are no alternating cycles in E_2^*, $p \geq 1$. this means that $z_{2r}z_{2p}$ is blue. By Assertion 2, with the cycle $(z_2 z_1 x_1 x_{2m+1} z_2)$ we have

$$\text{edge } z_2 x_{2m+1} \text{ is blue.} \tag{2.11}$$

If $p=1$, then by (2.9) and the fact that $2m+1 \leq 2r$, $(z_{2r} z_2 x_{2m+1} x_{2m} z_{2r})$ is a blue quadrangle with respect to F_0, contradicting Assertion 2. Thus we may assume that $p > 1$. Observe that in E_1, by the minimality of k, the vertex $x_5 \notin \{x_1, x_2, x_3, x_4\}$ even if $k=3$ and is clearly so if $k \geq 5$. If now $k > 5$, then by (2.3), (2.4) and (2.9) it follows that

$$T = (z_{2r} z_{2p} x_4 x_5 z_3 z_2 x_{2m+1} x_{2m} z_{2r})$$

is a blue exchangeable trail with respect to F_0. Let $F = F_0 \Delta T$. Then the alternating chain in F, namely

$$(z_{2r} x_{2m} x_{2m-1} \cdots x_5 z_3 z_4 \cdots z_{2p} x_4 x_3 x_2 x_1 x_{2\mu} x_{2\mu-1}$$
$$\cdots x_{2m+1} z_2 z_1 z_{2\nu} z_{2\nu-1} \cdots z_{2r+1} = z_{2p} z_{2p+1} z_{2p+2} \cdots z_{2r})$$

contains all the vertices of C_1 and C_2 implying that F has fewer components than F_0, a contradiction. Thus we may assume that $p > 1$ and $k \leq 5$.

Now as $p \geq 2$, so $l^*+1 = 2r+1 \geq 2(p+1)+1 \geq 7$. Hence by (2.8) and (2.9)

$$x_2 z_{2r},\ x_4 z_{2r} \quad \text{and} \quad x_6 z_{2r} \text{ are blue.} \tag{2.12}$$

Also as $x_1 z_{2p-1}$ is red by (2.10) and using (2.3) and Assertion 2 on $(x_1 z_{2p-1} z_{2p} x_2)$, we get

$$z_{2p} x_2 = z_{2r+1} x_2 \text{ is blue.} \tag{2.13}$$

If, moreover, $k = 2m+1 = 3$ then $z_2 x_3$ is blue by (2.11) and so by (2.4), (2.12) and (2.13) we have that $T = (z_{2r+1} x_2 x_3 z_2 z_3 x_5 x_4 z_{2r})$ is a blue exchangeable trail with respect to F_0 since $z_1, \ldots, z_{2r}$ are all distinct vertices and $x_2 x_3 \neq x_5 x_4$ since $x_4 = x_1$. Let $F = F_0 \Delta T$. Then F is an equitably colored f-factor and

$$(z_{2r+1} x_2 x_1 x_{2\mu} x_{2\mu-1} \cdots x_5 z_3 z_4 \cdots z_{2p} z_{2p+1} \cdots z_{2r} x_4 x_3 z_2 z_1 z_{2\nu} z_{2\nu-1} \cdots z_{2r+1})$$

is an alternating chain in F including all the vertices of C_1 and C_2 and as above we get a contradiction.

Thus we have proved that $p \geq 2$ and $k = 2m + 1 = 5$. This implies, by (2.11), that z_2x_5 is blue and hence, using (2.3), (2.4) and Assertion 2 first on $(z_3z_2x_5x_4)$ we get that z_3x_4 is red and then on $(z_4z_3x_4x_3)$ we get

$$z_4x_3 \text{ is blue.} \tag{2.14}$$

If $p = 2$, then $z_{2r}z_4$ is blue as $z_4 = z_{2p} = z_{2r+1}$. So by (2.4), (2.12) and (2.14) we see that $(z_{2r}z_4x_3x_2)$ is a blue quadrangle contradicting Assertion 2. So $p \geq 3$. Then by (2.4), (2.12), (2.13) and (2.14) we get

$$T = (z_4x_3x_2z_{2p} = z_{2r+1}z_{2r}x_6x_7z_5z_4)$$

is a blue exchangeable trail with respect to F_0 since $z_1, \ldots, z_{2r}$ are all distinct vertices and $x_2x_3 \neq x_6x_7$ since $x_6 = x_1$.

Let $F = F_0 \Delta T$. Then F is an equitably colored f-factor in which

$$(x_1x_{2\mu}x_{2\mu-1} \cdots x_7z_5z_6 \cdots z_{2r}x_6x_5x_4x_3z_4z_3z_2z_1z_{2\nu}z_{2\nu-1} \cdots z_{2r+1} = z_{2p}x_2x_1)$$

is a connected subgraph of F containing all the vertices of C_1 and C_2, and as above we get a contradiction.

This completes the proof of Assertion 5.

Define a directed graph D whose vertex set is $V(D) = \{C_1, \ldots, C_d\}$ and whose arc set is $E(D) = \{(C_i, C_j): 1 \leq i \neq j \leq d \text{ and } C_i \to C_j\}$.

Assertion 6. *D is a tournament without any cyclic triples.*

That D is a tournament follows from Assertion 5 and the fact that both $C_i \to C_j$ and $C_j \to C_i$ cannot hold. Let, if possible, $(C_iC_jC_kC_i)$ be a cyclic triple in D. Without loss of generality assume that $i = 1$, $j = 2$ and $k = 3$. Let $x_{11}x_{12}$, $x_{21}x_{22}$ and $x_{31}x_{32}$ be red edges of C_1, C_2 and C_3, respectively such that x_{i1} is joined in red to all vertices of $V(C_{i+1})$ where the suffix $i+1$ is to be taken modulo 3. Then $T = (x_{11}x_{22}x_{21}x_{32}x_{31}x_{12}x_{11})$ is a red exchangeable trail with respect to F_0 satisfying the condition of Assertion 1, contradicting its conclusion, this completes the proof of the assertion.

Assertion 6 implies that D is a transitively oriented tournament. So there exists a vertex of D; say C_1, such that $C_1 \to C_i$, for every i, $2 \leq i \leq d$. Let $Z = \bigcup_{i=2}^{d} V(C_i)$.

Assertion 7. *All the A_1Z-edges are of the same color and all B_1Z-edges are of the other color.*

Suppose that the contrary holds and without loss of generality assume that $A_1V(C_2)$-edges are red and $A_1V(C_3)$-edges are blue and $C_2 \to C_3$ with $A_2V(C_3)$-edges red. Then since $C_1 \to C_3$, $B_1V(C_3)$-edges are red. Let $x_{i1}x_{i2}$, $1 \leq i \leq 3$, be a red edge of C_i with $x_{i1} \in A_i$ and $x_{i2} \in B_i$, $1 \leq i \leq 3$. Then $T = (x_{11}x_{22}x_{21}x_{32}x_{31}x_{12}x_{11})$ is a red exchangeable trail with respect to F_0 satisfying the condition of Assertion 1, a contradiction.

Let $|A_1| = k_1$ and $|B_1| = K_2$. Then since $d \geq 2$ and $A_1 \cup B_1 = V(C_1)$ it follows that $k_1 + k_2 \leq n - 4$. Assume further that A_1Z-edges are red;

Assertion 8. *There exists a blue A_1A_1-edge or a red B_1B_1-edge (such an edge is not in F_0 as $A_1 \cap B_1 = \emptyset$).*

Suppose that this is not true. Then

$$\text{every blue (resp. red) edge with one end vertex in } A_1 \text{ (resp. } B_1\text{)} \\ \text{has the other end in } B_1 \text{ (resp. } A_1\text{)} \tag{2.15}$$

Let $u \in A_1$, $v \in Z$, $w \in B_1$. Then

$$r(u) = r_{K_n}(u) \geq k_1 - 1 + n - k_1 - k_2 = n - k_2 - 1,$$

and

$$r(v) = r_{K_n}(v) \leq n - k_1 - k_2 - 2 + k_1 = n - k_2 - 2,$$

since there is at least one ZZ-blue edge at v. Therefore $r(v) < r(u)$. Also since $k_1 + k_2 \leq n - 4$, $r(w) = r_{K_n}(w) \leq k_1 \leq n - k_2 - 2 < r(u)$. Therefore $A_1 = \{v_1, \ldots, v_{k_1}\}$ and $r_{k_1} > r_{k_1+1}$, and since $r_i + b_i = n - 1$, it follows that $b_{k_1} < b_{k_1+1}$. Similarly we get that $B_1 = \{v_{n+1-k_2}, v_{n+2-k_2}, \ldots, v_n\}$, $b_{n+1-k_2} > b_{n-k_2}$ and $r_{n-k_2} > r_{n+1-k_2}$. Since C_1 is a bipartite graph with the bipartition A_1, B_1, we have also that $\sum_{i=1}^{k_1} f_i = \sum_{i=1}^{k_2} f_{n+1-i}$. Then by (2.15) it follows that

$$\sum_{i=1}^{k_1} b_i + \sum_{i=1}^{k_2} r_{n+1-i} = k_1 k_2,$$

contradicting condition (b) of the Main Theorem (see (2.1)).

Now we are ready to complete the proof of the Main Theorem. By Assertion 8, we may without loss of generality assume that x_1x_{2p+1} is a blue edge for some positive integer p where $x_1 \in A_1$. Note that $x_1x_{2p+1} \notin F_0$. Let $E_1 = x_1x_2 \cdots x_{2\mu}x_1$ and $E_2 = y_1y_2 \cdots y_{2\nu}y_1$ be alternating eulerian trails in C_1 and C_2 respectively, with the edges x_1x_2, y_1y_2 blue. Then $T = x_1x_{2p+1}x_{2p+2}y_1y_2x_2x_1$, is a blue exchangeable trail with respect to F_0. Let $F = F_0 \Delta T$. Then F is an equitably colored f-factor in which $C_3, \ldots, C_d$ are unaltered and

$$S = x_1x_{2p+1}x_{2p} \cdots x_2y_2y_3 \cdots y_{2\nu}y_1x_{2p+2}x_{2p+3} \cdots x_{2\mu}x_1$$

is a connected subgraph of F containing all the vertices of C_1 and C_2. This implies that F has fewer components than d, a contradiction. □

From the above theorem we have the following corollary.

Corollary 2. *Under the notation of the Main Theorem, the colored complete graph K_n contains a connected equitable colored $\widetilde{2s}$-factor, where $\widetilde{2s} = (2s, \ldots, 2s)$ is of* *length n and s is a positive integer, if and only if condition* (a) *of the Main Theorem and condition* (C.1) *below are satisfied:*

(C.1) *If $n \geq 7$, then for every positive integer $k \leq \frac{1}{2}(n-4)$ we have*

$$\sum_{i=1}^{k} b_i + \sum_{j=1}^{k} r_{n+1-j} > k^2.$$

Proof. We need only note that (C.1) implies condition (b) of the Main Theorem and that the necessity of (C.1) has been established in the proof of the necessity of the Main Theorem.

Remark 3. It may be remarked that a connected equitably colored 2-factor of the given colored K_n is an alternating hamiltonian cycle and therefore conditions (a) and (C.1) of Corollary 2 are necessary and sufficient for the existence of an alternating hamiltonian cycle in the colored K_n, a result proved earlier by Bánkfalvi and Bánkfalvi [1] answering a problem of Erdös.

We further deduce the following corollary for self-complementary colorations of K_n. To this end, we need the definition of a particular self-complementary coloration of K_{4N}, denoted by K_{4N}^*. Let $V(K_{4N}) = \{v_1, \ldots, v_{4N}\}$. For distinct i, j, the edge v_iv_j is colored red if and only if $1 \leq i < j \leq 2N$; or $1 \leq i \leq N$ and $2N+1 \leq j \leq 3N$; or $N+1 \leq i \leq 2N$ and $3N+1 \leq j \leq 4N$; and all the other edges of K_{4N} are colored blue. It is easy to check that the permutation σ of $V(K_{4N})$, namely

$$\sigma = (v_1 v_{2N+1} v_{N+1} v_{3N+1} v_2 v_{2N+2} v_{N+2} v_{3N+2} \cdots v_N v_{3N} v_{2N} v_{4N}$$

is an isomorphism from the red subgraph onto the blue subgraph of K_{4N}^* and therefore K_{4N}^* is a self-complementary coloration of K_{4N}.

Corollary 4. *Given a self-complementary coloration of K_n, $n \geq 8$, different from K_{4N}^*, then under the notation of the Main Theorem, the following are equivalent:*

(1) *For every $k \leq \frac{1}{2}(n-4)$ with $r_k > r_{k+1}$,*

$$\sum_{i=1}^{k} r_i < k(n-k-1) + \sum_{j=1}^{k} r_{n+1-j}. \tag{2.16}$$

(2) *The red subgraph (or the blue subgraph) of K_n has a hamiltonian cycle.*

(3) *The given colored K_n has a connected equitably colored $\bar{4}$-factor, where $\bar{4} = (4, \ldots, 4)$ is of length n.*

Further if $n \equiv 0 \pmod 4$, then (1), (2), (3) *and* (4) *given below are equivalent.*

(4) *The given colored K_n has an alternating hamiltonian cycle.*

(*Also for K_{4N}^* $(N \geq 2)$,* (1), (3) *and* (4) *hold*).

Proof. That (1) imples (2) follows from the Main Theorem B of Rao [9]. To prove that (2) implies (3), let C be a red hamiltonian cycle in K_n. Then $\sigma(C)$

where σ is an isomorphism of the red subgraph of K_n onto the blue subgraph is a blue hamiltonian cycle. This implies that $C \cup \sigma(C)$ is a connected equitably colored $\bar{4}$-factor of the given colored K_n. That (3) implies (1) follows from (2.1) of the Main Theorem of this paper and the facts that in any self-complementary coloration $r_i = b_{n+1-i}$ and $r_i + b_i = n-1$, $1 \leq i \leq n$. To prove the equivalence of (1), (2), (3) and (4) when $n \equiv 0 \pmod 4$, we shall prove that (1) and (4) are equivalent. Already we know that (4) implies condition (b) of the Main Theorem for $f = (2, \ldots, 2)$, of length n, which proves that (1) holds since the coloration is self-complementary. To prove that (1) implies (4), we observe that (1) implies (2) and therefore the red subgraph (or the blue subgraph) of K_n has a red (blue) hamiltonian cycle and since the coloration is self-complementary, the blue subgraph also has a hamiltonian cycle. Since $n \equiv 0 \pmod 4$, so the red (resp. blue) subgraph has a 1-factor and hence condition (a) of the Main Theorem is satisfied with $f = (2, \ldots, 2)$ of length n. Also, since we have a self-complementary coloration, (2.15) implies that condition (b) of the Main Theorem holds. Hence by the Main Theorem the given colored K_n has an alternating hamiltonian cycle. □

We conjecture that the following are necessary and sufficient conditions for the existence of a connected equitably colored $\widetilde{2s}$-factor, where $\widetilde{2s} = (2s, \ldots, 2s)$ is of length n, in self-complementary colorations of K_n where $n \equiv 0$ or $1 \pmod 4$ (under the notation of the Main Theorem).

(1) $(r_1 - s, \ldots, r_n - s)$ is a graphic degree sequence.

(2) For every $k \leq \frac{1}{2}(n-4)$ with $r_k > r_{k+1}$,

$$\sum_{i=1}^{k} r_i < k(n-k-1) + \sum_{j=1}^{k} r_{n+1-j}.$$

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