MATHEMATICAE
FUNDAMENTA
XCIII (1976)

Selection theorems for partitions of Polish spaces

by

A. Maitra and B. V. Rao (Calcutta)

Abstract. In this paper we evaluate the (Borel or projective) class of selectors for partitions of Polish spaces into disjoint closed sets. In particular, we improve upon the results pertaining to α^- partitions which have been obtained recently by Kuratowski and Maitra.

1. Introduction. The problem of the existence of "topologically pleasant" selectors for partitions of a Polish space into disjoint, non-empty, closed sets, where the partitions themselves are "topologically pleasant", has been considered by several authors. We mention here the articles of Mazurkiewicz [8], Bourbaki [2], and Kuratowski and Maitra [7].

In this paper we shall be mainly concerned with the evaluation of the (Borel or projective) class of selectors. The first such result known to us was proved by Mazurkiewicz ([8] and [5], p. 389). He showed that any partition of a closed subset of the space of irrationals which is induced by a continuous function defined on it to a separable metric space admits a coanalytic selector. In the same spirit, Bourbaki proved that any upper semi-continuous partition of a Polish space into closed sets admits a G_δ selector ([2], Chap. 9, Ex. 9(a), p. 262). Kuratowski and Maitra [7] extended Bourbaki's result by showing that any α^+ or α^- partition of a Polish space into closed sets admits a selector of multiplicative class $(\alpha+1)$ (for definitions, see Section 2).

We shall establish in this paper some general results on the existence of selectors, from which it will follow that the results of Kuratowski and Maitra for α^- partitions can be improved at all levels $\alpha>0$. Indeed, if $\alpha>0$, we prove that any α^- partition of a Polish space admits a selector of multiplicative class α, and, moreover that, in general, a selector of lower class does not exist.

Our method of defining a selector is as follows. We first define a suitable linear order on each Polish space such that every non-empty closed set has a first element. We achieve this by using a result of Arhangel'skiĭ [1], which states that every Polish space is a continuous open image of the space of irrationals. Using such a continuous open function, we transfer the lexicographic order on the space of irrationals to the given Polish space. The selector is now taken to be the set of all first elements of members of the given partition. Our results on the existence of tractable linear orders on Polish spaces seem to be of independent interest and have some connections with the work of Engelking, Heath and Michael [3].

The paper is organized as follows. Section 2 contains the basic definitions and notation. Section 3 shows how Polish spaces can be linearly ordered. In Section 4 we prove the main selection theorems. Section 5 is devoted to examples which establish that some of our results cannot be further improved upon as far as the class of the selector is concerned.

2. Definitions and notation. Let X be a Polish space. By a *partition* of X is meant a family of disjoint, non-empty, closed subsets of X whose union is X. If Q is a partition of X, we write $x \sim y$ to mean that x and y belong to the same element of Q. For $A \subseteq X$, the *saturation* of A with respect to Q is the union of all elements of Q which have a non-empty intersection with A. A^* will denote the saturation of A.

Let L be a σ-additive lattice of subsets of X. $-L$ stands for the family of complements of sets belonging to L. A partition Q of X is said to be a L^- (resp. L^+) *partition* just in case the saturation of every open (resp. closed) subset of X with respect to Q belongs to L (resp. $-L$). In particular, if L is the σ-additive lattice of subsets of X of additive class α, a L^- (resp. L^+) partition of X will also be called a α^- (resp. α^+) *partition*. Note that 0^- (resp. 0^+) partitions of X are just the lower semi-continuous (resp. upper semi-continuous) partitions of X. A partition is *continuous* if it is both lower and upper semi-continuous. A partition Q of X is said to be *analytic* if the saturation of each open subset of X with respect to Q is analytic. A *selector* for a partition Q of X is a subset S of X such that S intersects each element of Q in a single point.

A *linear order* on X is an anti-reflexive, transitive and connected binary relation on X with field equal to X. Let R be a linear order on X. If A is a non-empty subset of X, then x is said to be the *R-first element* of A if $x \in A$ and $\forall y(y \in A$ and $y \neq x \rightarrow xRy)$. x is a *jump point* of R if x has an immediate successor. The immediate successor of x, if it exists, is clearly unique and is denoted by x^+.

Denote by N the set of positive integers. The usual order on N is denoted by $<$. For each $k \in N$, P_k is the set of finite sequences of positive integers of length k. Set $P = \bigcup_{k=1}^{\infty} P_k$. For $p \in P$ and $1 \leq i \leq \text{length } p$, p_i is the ith coordinate of p. We define a partial order $<_*$ on P as follows: for $p, q \in P$, $p <_* q$ if length $p \leq$ length q and there is i such that $1 \leq i \leq \text{length } p$, $p_j = q_j$ for $1 \leq j \leq i-1$, and $p_i < q_i$. Write $p \leq_* q$ to mean $p <_* q$ or $p = q$.

We set $\Sigma = N^N$, the set of infinite sequences of positive integers. Equipped with the product of discrete topologies on N, Σ becomes a homeomorph of the space of irrationals. If $\sigma \in \Sigma$ and $i \in N$, σ_i will denote the ith coordinate of σ. For $p \in P_k$, $\Sigma(p)$ is the set of all $\sigma \in \Sigma$ such that $(\sigma_1, \ldots, \sigma_k) = p$. The lexicographic order on Σ will be denoted by $<^*$ and is defined as follows: $\sigma <^* \tau$ if there is a $k \in N$ such that $\sigma_i = \tau_i$ for $1 \leq i \leq k-1$, and $\sigma_k < \tau_k$. $\sigma \leq^* \tau$ means that $\sigma <^* \tau$ or $\sigma = \tau$.

Throughout, the real line is assumed to be equipped with the usual topology, while subsets of the real line are endowed with the corresponding relativized topology.

If X is topological space, $F(X)$ denotes the collection of all non-empty closed subsets of X. $F(X)$ is endowed with the Vietoris topology.

3. Linear orders on Polish spaces. The present section deals with the problem of defining suitable linear orders on Polish spaces. The results will then be used in the next section to deduce selection theorems.

THEOREM 3.1. *Let X be a Polish space. Then there exists a linear order R on X satisfying the following conditions:*

(a) *each non-empty closed subset of X has an R-first element,*

(b) *for each $a \in X$, the set $\{x \in X\colon xRa\}$ is open in X,*

(c) *there is a countable set $D \subseteq X$ such that*

$$\forall x \forall y (xRy \to (\exists z \in D)(xRz \text{ and } \neg(yRz))).$$

Proof. According to a result of Arhangel'skiĭ ([1], Cor. 4.7), there is a continuous open function f on Σ onto X. Let

$$R = \{(x, y) \in X \times X\colon (\exists \sigma \in f^{-1}(\{x\}))(\forall \tau \in f^{-1}(\{y\}))(\sigma <^* \tau)\}.$$

It is easy to check that R is a linear order on X.

Let C be a non-empty closed subset of X. Then $f^{-1}(C)$ is a non-empty closed subset of Σ. Let σ_0 be the lexicographic minimum of $f^{-1}(C)$. If $x_0 = f(\sigma_0)$, we assert that x_0 is the R-first element of C. To see this, let $x \in C$ and $x \neq x_0$. Then $\sigma_0 \notin f^{-1}(\{x\})$ and $f^{-1}(\{x\}) \subset f^{-1}(C)$. Hence, for any τ such that $f(\tau) = x$, we have: $\sigma_0 \neq \tau$ and $\tau \in f^{-1}(C)$. Consequently, $\sigma_0 <^* \tau$, and thus, $x_0 R x$.

In order to establish (b) and (c), notice that

$$xRy \leftrightarrow (\exists p \in P)\big(x \in f(\Sigma(p)) \text{ and } (\forall q \in P)\big(q \leqslant_* p \to y \notin f(\Sigma(q))\big)\big),$$

so that

$$R = \bigcup_{p \in P} \Big(f(\Sigma(p)) \times \Big(X - \bigcup_{q \leqslant_* p} f(\Sigma(q))\Big)\Big).$$

Since f is open, $f(\Sigma(p))$ is open in X, and $X - \bigcup_{q \leqslant_* p} f(\Sigma(q))$ is closed in X.

Now, for fixed $a \in X$, the set $\{x \in X\colon xRa\}$ is the horizontal section at a of R. So, it follows from the above representation of R that $\{x \in X\colon xRa\}$ is a union of some of the sets $f(\Sigma(p))$ and, consequently, open in X.

For $p \in P$, set $E(p) = X - \bigcup_{q \leqslant_* p} f(\Sigma(q))$. Let

$$D = \{z \in X\colon (\exists p \in P)(z \text{ is the } R\text{-first element of } E(p))\}.$$

Plainly D is countable. Moreover, if xRy there is a $p \in P$ such that $(x, y) \in f(\Sigma(p)) \times E(p)$. Taking z to be the R-first element of $E(p)$, we get: xRz and $\neg(yRz)$. This completes the proof.

In Section 5 we shall give an example to show that it is not always possible to define a linear order R on a Polish space X so that each non-empty closed subset of X has an R-first element and also that the order topology induced on X by R is coarser than the given topology. However, as the next result shows, this is always possible for 0-dimensional Polish spaces.

THEOREM 3.2. *Let X be a 0-dimensional Polish space. Then there exists a linear order R on X satisfying the following conditions:*

(a) *each non-empty closed subset of X has an R-first element,*

(b) *the order topology induced by R on X is coarser than the topology of X,*

(c) *R admits at most countably many jump points,*

(d) *there is a countable set $D \subset X$ such that*

$$\forall x \forall y (xRy \;\&\; (\exists z)(xRz \;\&\; zRy) \rightarrow (\exists z \in D)(xRz \;\&\; zRy)).$$

Proof. Being a 0-dimensional Polish space, X can be regarded as a closed subset of Σ ([5], p. 348). Take R to be the restriction of the lexicographic order $<^*$ to X. It is well known that R satisfies conditions (a) and (b) above.

For $\sigma, \tau \in \Sigma$ such that $\sigma <^* \tau$, let $J(\sigma, \tau) = \{\varrho \in \Sigma \colon \sigma <^* \varrho <^* \tau\}$. Let D be the family of all intervals $J(\sigma, \tau)$ such that $\sigma, \tau \in X$, $\sigma <^* \tau$ and $J(\sigma, \tau) \cap X = \emptyset$. If $J(\sigma_1, \tau_1)$ and $J(\sigma_2, \tau_2)$ belong to D, then, as is easy to check, $(\sigma_1, \tau_1) = (\sigma_2, \tau_2)$ or $J(\sigma_1, \tau_1) \cap J(\sigma_2, \tau_2) = \emptyset$. Now each $J(\sigma, \tau) \in D$ is non-empty and open in Σ. Consequently, since Σ is separable, D is countable. The left end-points of intervals in D are just the jump points of R. Thus, we have checked condition (c).

Finally, for (d), it suffices to let D be any countable dense set in X. Such that a D works follows from (b). This completes the proof.

4. Selection theorems. Using the results on linear orders established in the previous section, we shall first prove two general selection theorems, from which the results on selectors for α^+ and α^- partitions will follow.

THEOREM 4.1. *Let X be a Polish space, and let L be a σ-additive lattice of subsets of X containing all closed subsets of X. If Q is an L^- partition of X, then there is a selector S for Q such that $S \in -L$.*

Proof. According to Theorem 3.1, there is a linear order R on X and a countable subset D of X satisfying conditions (a), (b) and (c) of Theorem 3.1. Define $S \subseteq X$ by:

$$x \in S \leftrightarrow (\forall y)(yRx \rightarrow \neg(x \sim y)).$$

In other words, S is the set of R-first elements of the members of the partition Q. That R-first elements of members of Q exist follows from condition (a) of Theorem 3.1 and the fact that members of Q are closed in X. Clearly, then, S is a selector for Q.

Next, notice that

$$x \notin S \leftrightarrow (\exists y)(y \sim x \;\&\; yRx) \leftrightarrow (\exists a \in D)(x \in \{z \colon zRa\}^* \;\&\; \neg(xRa)).$$

The last equivalence is by virtue of condition (c) of Theorem 3.1. Now observe that, for fixed $a \in X$, the set $\{z \colon zRa\}^*$, being the saturation of the open set $\{z \colon zRa\}$, belongs to L; moreover, the set $\{z \colon \neg(zRa)\}$ is closed and so also belongs to L. Since

$$X - S = \bigcup_{a \in D} (\{z \colon zRa\}^* \cap \{z \colon \neg(zRa)\}),$$

it follows that $X-S$ is a countable union of sets belonging to L, and, consequently, $X-S \in L$. This terminates the proof.

It should be mentioned that Theorem 4.1 has certain similarities with the main theorem of Section 3 in [7]. However, neither implies the other. On the other hand, we are able to deduce from our formulation sharper results about α^- partitions than in [7].

THEOREM 4.2. *Let X be a 0-dimensional Polish space, and let L be a σ-additive lattice of subsets of X containing all clopen subsets of X. If Q is a L^- partition of X, then there is a selector S for Q such that $S \in -L$.*

Proof. Use Theorem 3.2 to get a linear order R on X and a countable set $D \subseteq X$ satisfying conditions (a)-(d) of the same theorem. Denote by D' the set of jump points of R. According to condition (c) of Theorem 3.2, D' is countable. Now define $S \subseteq X$ as in the proof of Theorem 4.1. As before, S is easily seen to be a selector for Q.

To show $S \in -L$, observe that

$$\begin{aligned} x \notin S &\leftrightarrow (\exists y)(y \sim x \;\&\; yRx) \\ &\leftrightarrow [(\exists\, a \in D)(x \in \{z\colon zRa\}^* \;\&\; aRx) \\ &\qquad \text{or } (\exists\, a \in D')(x \in \{z\colon zRa^+\}^* \;\&\; aRx)]\,, \end{aligned}$$

so that

$$X-S = \bigcup_{a \in D}(\{z\colon zRa\}^* \cap \{z\colon aRz\}) \cup \bigcup_{a \in D'}(\{z\colon zRa^+\}^* \cap \{z\colon aRz\})\,.$$

Since the order topology induced by R is coarser than the topology of X, the sets $\{z\colon zRb\}$ and $\{z\colon bRz\}$ are open in X for any $b \in X$. Furthermore, as each open subset of X is a countable union of clopen sets, it follows that L contains all open subsets of X. $X-S$ is, therefore, a countable union of elements of L, and hence, belong to L. This completes the proof.

The proof of Theorem 4.2 suggests the following general selection theorem for lower semi-continuous partitions.

THEOREM 4.3. *Let X be a topological space such that there is a linear order R on X satisfying conditions* (a) *and* (b) *of Theorem* 3.2. *If Q is any lower semi-continuous partition of X, then there exists a closed selector for Q.*

Proof. Let D be the set of jump points of R. Define $S \subseteq X$ as in the proof of Theorem 4.1, so that S is a selector for Q. One easily checks that

$$X-S = \bigcup_{a \in X}(\{z\colon zRa\}^* \cap \{z\colon aRz\}) \cup \bigcup_{a \in D}(\{z\colon zRa^+\}^* \cap \{z\colon aRz\})\,.$$

Plainly, $X-S$ is open, which completes the proof.

COROLLARY 4.4. *Any analytic partition of a Polish space admits a coanalytic selector.*

Proof. The above result follows from Theorem 4.1 by taking L to be the family of analytic subsets of the given Polish space.

4*

Note that Corollary 4.4 generalizes the result of Mazurkiewicz which was quoted in the introduction.

COROLLARY 4.5. *If* $\alpha>0$, *any* α^- *partition of a Polish space admits a selector of multiplicative class* α.

Proof. This again follows from Theorem 4.1. This time one takes L to be the family of subsets of the Polish space which are of additive class α.

For any α, an α^+ partition of a Polish space is, plainly, an $(\alpha+1)^-$ partition. Consequently, an immediate consequence of Corollary 4.5 is

COROLLARY 4.6. *Any* α^+ *partition of a Polish space admits a selector of multiplicative class* $\alpha+1$.

Since any 0^- partition of a Polish space is also a 1^- partition, Corollary 4.5 yields

COROLLARY 4.7. *Any lower semi-continuous partition of a Polish space admits a* G_δ *selector.*

It should be noted that Corollaries 4.6 and 4.7 were first proved by Kuratowski and Maitra [7]. Their method of proof, however, is quite different from ours.

As will be seen in Section 5, a lower semi-continuous partition of a Polish space need not, in general, admit even a F_σ selector, so that Corollary 4.7 gives the best general result for lower semi-continuous partitions of a Polish space. However, in certain special cases, one can do better than Corollary 4.7. We now proceed to describe some of these special situations.

COROLLARY 4.8. *Each lower semi-continuous partition of a* 0*-dimensional Polish space admits a closed selector.*

Proof. Take L to be the family of open subsets of the Polish space and use Theorem 4.2.

Our next result is about partitions whose members are not necessarily closed. It is clear that one may define lower semicontinuity for such partitions just as before, i.e., by requiring that the saturation of each open set be open.

THEOREM 4.9. *Let* X *be a subset of the real line, and let* R *be the restriction to* X *of the usual order on the real line. Suppose that* Q *is a lower semi-continuous partition of* X *into arbitrary sets such that either* (i) *every element of* Q *has an* R*-first element, or* (ii) *every element of* Q *has an* R*-last element. Then there is a selector* S *for* Q *such that* S *is closed in* X.

Proof. Assume (i). Let S be the set of R-first elements of members of Q. Plainly, S is a selector for Q. To show that S is closed in X, it suffices now to imitate the proof of Theorem 4.3.

Now assume (ii). The proof is as above, except that we now work with the linear order R', which is defined by: $xR'y$ if yRx.

COROLLARY 4.10. *Let* X *be a subset of the real line. If* Q *is a lower semi-continuous partition of* X *into compact sets, then there is a selector for* Q *which is closed in* X.

The remaining results of this section are again about partitions, all of whose members are closed.

COROLLARY 4.11. *Let Q be a lower semi-continuous partition of the real line. Then there is a selector for Q which is simultaneously an F_σ and a G_δ subset of the real line.*

Proof. Denote by $Q(0)$ the (unique) member of Q containing 0. Let $X_1 = (0, \infty) - Q(0)$, so that X_1 is an open subset of the real line. Let Q_1 be the restriction of Q to X_1, i.e., let $Q_1 = \{E \cap X_1 : E \in Q \,\&\, E \cap X_1 \neq \emptyset\}$. It is easy to verify that Q_1 is a lower semi-continuous partition of X_1 into closed sets. Indeed, if $E \in Q$ and $E \cap X_1 \neq \emptyset$, then $E \cap X_1$ is a closed subset of the real line, for $E \cap X_1 = E \cap [0, \infty)$. Thus, each member of Q_1 is a non-empty, closed, lower-bounded subset of the real line, and consequently, has a minimum (relative to the usual order on the real line). Hence, by Theorem 4.9, there is a selector S_1 for Q_1 such that S_1 is closed in X_1.

Similarly, let $X_2 = (-\infty, 0) - Q(0)$ and let

$$Q_2 = \{E \cap X_2 : E \in Q \,\&\, E \cap X_2 \neq \emptyset\}.$$

By an argument similar to the above, one can show that there is a selector S_2 for Q_2 such that S_2 is closed in X_2.

Denote by T the saturation of $(0, \infty)$ with respect to Q. Let $S = S_1 \cup \cup (S_2 - T) \cup \{0\}$. It is clear that S is a selector for Q and that S is simultaneously an F_σ and a G_δ subset of the real line. This completes the proof.

COROLLARY 4.12. *Let X be the unit circle equipped with the usual topology. If Q is a lower semi-continuous partition of X, then there is a selector for Q which is simultaneously an F_σ and a G_δ subset of X.*

Proof. Remove an element of Q from X and imbed the remainder as a subset of the real line. The desired result now follows from Corollary 4.10.

5. Examples. In this section we present examples to show that several of our results are optimal.

EXAMPLE 1. Fix $\alpha \geqslant 0$. Choose $E \subseteq [0, 1]$ such that E contains the point $\frac{1}{2}$, E is symmetric about $\frac{1}{2}$, and such that E is of multiplicative class α but not of additive class α. Let

$$Q = \{\{x\} : x \in E\} \cup \{\{x, 1-x\} : x \in [0, 1] - E\}.$$

Claim. Q is a α^- partition of $[0, 1]$. For, if V is an open subset of $[0, 1]$, then $V^* = V \cup \varphi(V - E)$, where φ is the homeomorphism $x \to 1-x$. Since $V - E$ is of additive class α, so is $\varphi(V - E)$. Consequently, V^* is of additive class α.

Now assume that S is a selector for Q of additive class α. Since $E \subseteq S$, it follows that $[0, 1] - E = ([0, 1] - S)^* = ([0, 1] - S) \cup \varphi([0, 1] - S)$, so that E is of additive class α. Contradiction! Thus, there is no selector for Q of additive class α.

This shows that Corollary 4.5 cannot be improved upon as far as the class of the selector is concerned.

EXAMPLE 2. Let X be the real line. For $x, y \in X$, define $x \sim y$ if $x-y$ is an integer. Then $\sim$ is an equivalence relation on X. Let Q be the set of equivalence classes of $\sim$. As is easy to check, Q is a lower semi-continuous partition of X into closed sets.

Let S be a selector for Q. For distinct integers m, n, $(S+m) \cap (S+n) = \emptyset$, where $S+m = \{x+m: x \in S\}$. Moreover, $X = \bigcup_{n=-\infty}^{\infty} (S+n)$. Since X is connected, S is not open in X. Nor can S be closed in X. For, if it were, then the real line would be a denumerable union of disjoint, non-empty, closed sets, which is impossible ([4], p. 178).

This example shows that the condition regarding the compactness of elements of the partition in Corollary 4.10 cannot be relaxed. The question now arises if Corollary 4.10 extends to spaces other than subsets of the real line. The answer is no, as the following example shows.

EXAMPLE 3. Let X be the unit square $[0, 1] \times [0, 1]$, equipped with the usual topology. Let

$$Q = \{\{(x, y), (1-x, 1-y)\}: (x, y) \in X\}.$$

It can easily be checked that Q is a continuous partition of X. Denote the homeomorphism $(x, y) \to (1-x, 1-y)$ by φ.

Now let S be a closed selector for Q. Note that $S - \{(\frac{1}{2}, \frac{1}{2})\}$ and $\varphi(S - \{(\frac{1}{2}, \frac{1}{2})\})$ form a non-trivial disconnection of $X - \{(\frac{1}{2}, \frac{1}{2})\}$. But this is impossible, for, as is well known, $X - \{(\frac{1}{2}, \frac{1}{2})\}$ is connected. Thus, Q does not admit a closed selector.

Observe also that there does not exist a linear order on X satisfying conditions (a) and (b) of Theorem 3.2. This is now an easy consequence of Theorem 4.3.

It is also interesting to note that there is no continuous selection (in the sense of [3]) on $F(X)$. This follows from the fact that Q is a continuous partition and that Q admits no closed selector. Indeed, this proves more, viz., that there is no continuous selection on Q, where Q is now regarded as a subset of $F(X)$ and is endowed with the relative topology inherited from $F(X)$. This should be compared with Proposition 5.1 in [3].

EXAMPLE 4. A better example than the above — better because the underlying space is one-dimensional — from which the same negative consequences can be deduced is this. Take X to be the unit circle endowed with the usual topology. Let Q be the collection of all diametrically opposite two-point subsets of X. It is easy to check that Q is a continuous partition of X. That there is no closed selector for Q follows from the connectedness of X.

EXAMPLE 5. Start with circle group C as above. For x, y in the circle group, define $x \sim y$ if $x = y$ or $x = y + \pi$. $\sim$ is an equivalence relation and the collection of all equivalence classes of $\sim$ constitutes a continuous partition of the circle group (indeed, this was just the partition considered in Example 4). Define $F: C \to F(C)$ by: $F(x) = \{x, x+\pi\}$, so that F is continuous.

Now let X be the product of denumerably many copies of C and equip X with the product topology. Define $G: X \to F(X)$ by: $G((x_n)) = F(x_1) \times F(x_2) \times \ldots$ Since F is continuous, it follows that G is continuous. The last assertion can be established in a manner analogous to the proof of Theorem 12 in [6]. Letting

$$Q = \{G((x_n)): (x_n) \in X\},$$

we see that Q is a continuous partition of X.

Let S be a selector for Q. We now assert that, if K is a compact subset of S, then K^* is nowhere dense in X. To see this, consider the product $\{0,1\}^N \times X$, where $\{0,1\}^N$ is endowed with the product of discrete topologies. Define a function $\varphi: \{0,1\}^N \times X \to X$ by: $\varphi((\varepsilon_n), (x_n)) = (x_n + \varepsilon_n \pi)$. It is easily checked that φ is continuous and onto X. Next, if $E \subseteq X$, $E^* = \varphi(\{0,1\}^N \times E)$. Since $K \subseteq S$, the restriction of φ to $\{0,1\}^N \times K$ is one-one. Hence, as K is compact, the restriction of φ to $\{0,1\}^N \times K$ is a homeomorphism onto K^*. Now assume, by way of contradiction, that K^* is not nowhere dense. So there is a non-empty, connected, open subset V of X such that $V \subseteq K^*$. The set $\varphi^{-1}(V) \cap (\{0,1\}^N \times K)$ is, then, a non-empty, connected, open subset of $\{0,1\}^N \times K$. Projecting $\varphi^{-1}(V) \cap (\{0,1\}^N \times K)$ to the first coordinate, we get a non-empty, connected, open subset of $\{0,1\}^N$, which is clearly impossible. Thus, K^* is nowhere dense in X.

It follows immediately that S cannot be an F_σ in X. For, if it were, S would be equal to $\bigcup_{n=1}^{\infty} K_n$, where K_n's are compact. But $X = S^* = \bigcup_{n=1}^{\infty} K_n^*$, so that the above considerations imply that X is meagre. This contradicts the Baire Category Theorem.

We have thus proved that the continuous partition Q does not admit an F_σ selector. Consequently, our results regarding selectors for lower semi-continuous partitions (Corollary 4.7) as well as for upper semi-continuous partitions (Corollary 4.6) cannot be improved upon.

Example 6. The following is an example of an upper semi-continuous partition of a 0-dimensional Polish space which does not admit an F_σ selector. Take X to be the Cantor set. Define a function $f: X \to [0,1]$ by:

$$f\left(\sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n},$$

where $\varepsilon_n = 0$ or 1. As is well known, f is continuous and onto $[0,1]$. Let Q be the partition of X induced by f, i.e., let $Q = \{f^{-1}(\{y\}): y \in [0,1]\}$. Since f is a closed mapping, Q is upper semi-continuous.

Suppose now that S is a selector for Q. It is easily checked that both S and $X-S$ are dense in X and that $X-S$ is countable. Moreover, if K is compact and $K \subset S$, then K is nowhere dense. It follows that, if S is an F_σ in X, then S is meagre in X. Consequently, as $X-S$ is also meagre in X, it follows that X is meagre in itself, which contradicts the Baire Category Theorem.

Thus, results analogous to Corollaries 4.8–4.12 do not hold for upper semi-continuous partitions.

Example 7. In Example 1, take the set E to be of additive class α but not of multiplicative class α. It is still assumed that E is symmetric about $\frac{1}{2}$ and contains $\frac{1}{2}$.

The partition Q defined in Example 1 now becomes an α^+ partition. An argument similar to the one used in Example 1 now shows that Q does not admit a selector of multiplicative class α. Thus, an α^+ partition need not, in general, admit a selector of multiplicative class α.

EXAMPLE 8. Here is an example of an analytic partition of a Polish space which does not admit an analytic selector. The example is related to Sierpiński's example of a planar Borel set which cannot be uniformized by an analytic set ([9], p. 138).

Let f be a continuous function on Σ onto an analytic non-Borel subset Y of $[0, 1]$. Denote by Q the partition of Σ induced by f, i.e., $Q = \{f^{-1}(\{y\}): y \in Y\}$. As is easily checked, Q is an analytic partition. Suppose, by way of contradiction, that S is an analytic selector for Q. Define a function $g: \Sigma \to \Sigma$ by: $g(\sigma) =$ the unique element of $S \cap f^{-1}(\{f(\sigma)\})$. We now verify that g is Borel measurable. First, note that, for any subset E of Σ, $g^{-1}(E) = f^{-1}(f(E \cap S))$. Hence, if E is any Borel subset of Σ, then $g^{-1}(E)$ is analytic. So, in particular, both $g^{-1}(E)$ and $g^{-1}(\Sigma - E)$ are analytic, whenever E is a Borel subset of Σ. It now follows by a well known result of Souslin ([5], p. 395) that $g^{-1}(E)$ is a Borel subset of Σ, whenever E is a Borel subset of Σ. Thus, g is Borel measurable. An immediate consequence of this is that S is a Borel subset of Σ, for $S = \{\sigma \in \Sigma: g(\sigma) = \sigma\}$. Now the restriction of f to S is one-one and $f(S) = Y$. Hence, Y is a Borel subset of $[0, 1]$ ([5], p. 397), which contradicts our assumption that Y is non-Borel.

We therefore conclude that Corollary 4.4 is the best possible result concerning selectors for analytic partitions.

References

[1] A. V. Arhangel'skii, *Open and near open mappings. Connections between spaces*, Trudy Moskov. Mat. Obsc. 15 (1966), pp. 181–223 (in Russian), Trans. Moscow Math. Soc., pp. 204–250 (English translation).

[2] N. Bourbaki, *General Topology*, Part 2, Paris 1966.

[3] R. Engelking, R. W. Heath and E. Michael, *Topological well-ordering and continuous selections*, Inventiones Math. 6 (1968), pp. 150–158.

[4] F. Hausdorff, *Set Theory*, New York 1957.

[5] K. Kuratowski, *Topologie*, Vol. 1, Warszawa 1958.

[6] — *On set-valued B-measurable mappings and a theorem of Hausdorff*, Theory of Sets and Topology (pp. 355–362), Berlin 1972.

[7] — and A. Maitra, *Some theorems on selectors and their applications to semi-continuous decompositions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 22 (1974), pp. 877–881.

[8] S. Mazurkiewicz, *Sur une propriété des ensembles C(A)*, Fund. Math. 10 (1927), pp. 172–174.

[9] W. Sierpiński, *Sur l'uniformisation des ensembles mesurables* (B), Fund. Math. 16 (1930), pp. 136–139.

INDIAN STATISTICAL INSTITUTE
Calcutta, India

Accepté par la Rédaction le 30. 9. 1974