Econometrica, Vol. 51, No. 4 (July, 1983)

ON THE INFORMATIONAL SIZE OF MESSAGE SPACES FOR EFFICIENT RESOURCE ALLOCATION PROCESSES

BY PARKASH CHANDER[1]

This paper develops a framework of analysis for studying the informational properties of a certain class of "parametric" resource allocation processes. It is shown that the Taylor process (related to certain ideas for planning in the so-called socialist economies as put forward by Taylor [26]) is informationally efficient in the sense that any informationally decentralized resource allocation process which has similar (static) properties (Pareto optimality) must use a message space which is dimensionally at least as large as that of the Taylor process. We also show that in general greater informational decentralization can be achieved through parametric than through "nonparametric" processes.

1. INTRODUCTION

THE PURPOSE OF THIS PAPER is to study the informational properties, for example, the minimality of the dimension of the message space, of certain resource allocation processes. One such process, the Taylor process [6, 20, 28] (related to certain ideas for planning in the so-called socialist economies as put forward by Taylor [26]), is shown to be informationally efficient in the sense that any informationally decentralized resource allocation process which has similar (static) properties (Pareto optimality) must use a message space which is dimensionally at least as large as that of the Taylor process.

In a study like ours, the resource allocation process becomes the unknown of the problem, rather than the datum, and the significance of the result concerning the informational efficiency of the process, such as the Taylor process, becomes dependent upon the size of the domain of variation of the unknown. For example, if the domain is so small as to exclude every process other than the Taylor process, then the informational efficiency result follows as a triviality. It thus needs to be qualified that the informational efficiency of the Taylor process is proved over a sufficiently large class of processes. This is shown to be a fact by proving that the class includes several processes other than the Taylor process, for example, the Malinvaud–Taylor process [6, 20].

One motivation for undertaking this study is that the development of the theory of economic organization in the tradition of Lange, Lerner and others has focused on the Taylor process (see, for example, Lange [18], Schumpeter [25], Dobb [11], Arrow and Hurwicz [2], Hayek [12], Malinvaud [20], Ward [28], and Marglin [19]). Another motivation is that it leads to the development of a framework of analysis that may be of general interest (see, for example, Chander [9]).

In fact, our framework is more general than that of Hurwicz [13, 15] and Mount and Reiter [21], which was originally developed for the purpose of

[1] I would like to thank S. Chakravarty, Pradeep Dubey, Leonid Hurwicz, J. S. Jordan, and T. N Srinivasan for stimulating discussion, and an unknown *Econometrica* referee for several constructive remarks.

studying the informational properties of the competitive process in the context of classical environments. The Hurwicz and Mount and Reiter framework has numerous applications (see, for example, Hurwicz [14], Calsamiglia [4], Walker [27], Jordan [16], Osana [23], and Jordan [17]), and has stimulated this work, but it is not broad enough to enable us to inquire into the informational properties of certain type of processes. In particular, we cannot place the Taylor process within their framework (nor, for that matter, can we place a certain generalized formalization of the competitive process (see Chander [9])). This is mainly due to the following two difficulties.

First, the environment class, over which the Taylor process performs efficiently, is not the Cartesian product of the sets of characteristics of the participants in the process. Instead, owing to the feasibility requirement, it is a nondecomposable proper subset of the Cartesian product.[2] Thus, we develop a generalized definition of the uniqueness property and a generalized single-valuedness lemma that are suitable for this type of environment class.

Second, the Taylor process is parametric in the sense that its outcome function is parametric.[3] Traditionally, the parametric processes have been regarded as too general to be of interest. In particular, Hurwicz [13, 15] and Mount and Reiter [21] restrict themselves to nonparametric processes. The present study develops for the first time a framework for analyzing the informational properties of parametric processes.

The contents of this paper are as follows. Section 2 summarizes the notation and definitions concerning the environments, which consist of two or more producers and a single consumer. Sections 3 and 6 state, respectively, the Taylor and the Malinvaud–Taylor processes and prove the local threadedness of their equilibrium message correspondences and the lower hemicontinuity of their (lower) inverses. Section 4 gives two results: (a) The Taylor process is informationally efficient among the broad class of parametric processes (Theorem 2). (b) The Malinvaud–Taylor process is informationally efficient among the restricted class of nonparametric processes, but not among the broad class of parametric processes (Theorem 3). In each case an appropriate single-valuedness lemma is used (see Lemmas 2 and 3 below) and a suitable smoothness condition (local threadedness of the equilibrium message correspondence) is imposed. Section 5 gives all the proofs of the results in the above mentioned Sections as well as those of some auxiliary results. Section 7 presents the conclusion.

[2] In the case of classical environments the feasibility requirement that trades add to zero restricts trades to a hyperplane but such (nontrivial) feasible trades exist always and, thus, the environment class such that feasible trades exist is the Cartesian product of the sets of characteristics of the agents. But one can well imagine environments of a certain type in which the characteristics of the agents may be such that feasible trades do not necessarily exist. In such cases the feasibility requirement can lead to a nondecomposable environment class.

[3] The generalized concept of a parametric process as opposed to that of a "concrete" ("nonparametric") process is due originally to Hurwicz [14]. In a nonparametric process the final allocations of the agents are determined on the basis of the equilibrium message alone. Whereas in a parametric process the characteristics of the agents also enter the computations, though in a decentralized fashion.

2. THE ENVIRONMENTS

We shall consider the economies with n commodities and n agents (producers and consumers). The set of commodities as well as the set of agents are denoted by $N = \{1, \ldots, n\}$. Each agent i, $i = 1, \ldots, n-1$, is a producer and the nth agent is the sole consumer. Each producer is characterized by its technology set (a collection of vectors, each representing a basic activity). And the nth agent is characterized by its resource endowment and utility function.

Let C denote the set of all closed and strictly convex sets in the *commodity space* R^n (the n-dimensional Euclidean space). Then the *set of all conceivable technology sets* for agent i, $i = 1, \ldots, n-1$, is

$$E^i = \{ T^i \in C : \text{for every } a \in T^i, a_i = 1 \text{ and } a_k \leqq 0 \ \forall k \neq i;$$
$$\text{there is } \theta > 0 \text{ such that for every } a \in T^i, a_n \leqq -\theta \}.$$

These assumptions on the technology sets mean no joint production and indispensability of commodity n (labor) as a nonproducible factor of production.[4]

Let w^* denote the vector $(0, \ldots, 0, 1) \in R^n$; then the *set of all conceivable resource endowments* for agent n is denoted by

$$W^* = \{w^*\},$$

the *consumption set* by R^n_{++}, and the *set of all conceivable utility functions by*

$$U^* = \Big\{ u : R^n_{++} \to R : \text{there is some } \alpha \in R^n_{++} \text{ such that } \sum_i \alpha_i = 1 \text{ and}$$
$$u(x) = \prod_i x_i^{\alpha_i} \text{ for all } x \in R^n_{++} \Big\}.^5$$

Let

$$E^n = U^* \times W^*,$$

and let E^0 denote the Cartesian product of the E^i:

$$E^0 = E^1 \times \cdots \times E^n.$$

A generic element of E^0 will be denoted by $e = (e^1, \ldots, e^n)$. Note that each e^i ($i = 1, \ldots, n-1$) is a closed and strictly convex subset of R^n. The set E^i will be

[4]The model of technology that is adopted here was proposed by Samuelson [24] as a generalization of the Leontief model. It is worth noting that one can broaden the environment class by relaxing some of the assumptions of this model and still obtain similar results. First, the convexity restriction can be completely dispensed with. Second, the assumption of one primary input, "labor" in the model, can be dropped. Instead, there can be some or all commodities which can be purchased by the production system in any amount for a prescribed price. (Typically, these "exogenous commodities" might include labor services of varying skills, raw materials, and some fixed capital services). Such a model will approximate an open industrial complex in a partial equilibrium setting where the prices of purchased inputs can be treated as fixed. A further comment in this regard is deferred to Section 7.

[5]$R^n_+ = \{x \in R^n : x_i \geqq 0 \ \forall i\}$, and $R^n_{++} = \{x \in R^n : x_i > 0 \ \forall i\}$.

referred to as the *space of characteristics of agent i*, $i \in N$. Let

$$S^* = \{a \in R^n : a + w^* \in R^n_{++}\}$$

and

$$r(e) = \left\{(a^1, \ldots, a^n) \in e^1 \times \cdots \times e^{n-1} \times S^* : a^n = \sum_{i=1}^{n-1} a^i x_i \right.$$
$$\left. \text{for some } x_i > 0 \ (i = 1, \ldots, n-1)\right\}.$$

Then S^* denotes the set of *all feasible net trade vectors* of the nth agent and $r(e)$ denotes the set of all *feasible combinations of the activity vectors and net trade vectors* in e. Let E be the subset of E^0 defined as

$$E = \{e \in E^0 : r(e) \neq \emptyset\}.$$

Then E is the *environment class*. The environment class E has the following properties in relation to E^0. For each $E^* \subset E$ and $i \in N$, let $L^i(E^*)$ denote the ith *coordinate set* of E^*:

$$L^i(E^*) = \{e^i \in E^i : (e^1, \ldots, e^{i-1}, e^i, e^{i+1}, \ldots, e^n) \in E$$
$$\text{for some } (e^1, \ldots, e^{i-1}, e^{i+1}, \ldots, e^n)$$
$$\in E^1 \times \cdots \times E^{i-1} \times E^{i+1} \times \cdots \times E^n\}$$

and let $L(E^*)$ denote the Cartesian product $L^1(E^*) \times \cdots \times L^n(E^*)$.

PROPOSITION 1: *For each $i \in N$, $L^i(E) = E^i$ and $L(E) \not\subset E$.*

Let $\theta : E \to R^{n^2}$ be the correspondence[6] such that for each $(e^1, \ldots, e^n) \in E$,

$$\theta(e) = \{(a^1, \ldots, a^n) \in r(e) : u(a^n + w^*) \geqq u(b^n + w^*) \text{ for all}$$
$$(b^1, \ldots, b^n) \in r(e)\},^7$$

where $(u, w^*) = e^n$.

[6] Let X and Y be two nonempty sets. A *correspondence* F of X into Y (often written $F : X \to Y$) is a rule which associates with each element x of X a (possibly empty) subset $F(x)$ of Y. If $F(x)$ is one element set for every $x \in X$, then F is said to be *single valued*, and it is identified with the function $f : X \to Y$ such that $F(x) = \{f(x)\}$ for every $x \in X$. Given a nonempty subset A of X, the *image* $F(A)$ of A *under* F is defined as $F(A) = \{y \in Y : y \in F(x)$ for some $x \in A\}$. The *domain* and *range* of F are defined as dom $F = \{x \in X : F(x) \neq \emptyset\}$ and range $F = F(X)$.

[7] We find it convenient to refer to the ordered n-tuple $(a^1, \ldots, a^n)$ as the n^2-dimensional vector with a^i_j in the $(n(i-1)+j)$th component.

PROPOSITION 2: *The correspondence* $\theta: E \to R^{n^2}$ *is single valued.*

In view of Proposition 2, the correspondence θ will be referred to as the *choice function*. This refers to the fact that θ chooses exactly one *optimal* combination of the activity vectors and net trade vectors from among the feasible ones. Note that if $(a^1, \ldots, a^n) \in \theta(e)$, then, by definition, there exist $(x_1, \ldots, x_{n-1}) \geqq 0$ such that $\sum_{i=1}^{n-1} a^i x_i = a^n$. Thus, it may be more appropriate to define "optimality" in terms of the ordered n-tuple $(a^1 x_1, \ldots, a^{n-1} x_{n-1}, a^n)$ rather than in terms of $(a^1, \ldots, a^{n-1}, a^n)$. For the purpose of this study, however, it is quite unnecessary to make a distinction between $(a^1 x^1, \ldots, a^{n-1} x^{n-1}, a^n)$ and $(a^1, \ldots, a^{n-1}, a^n)$ as there is a one-one correspondence between the two (see, for example, the proof of Proposition 1 in Section 5 below).

PROPOSITION 3: *For every* $e, \bar{e} \in E$, *if* $\theta(e) \cap \theta(\bar{e}) \neq \emptyset$, *then every* $e(i) \in E$, *where* $e(i) = (e^1, \ldots, e^{i-1}, \bar{e}^i, e^{i+1}, \ldots, e^n), i \in N$.

As noted above in Proposition 1, E is not the Cartesian product of the E^i, i.e., $L(E) \not\subset E$. Proposition 3, however, enables us to introduce the following definition of the uniqueness property.

E is said to have the *uniqueness property* with respect to a function f, if (i) f is a function such that dom $f \subset E$, (ii) for every $e, \bar{e} \in$ dom f, if $\theta(e) \cap \theta(\bar{e}) \neq \emptyset$ then every $e(i) \in E$, where $e(i) = (e^1, \ldots, e^{i-1}, \bar{e}^i, e^{i+1}, \ldots, e^n)$, $i \in N$, and (iii) if $(\theta(e) \cap \theta(\bar{e})) \cap \bigcap_{i=1}^{n} \theta(e(i)) \neq \emptyset$ then $f(e) = f(\bar{e})$.

Our definition of the uniqueness property is weaker than that of Hurwicz [15] and Osana [23], in that we do not require that $L(\text{dom } f)$ ($= L^1(\text{dom } f) \times \cdots \times L^n(\text{dom } f)$) should be contained in E, but substitute a weaker condition. Later we shall be in fact dealing with an f for which $L(\text{dom } f) \not\subset E$. (It is worth noting that the space into which f maps is immaterial to the definition.)

We now endow E with a topological structure. Let d be the pseudo-metric on E defined as $d(e, \bar{e}) = |\theta(e) - \theta(\bar{e})|$ for every $e, \bar{e} \in E$, where $|\cdot|$ is the usual metric on R^{n^2}. Since $\theta(e)$ may be equal to $\theta(\bar{e})$ for some $e, \bar{e} \in E$, $e \neq \bar{e}$, d is a pseudo-metric. An open sphere of radius ϵ about e then contains all those $\bar{e}$ for which $\theta(\bar{e}) = \theta(e)$ as well as those for which $|\theta(e) - \theta(\bar{e})| < \epsilon$. As in the case of a metric, the class of open spheres in E as defined by the pseudo-metric d forms a base for a topology. We give E the topology generated by the class of open spheres in E. In this way E is endowed with a *pseudo-metric topology*.

PROPOSITION 4: *The choice function* $\theta: E \to R^{n^2}$ *is continuous and* $\theta^{-1}: \theta(E) \to E$ *is lower hemicontinuous.*[8]

[8] Let X and Y be topological spaces. A correspondence $F: X \to Y$ is said to be *lower hemicontinuous* if $\{x : F(x) \cap V \neq \emptyset\}$ is open in X for every open set $V \subset Y$.

3. THE TAYLOR PROCESS

We shall now state the Taylor process. We take commodity n as the numeraire and assume its price to be identically equal to unity. Prices of the remaining commodities are taken to be positive and the *set of all possible price vectors* is denoted by

$$P = \{ p \in R^n : p_i > 0 \ \forall i \in N \text{ and } p_n = 1 \}.$$

For each $p \in P$ and $e^i \in E^i$, let

$$f^i(p, e^i) = p_i - \underset{a^i \in e^i}{\text{maximum}}\ pa^i \quad \text{and}$$

$$g^i(p, e^i) = \underset{a^i \in e^i}{\text{maximizer}}\ pa^i \qquad (i = 1, \ldots, n-1).$$

Then $f^i(p, e^i)$ denotes the minimum cost of producing one unit amount of commodity i and $g^i(p, e^i)$ denotes the least cost activity vector. For each $p \in P$ and $e^n \in E^n$ define

$$c(p, e^n) = \underset{y \in \{z \in S^* : pz \leq 0\}}{\text{maximizer}}\ u(y + w^*),$$

where $(u, w^*) = e^n$. Note that $c : P \times E^n \to S^*$. Let

$$f^n(p, e^n) = 1 - pc(p, e^n)(= 1) \quad \text{and}$$

$$g^n(p, e^n) = c(p, e^n).$$

Then $f^n(p, e^n)$ denotes the value of the utility maximizing consumption bundle and $g^n(p, e^n)$ the utility maximizing net trade vector of agent n (subject to its budget constraint).

The functions $(f^1, \ldots, f^n)$ and $(g^1, \ldots, g^n)$ as defined above determine the nature of the following dynamic process:[9]

$$p_i(t+1) = f^i(p(t), e^i) \qquad (i = 1, \ldots, n;\ t = 0, 1, \ldots),$$

where $p^0 \in P$, so that in equilibrium

$$p_i = f^i(p, e^i), \qquad p \in P \qquad (i = 1, \ldots, n),$$

and there exist $(a^1, \ldots, a^n) \in e^1 \times \cdots \times e^{n-1} \times S^*$ such that

$$a^i = g^i(p, e^i) \qquad (i = 1, \ldots, n).$$

Let $\mu : E \to \mu(E)$ be the correspondence defined as

$$\mu(e) = \{ p \in P : p = f(p, e) \}, \qquad e \in E,$$

where $f = (f^1, \ldots, f^n)$. Let $g = (g^1, \ldots, g^n)$, where g^i is as defined above.

[9] For the convergence proof of this process, see [6, 7, 29].

The *Taylor process* is then defined as the ordered pair $[\mu, g]$. The correspondence μ will be referred to as the *equilibrium message correspondence* and g will be referred to as the ("parametric") *outcome function* (g is single-valued as will be shown below). This defines the Taylor process. We now state its properties.

THEOREM 1: *The Taylor process $[\mu, g]$ satisfies the following properties:*

PROPERTY (A): *The equilibrium message correspondence μ is single valued, g is single valued, and for every $e \in E$, $g(p,e) \in \theta(e)$ for very $p \in \mu(e)$.*

PROPERTY (B): *For every $e, \bar{e} \in E$, if $\mu(e) \cap \mu(\bar{e}) \neq \emptyset$, then each economy $e(i)$, $\bar{e}(i)$ belongs to E and $\mu(e(i)) \cap \mu(\bar{e}(i)) = \mu(e) \cap \mu(\bar{e})$ for every $i \in N$, where $e(i) = (e^1, \dots, e^{i-1}, \bar{e}^i, e^{i+1}, \dots, e^n)$ and $\bar{e}(i) = (\bar{e}^1, \dots, \bar{e}^{i-1}, e^i, \bar{e}^{i+1}, \dots, \bar{e}^n)$.*

PROPERTY (C): *$\mu: E \to \mu(E)$ is continuous and $\mu^{-1}: \mu(E) \to E$ is lower hemicontinuous.*

(Here μ is a function as claimed in Property (A) above and $\mu(E) \subset R^n$ is endowed with the Euclidean topology.)

COROLLARY TO THEOREM 1 (PROPERTY (C)): *The correspondence $\mu: E \to \mu(E)$ is locally threaded.*[10]

The proof of Property (C) (as well as of Proposition 4 above and Theorem 3 below) involves the following definition and lemma. Given two topological spaces X and Y, a correspondence $F: X \to Y$ is said to be *uniformly locally threaded* if for every $x_0 \in X$ and every $y_0 \in F(x_0)$ there exists an open neighborhood U_0 in X of x_0 and a continuous function $s_0: U_0 \to Y$ such that $s_0(x_0) = y_0$ and $s_0(x) \in F(x)$ for every $x \in U_0$.

LEMMA 1: *If $F: X \to Y$ is uniformly locally threaded, then F is lower hemicontinuous.*

4. RESOURCE ALLOCATION PROCESSES AND INFORMATIONAL EFFICIENCY

Given the environment class E, an ordered pair $[\nu, h]$ consisting of a correspondence ν and a function h is called a (parametric) resource allocation process for E, if (i) $E \subset \operatorname{dom} \nu$ and (ii) $\operatorname{dom} h \supset \{(q,e): q \in \nu(e) \text{ and } e \in E\}$. The correspondence ν will be referred to as the *equilibrium message correspondence*, the function h will be referred to as the (parametric) *outcome function*, and $\nu(E)$ will be referred to as the *message space* of the process.

A process $[\nu, h]$ is said to be *nonparametric* if for every $q \in \nu(E)$, $h(q,e) = h(q, \bar{e})$ for every $e, \bar{e} \in E$, i.e., h is independent of e so that, given the equilibrium

[10]Given two topological spaces X and Y, a correspondence $F: X \to Y$ is said to be *locally threaded* if for every $x_0 \in X$ there exists an open neighborhood U_0 in X of x_0 and a continuous function $s_0: U_0 \to Y$ such that $s_0(x) \in F(x)$ for every $x \in U_0$ (cf. Mount and Reiter [21]).

message, an outside agency with no knowledge of the environment could determine the outcome.

A process $[\nu, h]$ is said to be *decisive* on E, if $\nu(e) \neq \emptyset$ for every $e \in E$; $[\nu, h]$ is said to be *nonwasteful* on E, if for every $e \in E$, $h(q, e) \in \theta(e)$ for every $q \in \nu(e)$; $[\nu, h]$ is said to be *smooth* on E, if ν is a locally threaded correspondence; $[\nu, h]$ is said to be privacy preserving[11] on E, if (i) for every pair of economies $e, \bar{e} \in E$, $\nu(e) \cap \nu(\bar{e}) \neq \emptyset$ implies that $\nu(e(i)) \cap \nu(\bar{e}(i)) = \nu(e) \cap \nu(\bar{e})$ for *any* pair of economies $e(i), \bar{e}(i)$ that *may* belong to E, where $e(i) = (e^1, \ldots, e^{i-1}, \bar{e}^i, e^{i+1}, \ldots, e^n)$, and $\bar{e}(i) = (\bar{e}^1, \ldots, \bar{e}^{i-1}, e^i, \bar{e}^{i+1}, \ldots, \bar{e}^n)$, $i \in N$, and (ii) the outcome function is privacy preserving in the sense that $h = (h^1, \ldots, h^n)$ such that $\operatorname{dom} h^i = \nu(E) \times L^i(E)$, $i \in N$, so that for every $e \in E$, $h(q, e) = (h^1(q, e^1), \ldots, h^n(q, e^n))$ for every $q \in \nu(e)$.

In what follows we shall restrict ourselves to resource allocation processes whose message spaces are subsets of a Euclidean space and prove that the message space of the Taylor process is minimal among the message spaces for the general class of parametric resource allocation processes which are privacy preserving and nonwasteful. We shall use the dimension of the message space as the measure of its informational size. To make dimensionality a genuine concept of the informational size of the message space we shall require the process to be smooth: the local threadedness of the equilibrium message correspondence.[12]

Is there any resource allocation process which is decisive, nonwasteful, privacy preserving, smooth, and uses a message space of smaller size than the Taylor process? Results below show that the answer is in the negative. To highlight the role of the parametric outcome function, these results are proved at two different levels of generality, first for the class of nonparametric processes and then for the more general class of parametric processes.

Theorem 2 below will use the following general lemma. Since our definition of the uniqueness property is weaker, this lemma is more general than the well known singlevaluedness lemma of Hurwicz [15].

[11] This definition of privacy is more general than that of Mount and Reiter [21], in that it is not based on the assumption that the environment class is decomposable. It is worth noting, however, that the Mount and Reiter definition is given in terms of the "coordinate correspondences" and the "crossing condition" is shown to be necessary and sufficient [21, Definition 3 and Lemma 5] whereas our definition is based directly on the crossing condition, which is, as can be shown easily, necessary but not sufficient (not unless the environment class is decomposable). In fact, if the environment class is decomposable, then (on account of the Lemma referred to above) the two definitions are equivalent.

[12] The smoothness of the process rules out certain encoding procedures by means of which the information contained in a two dimensional message can be compressed into a one dimensional message. Hurwicz [14, 15] and Mount and Reiter [21] provide some examples of such procedures related to the space filling Peano curve which maps the unit interval $[0, 1]$ onto the unit square $[0, 1] \times [0, 1]$. If we were to consider resource allocation processes that use general topological message spaces, then we can use a generalized concept of the informational size of message spaces as developed by Mount and Reiter [21]. In that case, of course, the process will be required to satisfy an additional smoothness condition similar to that of the Taylor process: the lower hemicontinuity of the (lower) inverse of the equilibrium message correspondence. For simplicity, however, we restrict ourselves to Euclidean message spaces only.

LEMMA 2: *Let* $[\nu, h]$ *be a decisive, nonwasteful, and privacy preserving nonparametric resource allocation process for* E. *If* E *has the uniqueness property with respect to the function* f *and* f *is one-one, then* ν *is injective over* dom f.[13]

Let X and Y be Euclidean spaces. Then "$X \geqq_D Y$" means "dimension of X is greater than or equal to dimension of Y."

THEOREM 2: *Let* $[\nu, h]$ *be a nonparametric resource allocation process for* E *which is decisive, nonwasteful, privacy preserving, and smooth. If its message space* $\nu(E)$ *is Euclidean, then* $\nu(E) \geqq_D \theta(E)$.

THEOREM 3: *Let* $[\nu, h]$ *be a (parametric) resource allocation process for* E *which is decisive, nonwasteful, privacy preserving, and smooth. If its message space* $\nu(E)$ *is Euclidean, then* $\nu(E) \geqq_D \mu(E)$ *(where* $\mu(E)$ *is the message space of the Taylor process).*

It may be noted that $\theta(E)$ is of dimension $n(n-1)$, but $\mu(E)$ is of dimension $n-1$ only.

Interestingly, the Malinvaud–Taylor process to be discussed in Section 6 is nonparametric and the dimension of its message space is $n(n-1)$ equal to that of $\theta(E)$, i.e., the Malinvaud–Taylor process is informationally efficient for the restricted class of nonparametric processes, but not for the general class of parametric processes. This should illustrate the role of the parametric outcome function in relation to the informational efficiency of the process.

The proof of Theorem 3 will use the following definition and lemma.

We shall adopt the following notation concerning n-tuples. For any two n-tuples $x = (x^1, \ldots, x^n)$ and $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$ we shall denote by $x(i)$, $i = 1, \ldots, n$, the n-tuple $(x^1, \ldots, x^{i-1}, \bar{x}^i, x^{i+1}, \ldots, x^n)$.

The environment class E is said to have the *asymmetry property* with respect to a function f, if (i) f is a function such that dom $f \subset E$ and $L(\text{dom } f) \subset E$, and (ii) for every $e = (e^1, \ldots, e^n)$ and $\bar{e} = (\bar{e}^1, \ldots, \bar{e}^n)$ (in dom f), if there are n-tuples a, $\bar{a}$, and $a(i)$ $(i \in N)$ such that $a \in \theta(e)$, $\bar{a} \in \theta(\bar{e})$, and $a(i) \in \theta(e(i))$ for every $i \in N$, then $f(e) = f(\bar{e})$.

Note that the space into which f maps is immaterial to the definition.

LEMMA 3: *Let* $[\nu, h]$ *be a decisive, nonwasteful, and privacy preserving (parametric) resource allocation process for* E. *If* E *has the asymmetry property with respect to a function* f *and* f *is one-one, then* ν *is injective over* dom f.

It is worth noting that for every f, if E has the asymmetry property with respect to f, then it also has the uniqueness property with respect to f. The

[13] A correspondence $F: X \to Y$ is said to be injective over a subset $X^* \subset X$ if for every $x, \bar{x} \in X^*$, $F(x) \cap F(\bar{x}) \neq \emptyset$ implies $x = \bar{x}$.

converse, however, is not true, that is, there may be a function f such that e has the uniqueness property with respect to f, but it does not have the asymmetry property with respect to f. The asymmetry property is thus stronger than the uniqueness property. There may be special situations, however, where the two properties are equivalent in the sense that one implies the other. One such situation arises when the environments are of the classical pure-exchange variety and the choice function is the Pareto correspondence. This, however, is not the situation in the present case as we show by means of an example.[14] Let S denote the set of all singleton sets. Let

$$E^{*i} = \{e^i \in E^i : e^i \in S\} \qquad (i = 1, \ldots, n-1),$$

$$E^{*n} = E^n, \quad \text{and}$$

$$E^* = \{(e^1, \ldots, e^n) \in E : e^i \in E^{*i}\ (i = 1, \ldots, n)\}.$$

Then $E^* \subset E$ is the set of all environments that lack technical substitution possibilities.

PROPOSITION 5: *The environment class E has the uniqueness property with respect to the identity function f^* on E^*, but it does not have the asymmetry property with respect to f^*.*

5. PROOF OF THE RESULTS[15]

PROOF OF PROPOSITION 1: We first prove that $L^i(E) = E^i$ for each $i \in N$. Without loss of generality take $i = 1$. Let $\bar{e}^1 \in E^1$. We show that there exist $\bar{e}^i \in E^i$ $(i = 2, \ldots, n-1)$ such that $(\bar{e}^1, \ldots, \bar{e}^{n-1}, e^n) \in E$ for every $e^n \in E^n$.

Given some $\bar{a}^1 \in \bar{e}^1$, there exists a (normalized) vector $\bar{p} \in R^n_{++}$ such that $\bar{p}_n = 1$ and $\bar{p}\bar{a}^1 = 0$. Let $\bar{a}^2, \ldots, \bar{a}^{n-1}$ be some n-dimensional vectors such that $\bar{a}^i_i = 1$, $\bar{a}^i_j \leqq 0$ for $j \neq i$ $\bar{a}^i_n < 0$, and $\bar{p}\bar{a}^i = 0$ $(i = 2, \ldots, n-1)$. Let $\bar{e}^i = \{\bar{a}^i\}$ $(i = 2, \ldots, n-1)$. Let $\bar{a}^n \in S^*$ be such that $\bar{p}\bar{a}^n = 0$. We prove that $(\bar{a}^1, \ldots, \bar{a}^n) \in r(\bar{e}^1, \ldots, \bar{e}^{n-1}, e^n)$ for every $e^n \in E^n$.

Let $A = (a_{ij})$ be the $(n-1) \times (n-1)$ matrix such that $a_{ij} = \bar{a}^j_i$ $(i, j = 1, \ldots, n-1)$. Then $a_{jj} = 1$ and $a_{ij} \leqq 0$ for $i \neq j$. Since $(\bar{p}_1, \ldots, \bar{p}_{n-1})A + (\bar{a}^1_n, \ldots, \bar{a}^{n-1}_n) = 0$ and $\bar{a}^j_n < 0$ for $j = 1, \ldots, n-1$, $(\bar{p}_1, \ldots, \bar{p}_{n-1})A \gg 0$. Since $(\bar{p}_1, \ldots, \bar{p}_{n-1}) \gg 0$, it follows from [22, Theorem 6.2] that A is nonsingular and $A^{-1} \geqq 0$. Let $(\bar{x}_1, \ldots, \bar{x}_{n-1}) = A^{-1}(\bar{a}^n_1, \ldots, \bar{a}^n_{n-1})$. Since $A^{-1} \geqq 0$ and $\bar{a}^n_i$

[14] Another example where uniqueness is not equivalent to asymmetry is the class of environments that were introduced by Hurwicz [14] in his examples A and B.

[15] We shall follow the following notation concerning the vectors. Given any two vectors $x = (x_i)$ and $y = (y_i)$,

$x \gg y$ implies $x_i > y_i$ for all i,

$x > y$ implies $x_i \geqq y_i$ for all i and $x_i > y_i$ for at least one i,

$x \geqq y$ implies $x_i \geqq y_i$ for all i.

> 0 $(i = 1, \ldots, n-1)$, $(\bar{x}_1, \ldots, \bar{x}_{n-1}) \gg 0$. We claim that $\sum_{i=1}^{n-1} \bar{a}^i \bar{x}_i = \bar{a}^n$. The first $n-1$ of these n equalities follow from the definition of the $\bar{x}_i$ $(i = 1, \ldots, n-1)$. The nth equality follows from the fact that $\bar{p}\bar{a}^i = 0$ $(i = 1, \ldots, n)$. This proves that $(\bar{a}^1, \ldots, \bar{a}^n) \in r(\bar{e}^1, \ldots, \bar{e}^{n-1}, e^n)$ for every $e^n \in E^n$. Hence $(\bar{e}^1, \ldots, \bar{e}^{n-1}, e^n) \in E$ for every $e^n \in E^n$.

We now show that $L(E) \not\subset E$. Without loss of generality take $n = 3$. Let $\bar{a}^1 = (1, -2, -\frac{1}{2})$, $\bar{a}^2 = (-\frac{1}{4}, 1, -\frac{1}{2})$, $\tilde{a}^1 = (1, -\frac{1}{4}, -\frac{1}{2})$, and $\tilde{a}^2 = (-2, 1, -\frac{1}{2})$. Let $\bar{e}^i = \{\bar{a}^i\}$ and $\tilde{e}^i = \{\tilde{a}^i\}$, $i = 1, 2$. Then for every $e^3 \in E^3$, $(\bar{e}^1, \bar{e}^2, e^3) \in E$, $(\tilde{e}^1, \tilde{e}^2, e^3) \in E$, but $(\bar{e}^1, \tilde{e}^2, e^3) \notin E$ for any $e^3 \in E$. *Q.E.D.*

Define

$$P = \{ p \in R^n_{++} : p_i > 0 \; \forall i \in N \text{ and } p_n = 1 \}.$$

Then P will be referred to as the *set of all possible (normalized) price vectors*. Let $c : P \times E^n \to S^*$ be the function such that for each $p \in P$ and $e^n \in E^n$,

$$c(p, e^n) = \underset{y \in \{z \in S^* : pz \leq 0\}}{\text{maximizer}} \; u(y + w^*),$$

where $(u, w^*) = e^n$. Note that $pc(p, e^n) = 0$.

Proof of Proposition 2 will use the following well-known result (see, for example, [1, 5, 24]).

THE SUBSTITUTION THEOREM: *If $e \in E$, then there exists a unique equilibrium price vector $p \in P$ such that for every $(a^1, \ldots, a^{n-1}) \in e^1 \times \cdots \times e^{n-1}$, $pa^i \leq 0$ $(i = 1, \ldots, n-1)$ and for some $(\bar{a}^1, \ldots, \bar{a}^{n-1}) \in e^1 \times, \ldots, \times e^{n-1}$, $p\bar{a}^i = 0$ $(i = 1, \ldots, n-1)$.*

PROOF OF PROPOSITION 2: Let $\bar{e} \in E$. We first prove that $\theta(\bar{e}) \neq \emptyset$ and then show that $\theta(\bar{e})$ is a singleton.

Since $\bar{e} \in E$, (by the substitution theorem) there exists a normalized price vector $\bar{p} \in P$ and $(\bar{a}^1, \ldots, \bar{a}^{n-1}) \in \bar{e}^1 \times \cdots \times \bar{e}^{n-1}$ such that $\bar{p}\bar{a}^i = 0$ $(i = 1, \ldots, n-1)$. Let $\bar{a}^n = c(\bar{p}, \bar{e}^n)$. Then as in the proof of Proposition 1, $(\bar{a}^1, \ldots, \bar{a}^n) \in r(\bar{e})$. Hence $\theta(\bar{e}) \neq \emptyset$.

Suppose that $(\bar{b}^1, \ldots, \bar{b}^n) \in \theta(\bar{e})$. Then, by definition, there exist $(\bar{y}_1, \ldots, \bar{y}_{n-1}) \gg 0$ such that $\sum_{i=1}^{n-1} \bar{b}^i \bar{y}_i = \bar{b}^n$. The substitution theorem implies that $\bar{p}\bar{b}^i \leq 0$ $(i = 1, \ldots, n-1)$. Thus $\bar{p}\bar{b}^n \leq 0$ and $(\bar{b}^1, \ldots, \bar{b}^n) \in \theta(\bar{e})$. This together with the fact that $\bar{a}^n = c(\bar{p}, \bar{e}^n)$ implies that $\bar{b}^n = \bar{a}^n$ and thus $\bar{p}\bar{b}^n = 0$. If $(\bar{b}^1, \ldots, \bar{b}^{n-1}) \neq (\bar{a}^1, \ldots, \bar{a}^{n-1})$ then $\bar{p}\bar{b}^i < 0$ for some $i (= 1, \ldots, n-1)$ because of the strict convexity assumption. This, however, contradicts that $\bar{p}\bar{b}^n = 0$ and $\bar{b}^n = \sum_{i=1}^{n-1} \bar{b}^i \bar{y}_i$. Hence $(\bar{b}^1, \ldots, \bar{b}^n) = (\bar{a}^1, \ldots, \bar{a}^n)$. This proves that θ is single-valued.

PROPOSITION 2′: *An ordered n-tuple $(\bar{a}^1, \ldots, \bar{a}^n) \in \theta(\bar{e})$, if and only if,* (i) $(\bar{a}^1, \ldots, \bar{a}^n) \in \bar{e}^1 \times \cdots \times \bar{e}^{n-1} \times S^*$, (ii) *there exists a price vector $\bar{p} \in P$ such* *that* $\bar{p}\bar{a}^i = 0$ $(i \in N)$ *and* $\bar{a}^n = c(\bar{p}, \bar{e}^n)$, *and* (iii) $\bar{p}a^i \leqq 0$ $(i = 1, \ldots, n-1)$ *for every* $(a^1, \ldots, a^{n-1}) \in e^1 \times \cdots \times e^{n-1}$.

PROOF: Sufficiency follows from the proof of Proposition 2. We prove the necessity.

Suppose that $(\bar{a}^1, \ldots, \bar{a}^n) \in \theta(\bar{e})$. By definition, $(\bar{a}^1, \ldots, \bar{a}^n) \in \bar{e}^1 \times \cdots \times \bar{e}^{n-1} \times S^*$ and there exist $(\bar{x}_1, \ldots, \bar{x}_{n-1}) \gg 0$ such that $\bar{a}^n = \sum_{i=1}^{n-1} \bar{a}^i \bar{x}_i$. Let $A = (a_{ij})$ be the $(n-1) \times (n-1)$ matrix such that $a_{ij} = \bar{a}_i^j$ $(i, j = 1, \ldots, n-1)$. Then $a_{ii} = 1$ and $a_{ij} \leqq 0$ for $j \neq i$ $(i = 1, \ldots, n-1)$. Since $\bar{a}^n \in S^*$, $\bar{a}_i^n > 0$ $(i = 1, \ldots, n-1)$. A well-known result (see, for example, [22, Theorem 6.2]) then implies that A is nonsingular and $A^{-1} \geqq 0$. Let $(\bar{p}_1, \ldots, \bar{p}_{n-1}) = -(\bar{a}_n^1, \ldots, \bar{a}_n^{n-1})A^{-1}$. Since $\bar{a}_n^i < 0$ $(i = 1, \ldots, n-1)$, $(\bar{p}_1, \ldots, \bar{p}_{n-1}) \gg 0$. Let $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_{n-1}, 1)$. Then $\bar{p} \in P$ and $\bar{p}\bar{a}^i = 0$ $(i = 1, \ldots, n)$. It is also clear from above that $\bar{a}^n = c(\bar{p}, \bar{e}^n)$.

Next we prove that for every $(a^1, \ldots, a^{n-1}) \in \bar{e}^1 \times \cdots \times \bar{e}^{n-1}$, $\bar{p}a^i \leqq 0$ $(i = 1, \ldots, n-1)$. Without loss of generality take $i = 1$ and suppose contrary to the assertion that $\bar{p}a^1 > 0$ for some $a^1 \in e^1$. Let $(\bar{b}^1, \ldots, \bar{b}^{n-1})$ be defined as $\bar{b}^1 = a^1$ and $\bar{b}^i = \bar{a}^i$ $(i = 2, \ldots, n-1)$. Let $B = (b_{ij})$ be the $(n-1) \times (n-1)$ matrix such that $b_{ij} = \bar{b}_i^j$ $(i, j = 1, \ldots, n-1)$. Then $(\bar{p}_1, \ldots, \bar{p}_{n-1})B \gg 0$. Since $(\bar{p}_1, \ldots, \bar{p}_{n-1}) \gg 0$, it follows from [22, Theorem 6.2] that B is nonsingular and $B^{-1} \geqq 0$. Let $(\bar{y}_1, \ldots, \bar{y}_{n-1}) = B^{-1}(\bar{a}_1^n, \ldots, \bar{a}_{n-1}^n)$. Since $B^{-1} \geqq 0$ and $\bar{a}_i^n > 0$ $(i = 1, \ldots, n-1)$, $(\bar{y}_1, \ldots, \bar{y}_{n-1}) \gg 0$. We claim that $\sum_{i=1}^{n-1} \bar{b}^i \bar{y}_i = \bar{a}^n$. Let $\bar{b}^n = \sum_{i=1}^{n-1} \bar{b}^i \bar{y}_i$. Since $\bar{p}\bar{b}^1 > 0$ and $\bar{p}\bar{b}^i = 0$ $(i = 2, \ldots, n-1)$, $\bar{p}\bar{b}^n > 0$. Now $\bar{p}\bar{a}^n = 0$ and, by definition of $(\bar{y}_1, \ldots, \bar{y}_{n-1})$, $\bar{b}_i^n = \bar{a}_i^n$ $(i = 1, \ldots, n-1)$. Therefore, $\bar{b}^n > \bar{a}^n$. This contradicts that $(\bar{a}^1, \ldots, \bar{a}^n) \in \theta(\bar{e})$. Hence our supposition is wrong. This proves our assertion.

COROLLARY TO PROPOSITION 2′: *Let* $e \in E^0$. *Then* $e \in E$, *if and only if, there exists an equilibrium price vector* $p \in P$ *such that for every* $(a^1, \ldots, a^{n-1}) \in e^1 \times \cdots \times e^{n-1}$, $pa^i \leqq 0$ $(i = 1, \ldots, n-1)$ *and for some* $(\bar{a}^1, \ldots, \bar{a}^{n-1}) \in e^1 \times \cdots \times e^{n-1}$, $p\bar{a}^i = 0$ $(i = 1, \ldots, n-1)$.

PROOF OF PROPOSITION 3: It is evident that if $\theta(e) \cap \theta(\bar{e}) \neq \emptyset$, then for each of the economies e, $\bar{e}$, and $e(i)$ $(i \in N)$, the equilibrium price vector must be the same. The proof then follows from the corollary to Proposition 2′.

PROOF OF THEOREM 1 (PROPERTY (A)): By definition of μ and Proposition 2′, $p \in \mu(e)$ only if there exists an $(\bar{a}^1, \ldots, \bar{a}^n) \in \theta(e)$ such that $\bar{p}\bar{a}^i = 0$ $(i \in N)$. Since θ is single valued (by Proposition 2) and the system of equations $pa^i = 0$ $(p \in P$ and $i \in N)$ admits only one solution, μ is single valued. It is also clear from Proposition 2′ that $g(p, e) \in \theta(e)$ for $p \in \mu(e)$. Because μ and θ are single valued, therefore, g must be single valued.

PROOF OF THEOREM 1 (PROPERTY (B)): Evidently, if $p \in \mu(e) \cap \mu(\bar{e})$ then $p \in \mu(e(i))\mu(\bar{e}(i))$, $i \in N$. Corollary to Proposition 2′ then implies that $e(i)$ and $\bar{e}(i)$ belong to E for each $i \in N$. Hence the proof.

PROOF OF PROPOSITION 4: Since $\theta: E \to \theta(E)$ is single-valued and $d(e, \bar{e}) = |\theta(e) - \theta(\bar{e})|$ for all $e, \bar{e} \in E$, θ is an isometry (a d-$|\cdot|$ isometry). This proves that $\theta: E \to \theta(E)$ is continuous.

Let $s: \theta(E) \to E$ be some global thread of $\theta^{-1}: \theta(E) \to E$, i.e., $s(a) \in \theta^{-1}(a)$ for every $a \in \theta(E)$. Then s is one-one and the restriction of the metric d to the points in $s(\theta(E))$ is a metric on $s(\theta(E))$, i.e., $s(\theta(E))$ is a metric subspace of the pseudo-metric space E and s is an isometry. This implies that s is a homeomorphism. This proves that each (global) thread of $\theta^{-1}: \theta(E) \to E$ is a continuous function, i.e., θ^{-1} is uniformly globally threaded. Using Lemma 1, it follows that θ^{-1} is lower hemicontinuous.

PROOF OF LEMMA 1: Let V be open in Y. We must show that $G = \{x : F(x) \cap V \neq \emptyset\}$ is open in X. For each $x_0 \in G$, pick a $y_0 \in F(x_0) \cap V$. Then by assumption there exists an open neighborhood U_0 containing x_0 and a continuous function $s_0: U_0 \to Y$ such that $s_0(x_0) = y_0$. Let $\bar{U}_0 = U_0 \cap \{x : s_0(x) \in V\}$. Then $\bar{U}_0$ is an open set containing x_0, which is contained in G. Hence G is open. This proves that F is lower hemicontinuous.

PROOF OF THEOREM 1 (PROPERTY (C)): The function $\mu: E \to \mu(E)$ is the composition of the function $\theta: E \to \theta(E)$ and a function $\eta: \theta(E) \to \mu(E)$ defined as follows. For each $(a^1, \ldots, a^n) \in \theta(E)$ identify the ordered n-tuple $(a^1, \ldots, a^n)$ with the $n \times n$ matrix whose columns are $a^1, \ldots, a^n$, i.e., with $a = [a^1, \ldots, a^n]$. Let $\eta: \theta(E) \to P$ be the function such that for each $a \in \theta(E)$, $\eta(a) = p$, where p is such that $pa = 0$. Evidently, η is single-valued and $\eta \circ \theta = \mu$.

By Proposition 4, we know that θ is continuous and θ^{-1} is lower hemicontinuous. We show that η is continuous and η^{-1} is lower hemicontinuous. It then follows that μ is continuous and μ^{-1} is lower hemicontinuous (cf., Berge [3, Theorem 1 Section 2, Chapter VI]).

First we prove that η^{-1} is uniformly globally threaded such that each of its threads is an open and continuous function.

Let $\bar{p} \in \eta(\theta(E))$ and let $\bar{a} \in \eta^{-1}(\bar{p})$. Then $\bar{p}\bar{a} = 0$ and there exists an $\bar{e} \in E$ such that $\bar{p} = \mu(\bar{e})$ and $\bar{a} = \theta(\bar{e})$. Let $\bar{e}^n = (\bar{u}, w^*)$. Then, there exist $(\bar{\alpha}_1, \ldots, \bar{\alpha}_n) \in R^n_{++}$ such that $\bar{u}(x) = \prod x_i^{\bar{\alpha}_i}$ for all $x \in R^n_+$. Since $\bar{a} = \theta(\bar{e})$ and $\bar{p} = \mu(\bar{e})$, $\bar{\alpha}_i = \bar{p}_i(\bar{a}^n_i + w^*_i)$ $(i = 1, \ldots, n)$ where $\bar{a}^n$ is the nth column of $\bar{a}$.

We construct a function $\bar{s}: \mu(E) \to \theta(E)$ such that $\bar{s}(\bar{p}) = \bar{a}$ and $\bar{s}(p) \in \eta^{-1}(p)$ for every $p \in \eta(\theta(E))$, i.e., $\bar{s}$ is a global thread of η^{-1} passing through $\bar{a}$. For every $p \in \eta(\theta(E))$, define a $n \times n$ diagonal matrix $Q = \operatorname{diag}(\bar{p}_i/p_i)$. For every $p \in \eta(\theta(E))$, let

$$\bar{s}(p) = Q\bar{a}Q^{-1} = a, \quad \text{say}.$$

Evidently, $pa = 0$ and $p_i(a^n_i + w^*_i) = \bar{p}_i(\bar{a}^n_i + w^*_i) = \bar{\alpha}_i$, i.e., $a^n = c(p, \bar{e}^n)$, where a^n is the nth column of a. Let $e = (e^1, \ldots, e^n)$ be such that $e^i = \{a^i\}$ $(i = 1, \ldots, n-1)$ and $e^n = \bar{e}^n = (\bar{u}, w^*)$. Then $e \in E$, (by Proposition 2′) $a = \theta(e)$ and $p = \mu(e)$. Thus, $a \in \eta^{-1}(p)$, since $pa = 0$. This proves that $\bar{s}(p) \in \eta^{-1}(p)$.

From the definition of $\bar{s}$ it is clear that $\bar{s}: \eta(\theta(E)) \to \bar{s}(\eta(\theta(E)))$ is one-one open and continuous. Since $\bar{p}$ and $\bar{a}$ are arbitrary, this proves that η^{-1} is uniformly globally threaded such that each of its threads is open and continuous. Let S be the collection of all global threads of η^{-1}. Then for any set $G \subset \eta(\theta(E))$,

$$\eta^{-1}(G) = \bigcup_{s \in S} s(G).$$

Since each function $s \in S$ is open, $s(G)$ is open for every open set $G \subset \eta(\theta(E))$. This implies that $\eta^{-1}(G)$ is open for G open. Hence η is continuous. Also, since η^{-1} is uniformly globally threaded, Lemma 1 implies that η^{-1} is lower hemicontinuous.

PROOF OF PROPOSITION 5: Clearly, dom $f^* = E^*$. Using Proposition 2′ and the single-valuedness of θ, for every $\bar{e} \in E^*$, $(\bar{a}^1, \ldots, \bar{a}^n) = \theta(\bar{e})$ if and only if

(a) $\bar{e}^i = \{\bar{a}^i\}$ $\quad (i = 1, \ldots, n-1)$;

(b) there is an equilibrium price vector $\bar{p} \in P$ such that $\bar{p}\bar{a}^i = 0$ $\quad (i = 1, \ldots, n)$;

(c) $\bar{e}^n = (\bar{u}, w^*)$, where $\bar{u}(x) = \prod x_i^{\bar{\alpha}_i}$ for all $x \in R^n_+$ and $\bar{\alpha}_i = \bar{p}_i(\bar{a}^n_i + w^*_i)$.

These conditions imply that for every $e, \bar{e} \in E^*$, $\theta(e) = \theta(\bar{e})$ only if $e = \bar{e}$. This proves that E has the uniqueness property with respect to f^*.

That E does not have the asymmetry property with respect to f^* can be proved as follows. Let $e, \bar{e} \in E^*$ be such that $\theta(e) \neq \theta(\bar{e})$ but there is a price vector $p \in P$ such that

(d) $pa^i = p\bar{a}^i = 0, \quad i \in N,$

where $(a^1, \ldots, a^n) = \theta(e)$ and $(\bar{a}^1, \ldots, \bar{a}^n) = \theta(\bar{e})$. Clearly, such e and $\bar{e}$ exist. Let $a(i) = (a^1, \ldots, a^{i-1}, \bar{a}^i, a^{i+1}, \ldots, a^n)$ and $e(i) = (e^1, \ldots, e^{i-1}, \bar{e}^i, e^{i+1}, \ldots, e^n)$, $i \in N$. Then conditions (a), (b), and (c) as applied to e, $\bar{e}$, and $e(i)$, $i \in N$, and (d) imply that $a(i) = \theta(e(i))$ for each $i \in N$. Since $e \neq \bar{e}$, the proof follows.

PROOF OF LEMMA 3: We have to show that given two arbitrary elements $e, \bar{e} \in$ dom f, $\nu(e) \cap \nu(\bar{e}) \neq \emptyset$ implies $e = \bar{e}$. Let $m \in \nu(e) \cap \nu(\bar{e})$ and let $h(m, e) = a$ and $h(m, \bar{e}) = \bar{a}$. Because $[\nu, h]$ is given to be privacy preserving, therefore, (i) if $m \in \nu(e) \cap \nu(\bar{e})$ then $m \in \nu(e(i)) \cap \nu(\bar{e}(i))$ for every $i \in N$, and (ii) if $m \in \nu(e(i))$ for every $i \in N$, $a = h(m, e)$, and $\bar{a} = h(m, \bar{e})$, then $a(i) = h(m, e(i))$ for every $i \in N$, where $e(i)$ and $a(i)$ are as defined in the definition of the asymmetry property. (i) and (ii) together imply that $a \in \theta(e)$, $\bar{a} \in \theta(\bar{e})$, and $a(i) \in \theta(e(i))$ for every $i \in N$, since we are given that $[\nu, h]$ is nonwasteful for E and that dom $f \subset L$(dom f) $\subset E$. The fact that $a \in \theta(e)$, $\bar{a} \in \theta(\bar{e})$, and $a(i) \in \theta(e(i))$ for every $i \in N$, together with the fact that E has the asymmetry property with respect to f implies that $f(e) = f(\bar{e})$. Since f is one-one, $e = \bar{e}$.

PROOF OF THEOREM 3: First we construct a subclass of environments $E^{**} \subset E$ such that E has the asymmetry property with respect to the identity function on E^{**} and $\mu(E^{**}) = \mu(E)$. There are several such subclasses. We shall take up the one that is technically easier to handle. Let

$$E^{**i} = \{ e^i \in E^{*i} : \text{if } a \in e^i \text{ then } a_j = 0 \text{ for } j \neq i, n \} \qquad (i = 1, \ldots, n-1)$$

where $E^{*i} (i = 1, \ldots, n-1)$ are as defined in Proposition 5. Let

$$E^{**n} = U^{**} \times W^*, \qquad \text{where}$$

$$U^{**} = \{ u \in U^* : u(x) = \textstyle\prod x_i^{1/n} \text{ for all } x \in R_+^n \}.$$

Clearly, U^{**} is a singleton consisting of the utility function $u^* : R_+^n \to R$ such that $u^*(x) = \prod x_i^{1/n}$ for all $x \in R_+^n$.

Let $E^{**} = E^{**1} \times \cdots \times E^{**n}$. Evidently, $L(E^{**}) = E^{**}$ and $E^{**} \subset E^*$. We prove that E has the asymmetry property with respect to the identity function on E^{**}. For every $\tilde{e} \in E^{**}$, $(\tilde{a}^1, \ldots, \tilde{a}^n) \in \theta(\tilde{e})$, if and only if

$$\{\tilde{a}^i\} = \tilde{e}^i \quad \text{and}$$

$$(1) \qquad -\tilde{a}_n^i(\tilde{a}_i^n + w_i^*) = \frac{1}{n} \qquad (i = 1, \ldots, n-1).$$

Let $e, \bar{e} \in E^{**}$. If $(a^1, \ldots, a^n) \in \theta(e)$ and $(\bar{a}^1, \ldots, \bar{a}^n) \in \theta(\bar{e})$, then (1) implies that

$$(2) \qquad -a_n^j(a_j^n + w_j^*) = \frac{1}{n} \qquad (j = 1, \ldots, n-1)$$

and

$$(3) \qquad -\bar{a}_n^j(\bar{a}_j^n + w_j^*) = \frac{1}{n} \qquad (j = 1, \ldots, n-1).$$

Let $a(i)$ and $e(i)$, $i \in N$, be as defined in the definition of the asymmetry property. Then, by (1), $a(i) \in \theta(e(i))$, $i \in N$, only if

$$(4) \qquad -\bar{a}_n^i(a_i^n + w_i^*) = \frac{1}{n}.$$

Clearly, (2), (3), and (4) imply that $\bar{a}_n^i = a_n^i$, i.e., $\bar{a}^i = a^i$. This means that $a(i) = a$ and so $a \in \theta(e(i))$. But if $a \in \theta(e(i))$ then (1) implies that $e(i) = e$. This proves that E^{**} has the asymmetry property with respect to the identity function on E^{**}.

Let $\hat{\mu}$ be the restriction of μ to E^{**}, i.e., $\hat{\mu} : E^{**} \to \mu(E)$. We prove that $\hat{\mu}(E^{**}) = \mu(E)$ and that $\hat{\mu}^{-1} : \mu(E) \to E^{**}$ is a continuous bijection. Let $p \in \mu(E)$. Then $p \in R_{++}^n$ and $p_n = 1$. For $i = 1, \ldots, n-1$, let a^i be the n-dimensional vector which has unity in the ith place, $-p_i$ in the nth place, and zero elsewhere. Let $e^i = \{a^i\}$ $(i = 1, \ldots, n-1)$ and $e^n = (u^*, w^*)$. Then $pa^i = 0$ $(i = 1, \ldots, n-1)$. Corollary to Proposition 2′ and the definition of E^{**} imply that $(e^1, \ldots, e^n) \in E^{**}$. Definition of μ implies that $\mu(e) = p$. Thus, $p \in \mu(E^{**})$. This implies that $\mu(E^{**}) \supset \mu(E)$; thus $\mu(E^{**}) = \mu(E)$. Moreover, it is clear that $\hat{\mu} : E^{**} \to \mu(E)$ is one-one. Thus,

$$\hat{\mu} : E^{**} \to \mu(E) \quad \text{is a bijection.}$$

Let $\hat{\theta}$ denote the restriction of θ to E^{**}, i.e., $\hat{\theta} : E^{**} \to \hat{\theta}(E^{**})$. Since $E^{**} \subset E^*$, $\hat{\theta}$ is injective over E^{**} (by Proposition 5). Since $\hat{\theta}$ is single-valued, the restriction of the pseudo-metric d to the points on E^{**} is a metric on E^{**}, i.e., E^{**} is a metric subspace of the pseudo-metric space E. So $\theta : E^{**} \to \hat{\theta}(E^{**})$ is an isometry, i.e., $\hat{\theta}$ is a homeomorphism. This means

$$\hat{\theta}^{-1} : \hat{\theta}(E^{**}) \to E^{**} \quad \text{is a continuous bijection.}$$

Let $\eta : \theta(E) \to \mu(E)$ be the continuous surjection as defined in the proof of Theorem 1 (Property (C)). Let $\hat{\eta} : \hat{\theta}(E^{**}) \to \mu(E)$ be the restriction of η to $\hat{\theta}(E^{**})$. It is evident from the proof of Theorem 1 (Property (C)) that $\hat{\eta}$ is a homeomorphism and that $\hat{\mu}^{-1} = \hat{\theta}^{-1} \circ \hat{\eta}^{-1}$. Since $\hat{\theta}^{-1}$ is a continuous bijection,

$$\hat{\mu}^{-1} : \mu(E) \to E^{**} \quad \text{is a continuous bijection.}$$

Given that $[\nu, h]$ is decisive, nonwasteful, and privacy preserving, Lemma 3 implies that ν is injective over E^{**}. Since $\mu(E)$ is Euclidean, let p^0 be an interior point of $\mu(E)$. Let $e^0 = \hat{\mu}^{-1}(p^0)$. Then by the definition of $\hat{\mu}$, $e^0 \in E^{**}$. We are given that $[\nu, h]$ is smooth over E; therefore, there exists an open neighborhood W_0 in E of e^0 and a continuous function $s_0 : W_0 \to \nu(E)$ such that $s_0(e) \in \nu(e)$ for every $e \in W_0$. Let $W^*_0 = W_0 \cap E^{**}$ and $s_0^* = s_0 | W_0^*$. Then W_0^* is a nonempty open neighborhood in E^{**} of e^0 and $s_0^* : W_0^* \to \nu(E)$ is continuous. Moreover, s_0^* is one-one, since for every $e, \bar{e} \in W_0^*$, $s_0^*(e) = s_0^*(\bar{e})$ implies $\nu(e) \cap \nu(\bar{e}) \neq \emptyset$ in contradiction to Lemma 3. Let $U_0 = \hat{\mu}(W_0^*)$. Then U_0 is an open neighborhood in $\mu(E)$ of p^0 (since $\hat{\mu}^{-1} : \mu(E) \to E^{**}$ is a continuous bijection and W_0^* is open in E^{**}). Let $\hat{\mu}_0^{-1} = \hat{\mu}^{-1} | U_0$. Since s_0^* and $\hat{\mu}_0^{-1}$ are both continuous and one-one functions, the composition $s_0^* \circ \hat{\mu}_0^{-1} : U_0 \to \nu(E)$ is continuous and one-one. Therefore $\nu(E) \geqq_D U_0$ (cf. [8, Corollary 1.1]). However, the dimension of U_0 is equal to the dimension of $\mu(E)$, since U_0 is open in $\mu(E)$. Hence $\nu(E) \geqq_D \mu(E)$.

Proof of Lemma 2: We have to show that given two arbitrary elements $e, \bar{e} \in \operatorname{dom} f$, $\nu(e) \cap \nu(\bar{e}) \neq \emptyset$ implies $e = \bar{e}$. Let $m \in \nu(e) \cap \nu(\bar{e})$ and let $\underline{a} = h(m, e)$. Given that $[\nu, h]$ is nonparametric, $\underline{a} = h(m, e')$ for every $e' \in E$. In particular, $\underline{a} = h(m, \bar{e})$. Since $[\nu, h]$ is given to be nonwasteful for E, $\underline{a}$ belongs to both $\theta(e)$ and $\theta(\bar{e})$, i.e., $\underline{a} \in \theta(e) \cap \theta(\bar{e})$.

Since $e, \bar{e} \in \operatorname{dom} f$ and $\theta(e) \cap \theta(\bar{e}) \neq \emptyset$, it follows from the definition of the uniqueness property that $e(i), \bar{e}(i) \in E$ for every $i \in N$ (where $e(i)$ and $\bar{e}(i)$ are as defined in the definition). Given that $[\nu,h]$ is privacy preserving and that $e(i)$, $\bar{e}(i) \in E$ for every $i \in N$, $\nu(e) \cap \nu(\bar{e}) = \nu(e(i)) \cap \nu(\bar{e}(i))$ for every $i \in N$. This implies that $m \in \nu(e) \cap \nu(\bar{e}) \cap \bigcap_{i=1}^{n} \nu(e(i))$. Since $[\nu,h]$ is nonwasteful for E and $e(i) \in E$ for every $i \in N$, $\underline{a} \in \theta(e) \cap \theta(\bar{e}) \cap \bigcap_{i=1}^{n} \theta(e(i))$ (because $\underline{a} = h(m, e(i))$ for every $e(i) \in E$). But the fact that this intersection is nonempty, together with the fact that E has the uniqueness property with respect to f, implies that $f(e) = f(\bar{e})$. Since f is one-one, $e = \bar{e}$.

Proof of Theorem 2: First we construct an environment class $E^* \subset E$ such that E has the uniqueness property with respect to the identity function on E^* and $\theta(E^*) = \theta(E)$. Let E^* be the class of environments as defined in Proposition 5. As proved in Proposition 5, E has the uniqueness property with respect to the identity function on E^*.

Let θ^* denote the restriction of θ to E^*. Then $\operatorname{dom}\theta^* = E^*$ and $\theta^*(E^*) \subset \theta(E)$. We prove that $\theta^*(E^*) \supset \theta(E)$, thus, $\theta^*(E^*) = \theta(E)$. Let $(\bar{a}^1, \ldots, \bar{a}^n) \in \theta(E)$. Then, by Proposition 2′, there exists a price vector $p \in P$ such that $p\bar{a}^i = 0$ for all $i \in N$. Define $\bar{\alpha}_i = p_i(\bar{a}_i^n + w_i^*)$. Then $\sum \bar{\alpha}_i = 1$. Let $\bar{u} : R_+^n \to R$ be the utility function defined as $\bar{u}(x) = \prod x_i^{\bar{\alpha}_i}$ for all $x \in R_+^n$. Then $\bar{u} \in U^*$ and $\bar{a}^n = c(p, \bar{e}^n)$ for $\bar{e}^n = (\bar{u}, w^*)$. Let $\bar{e}^i = \{\bar{a}^i\}$ $(i = 1, \ldots, n-1)$. Proposition 2′ implies that $(\bar{a}^1, \ldots, \bar{a}^n) \in \theta(\bar{e})$. Definition of E^* implies that $\bar{e} \in E^*$. This proves that $\theta(E) \subset \theta^*(E^*)$. Hence $\theta^*(E^*) = \theta(E)$.

By Proposition 2, θ^* is single-valued and as proved in Proposition 5, θ^* is injective over E^*. As in the proof of Theorem 3, E^* is a metric subspace of E and $\theta^* : E^* \to \theta(E)$ is a homeomorphism. This means

$$\theta^{*-1} : \theta(E) \to E^* \quad \text{is a continuous bijection.}$$

Given that $[\nu,h]$ is a nonparametric resource allocation process, which is decisive, nonwasteful and privacy preserving, Lemma 2 implies that ν is injective over E^*. We can now follow a similar line of proof as in Theorem 3 and prove that there are functions s_0^* and θ_0^{*-1} such that the composition $s_0^* \circ \theta_0^{*-1} : U_0 \to \nu(E)$ is continuous and one-one for some open neighborhood U_0 in $\nu(E)$. It then follows that $\nu(E) \geqq_D \theta(E)$.

6. THE MALINVAUD–TAYLOR PROCESS

This section has a dual purpose: (i) to show that the class of processes considered in Theorem 3 above includes processes other than the Taylor process, and (ii) to highlight the role of the parametric (or nonparametric) outcome function in relation to the informational efficiency of the process. A natural choice for this is the Malinvaud–Taylor process [8, 20, 28] which is decisive, nonwasteful, privacy preserving, smooth, nonparametric, and informationally efficient among the class of nonparametric resource allocation processes (for this we use Theorem 2), but not among the general class of parametric resources allocation processes (for this we use Theorem 3).

Let $\beta:\theta(E)\to P$ be the function such that for each $(a^1,\ldots,a^n)\in\theta(E)$, $\beta(a^1,\ldots,a^n)=p$, where $p\in P$ is such that $pa^i=0$ for all $i\in N$. For each $a\in\theta(E)$ and $e^i\in E^i$, let

$$g^i(a,e^i)=\underset{b^i\in e^i}{\text{maximizer}}\ \beta(a)b^i \qquad (i=1,\ldots,n-1)$$

and

$$g^n(a,e^n)=\underset{y\in\{z\in S^*:\beta(a)z\leq 0\}}{\text{maximizer}} u(y+w^*), \qquad \text{where}\quad e^n=(u,w^*).$$

The functions $(g^1,\ldots,g^n)$ as defined above determine the nature of the following dynamic process:

$$a^i(t+1)=g^i\big(\beta(a^1(t),\ldots,a^n(t)),e^i\big) \qquad (i\in N; t=0,\ldots),$$

where $(a^1(0),\ldots,a^n(0))\in e^1\times\cdots\times e^{n-1}\times S^*\cap\theta(E)$. So in equilibrium

$$\bar a^i=g^i\big(\beta(\bar a^1,\ldots,\bar a^n);e^i\big) \qquad (i=1,\ldots,n).$$

Let $\nu:E\to\nu(E)$ be the correspondence defined as

$$\nu(e)=\big\{(a^1,\ldots,a^n)\in e^1\times\cdots\times e^{n-1}\times S^*:(a^1,\ldots,a^n)$$
$$=g\big(\beta(a^1,\ldots,a^n),e\big)\big\}$$

where $g=(g^1,\ldots,g^n)$.

The Malinvaud–Taylor process is then defined as the ordered pair $[\nu, I]$ where I is the identity function on $\theta(E)$. The static properties of the Malinvaud–Taylor process follow from the fact that $\nu=\theta$. Its smoothness follows from Proposition 4. That it is informationally efficient among the class of *nonparametric* resource allocation processes follows from Theorem 2, and that it is informationally inefficient among the more general class of parametric processes from Theorem 3 and the fact that $\theta(E)\geqq_D\mu(E)$ but $\mu(E)\ngeqq_D\theta(E)$, where $\mu(E)$ is the message space of the Taylor process.

7. CONCLUSION

We have proved above the informational efficiency of the Taylor process among a certain class of *parametric* resource allocation processes. For this we have developed an analytic framework that may be of more general applicability. The definition of the asymmetry property, Lemma 3, and Theorem 3 in comparison to that of the uniqueness property, Lemma 2, and Theorem 2, respectively, are of particular interest from this point of view. Our results bring into focus the following aspects of the general theory of resource allocation processes.

(i) Greater informational decentralization can be achieved through parametric than through nonparametric processes. (ii) For any environment class for which the uniqueness property is equivalent to the asymmetry property, greater informational decentralization cannot be achieved through parametric processes.

Point (ii) above offers a useful and easily testable criterion. We give here two examples. For the classical pure-exchange environment class the uniqueness property is equivalent to the asymmetry property (see Chander [9]). But for the environment classes that were introduced by Hurwicz [14] in his examples A and B (concerning the possibility of informational decentralization in production economies with externalities) uniqueness is not equivalent to asymmetry (see Chander [10]).

Finally, we end this paper with a comment concerning the environment class considered. One natural question may be: can we broaden the environment class and still find (informationally decentralized) processes which will possess desired (static) optimality properties (for example, Pareto optimality of the equilibrium), and possibly some dynamic (stability) properties, and which will be amenable to the same type of analysis as above? The answer to this question is, of course, in the affirmative. After all, the environments considered in this paper constitute only a special case of the classical environments with production. We restrict to a narrow class, however, because it enables us to focus on some particular aspects (as noted in the two preceding paragraphs) of the general theory of resource allocation processes.

Indian Statistical Institute

Manuscript received November, 1981; revision received July, 1982.

REFERENCES

[1] ARROW, K. J.: "Alternative Proof of the Substitution Theorem for Leontief Models in the General Case," in *Activity Analysis of Production and Allocation*, ed. by T. C. Koopmans. New York: John Wiley and Sons, 1951, pp. 155–164.

[2] ARROW, K. J., AND L. HURWICZ: "Decentralization and Computations in Resource Allocation," in *Essays in Economics and Econometrics*, ed. by R. W. Pfouts. Chapel Hill, North Carolina: University of North Carolina Press, 1960, pp. 34–104.

[3] BERGE, C.: *Topological Spaces*. New York: Macmillan, 1963.

[4] CALSAMIGLIA, S.: "Decentralized Resource Allocation and Increasing Returns," *Journal of Economic Theory*, 14(1977), 263–283.

[5] CHANDER, P.: "A Simple Proof of the Nonsubstitution Theorem," *Quarterly Journal of Economics*, 88(1974), 698–701.

[6] ———: "On a Planning Process due to Taylor," *Econometrica*, 46(1978), 761–777.

[7] ———: "The Computation of Equilibrium Prices," *Econometrica*, 46(1978), 723–726.

[8] ———: "Informational Requirements for Efficient Resource Allocation Processes," paper presented at the Econometric Society World Congress, 1980, Aix, France, (Revised) January, 1981.

[9] ———: "On the Informational Efficiency of the Competitive Resource Allocation Process," Discussion Paper No. 8103, Indian Statistical Institute, forthcoming in *Journal of Economic Theory*, 1983.

[10] ———: "Informational Decentralization and Production Economies Involving Externalities," Discussion Paper No. 8104, Indian Statistical Institute, June, 1981.

[11] DOBB, MAURICE: *On Economic Theory and Socialism*. London: Routledge and Kegan Paul Ltd., 1955, pp. 239–246.

[12] HAYEK, F. A.: "The Present State of the Debate," in *Collectivist Economic Planning*, ed. by F. A. Hayek. London: Routledge and Kegan Paul Ltd., 1963, pp. 207–214.

[13] HURWICZ, L.: "Optimality and Informational Efficiency in Resource Allocation Processes," in *Mathematical Methods in the Social Sciences*, K. J. Arrow, S. Karlin, and P. Suppes. Stanford: Stanford University Press, 1959, pp. 27–46.

[14] ———: "On Informationally Decentralized Systems," in *Decision and Organization* ed. by C. B. McGuire and R. Radner. Amsterdam: North Holland, 1972, Chapter 14, pp. 297–336.

[15] ———: "On the Informational Requirements of Informationally Decentralized Pareto-satisfactory Processes," (mimeographed) paper presented at the Conference Seminar on Decentralization, Northwestern University, 1972; reproduced in *Studies in Resource Allocation Processes*, ed. by K. J. Arrow and L. Hurwicz. Cambridge: Cambridge University Press, 1977, pp. 413–424.

[16] JORDAN, J: "Expectations Equilibrium and Informational Efficiency for Stochastic Environments," *Journal of Economic Theory*, 16(1977), 354–372.

[17] ———: "The Competitive Process is Informationally Efficient Uniquely," *Journal of Economic Theory*, 28(1982), 1–18.

[18] LANGE, O.: "On the Economic Theory of Socialism," in *The Economic Theory of Socialism*, ed. by B. E. Lippincott. Minneapolis: University of Minnesota Press, 1938, pp. 57–141.

[19] MARGLIN, S. A.: "Information in Price and Command Systems in Planning," in *Public Economics*, ed. by J. Margolis and H. Guitton. London: Macmillan, 1969, pp. 54–57.

[20] MALINVAUD, E.: "Decentralized Procedures for Planning," in *Activity Analysis in the Theory of Growth and Planning*, ed. by M. O. L. Bacharach and E. Malinvaud. London: Macmillan, 1967, pp. 170–208.

[21] MOUNT, K., AND S. REITER: "The Informational Size of Message Spaces," *Journal of Economic Theory*, 6(1974), 161–192.

[22] NIKAIDO, H.: *Convex Structures and Economic Theory*. New York and London: Academic Press, 1968, pp. 87–148.

[23] OSANA, H.: "On the Informational Size of Message Spaces for Resource Allocation Processes," *Journal of Economic Theory*, 17(1978), pp. 66–78.

[24] SAMUELSON, P. A.: "Abstract of a Theorem Concerning Substitutability in Open Leontief Models," in *Activity Analysis of Production and Allocation*, ed. by T. C. Koopmans, 1951, pp. 142–146.

[25] SCHUMPETER, J. A.: *Capitalism, Socialism and Democracy*. New York: Harper and Row, Third Edition, 1949, pp. 172–199.

[26] TAYLOR, F. M.: "The Guidance of Production in a Socialist State," *American Economic Review*, 19(1929), 1–8, reproduced in *Economic Theory of Socialism*, ed. by B. E. Lippincott. Minneapolis: University of Minnesota Press, 1938.

[27] WALKER, M.: "On the Informational Size of Message Spaces," *Journal of Economic Theory*, 15(1977), 366–375.

[28] WARD, B.: *The Socialist Economy*. New York: Random House, 1967.

[29] WEITZMAN, M. L.: "On Choosing an Optimal Technology," *Management Science*, 13(1967), 413–428.