POST-PROJECT RESEARCH CONTRIBUTION TO THE COMPLEMENTARITY PROBLEM

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Introduction

The project titled Optimization and Reliability Modeling was started with the objective of studying various optimization models following a unified approach with a solution procedure and applying optimization techniques in different areas of reliability as well as life testing problems so that a general-purpose optimization method could be devised for a group of optimization problems that arise in real-life applications and codes could be developed for the theoretically best-known algorithms for obtaining optimal solutions. The complementarity model provides a unifying framework for several optimization problems. The project was funded by the Indian Statistical Institute for the period from April 2012 to March 2016. The study was conducted based on various optimization problems following the approach of Professor R. W. Cottle of Stanford University, well known for his contributions to optimization theory and algorithms. The motivation for this project on optimization originated in the early work of Professor George B. Dantzig, the father of linear programming and the simplex algorithm, and in the work of the systems optimization group at Stanford University. Based on the outcome of this project, various research works were initiated in the area of complementarity problems. For further details of the author, the project report, and the post-project research publications, visit https://sites.google.com/view/abhilashaeshaarupdas/home.

Complementarity Problem and its Applications

The linear complementarity problem (LCP) is a combination of linear and nonlinear systems of inequalities and equations. It is an important problem in operations research and in other fields. The problem has undergone several name changes, from composite problem to fundamental problem to complementary pivot problem. The current name, linear complementarity problem, was proposed by R. W. Cottle. LCP is normally identified as a part of optimization theory and equilibrium problems. The algorithm presented by Lemke and Howson to compute an equilibrium pair of strategies for a bimatrix game, later extended by Lemke to solve an LCP(q, A), contributed significantly to the development of linear complementarity theory and brought the LCP into the limelight. Ever since, the subject has been making great strides and has been a fertile field for practitioners and researchers. The role of the complementary slackness principle is well-known in optimization theory. This principle holds not only for the linear programming problem but also for more general programming problems. In particular, it is useful for developing algorithms for convex quadratic programming problems. The complementary slackness principle for more general programming problems is based on the Karush-Kuhn-Tucker optimality conditions. In fact, various classes of optimization problems can be formulated as complementarity problems. More specifically, the problems that can be posed as an LCP include linear programming, linear fractional programming, convex quadratic programming, and the bimatrix game problem. It is

well studied in the literature on mathematical programming and arises in several applications in operations research, control theory, mathematical economics, geometry, and engineering. For recent books on this problem and applications, see Cottle, Pang and Stone, and Murty. The novelty of the problem is that it unifies several optimization problems in the field, such as mathematical programming, game theory, the theory of equilibrium in a competitive economy, equilibrium of traffic flows, mechanics, engineering, lubricant evaporation in the cavity of a cylindrical bearing, elastic theory, fluid flow through a semipermeable membrane, maximizing oil production, computation of fixed points, and related areas. The absolute value equation and interval matrices are closely related to the theory of complementarity.

Matrix classes play an important role in the study of LCP theory and algorithms. The study of special properties of the data matrix has historically been an important part of LCP research. A variety of classes of matrices are introduced in the context of the linear complementarity problem. Many of the matrix classes encountered in the context of LCP are commonly found in several applications. Some of these matrix classes are of interest because they characterize certain properties of the LCP, and they offer certain nice features from the viewpoint of algorithms. Several algorithms have been designed to solve the linear complementarity problem. These methods are matrix class dependent, i.e., they can find a solution to this problem only for certain matrix classes and provide no information otherwise. The matrix class dependence of the LCP algorithms has led to considering the existence of solutions in terms of matrix classes. It is useful to review some matrix classes and their properties, which will form the basis for further discussions. Several algorithms have been designed for the solution of the complementarity problem. These methods are matrix class dependent, i.e., they can find a solution to this problem only for special classes of matrices and provide no information otherwise. This sort of application, and potential future applications, has motivated the study of optimization theory, especially algorithms and matrix classes. In fact, much of the theory and algorithms for its solution are based on certain matrix classes. The connection between affine pseudo-monotone mapping, affine quasi-monotone mapping, positive subdefinite matrices, generalized positive subdefinite matrices, and the linear complementarity problem is important for the characterization of certain matrix classes. The concept of principal pivot transform is a fundamental concept for developing many theories and algorithms in optimization theory. The notion of principal pivot transform is originally motivated by the well-known linear complementarity problem.

Complementarity Problems with Game Theory

The linear complementarity problem and the matrix game have some important connections. Many of the results of LCP can be stated in terms of the value of a matrix game. In this connection, Kaplansky's result on matrix games is useful for deriving certain results. A bimatrix game is a non-cooperative nonzero-sum two-person game in which each player has a finite number of actions (called pure strategies). This problem can be formulated as a linear complementarity problem. Lemke and Howson proposed an algorithm to compute a Nash equilibrium pair of strategies for a bimatrix game.

Stochastic games were first formulated by Shapley in 1953. The game considered by Shapley is called two-person zero-sum discounted game with finite state and action space. In this fundamental paper, Shapley proved the existence of a value and optimal stationary strategies for the discounted case, thereby providing a method for the iterative computation of the value of a stochastic game with discounted payoff. Gillette studied the undiscounted case, or limiting average payoff case. Since then, most researchers in stochastic games have focused on finding

sufficient conditions for the existence of a value and optimal strategies. Shapley's stochastic game was also extended to non-zero-sum games by many researchers. The theory of stochastic games has been applied to study many practical problems like search problems, military applications, advertising problems, the traveling inspector model, and economic applications.

Another major area of research in this field is to identify those classes of zero-sum stochastic games for which there is a possibility of obtaining a finite-step algorithm to compute a solution. Many of the results in this area concern zero-sum games with special structures, treated as structured stochastic games. The class of structured stochastic games contains single control game, switching control game, game with state independent transitions and separable rewards, and game with additive rewards and transitions. For each of these structured classes of games, it is known that optimal stationary strategies exist and that the value vector and optimal stationary strategies satisfy the order field property, i.e., the games' solutions lie in the same ordered field as the game data (e.g., rational). Many researchers have attempted either to give a finite-step method for computing a value and optimal strategies, or at least to give a constructive proof of their existence. Filar and Schultz observed that an undiscounted zero-sum stochastic game possesses optimal stationary strategies if and only if a global minimum with optimum value zero can be found to an appropriate linearly constrained nonlinear program. Perhaps a more interesting problem is reducing these nonlinear programs to a linear complementarity problem or a linear program. The linear complementarity problem arises in certain classes of stochastic games. The problem of computing the value vector and optimal stationary strategies for zero-sum undiscounted switching control and zero-sum ARAT stochastic games is formulated as a linear complementarity problem.

Complementarity Problems with Large and Sparse Matrices

Large and sparse matrices are matrices that have a large number of rows and columns but a small number of non-zero elements. In other words, they are matrices where the majority of the elements are zero. Sparse matrices are commonly used to represent complex systems or large data sets in fields such as computer science, mathematics, physics, and engineering. Several classes of matrix-splitting methods for solving the complementarity problem with large, sparse matrices are considered. Some generalizations of the linear complementarity problem, namely the vertical, horizontal, and implicit linear complementarity problems, are also included in this context.

Absolute Value Equation and Interval Matrices

The absolute value equation (AVE) is a class of nonlinear systems. Despite its simple appearance, the AVE is computationally challenging and deeply connected to other fundamental problems in optimization and numerical analysis. The AVE is mathematically equivalent to the LCP. This equivalence provides a bridge to the vast body of theory and algorithms developed for LCP. Despite its simple form, solving the general AVE problem is NP-hard. This means no known polynomial-time algorithm exists for the general case, which explains the focus on finding sufficient conditions for unique solvability and developing specialized algorithms.

The connection between absolute value equations and interval matrices is a deep and well-studied relationship, primarily rooted in the concept of regularity. In essence, the theory of interval matrices provides the fundamental criteria for determining when an AVE is guaranteed

to have a unique solution. Conversely, AVEs serve as powerful computational tools for analyzing the solution sets of interval linear systems. The relationship is also practical in the other direction. AVEs are used to efficiently compute the hull of the solution set for systems of interval linear equations. The close relationship also means they share computational challenges. The problem of checking the regularity of an interval matrix is NP-hard. Consequently, verifying the unique solvability of a general AVE is also NP-hard.

Nonlinear Complementarity Problem

The nonlinear complementarity problem (NCP) is an interesting class of complementarity problems. It can further be seen as an extension of the linear complementarity problem. Thus, using NCP algorithms, one can also solve LCP problems. This makes techniques for solving nonlinear complementarity problems of particular interest among researchers in this field. Several techniques have emerged in recent years, but implementing them is challenging. As many of these methods are complex and fail to work across multiple problem types, ongoing efforts focus on iterative techniques to address these issues. Efforts by leading researchers such as Kanzow, Yamashita, Fukushima, Fischer, Peng, and Ferris have made significant contributions to this field.

The idea of NCP functions, along with merit functions, the Penalized Newton method, and the method of subdifferentials, are other approaches to solve nonlinear complementarity problems. Recently, in multilinear algebra, several topics including tensors, have become popular for further study. The corresponding tensor complementarity problem was introduced as a subclass of nonlinear complementarity problems, involving a particular class of homogeneous polynomial functions expressed in terms of a tensor. By reformulating the multilinear game as a tensor complementarity problem, a bridge between the two is established.

Semidefinite Linear Complementarity Problem

The semidefinite linear complementarity problem (SDLCP) has a wide range of applications in control theory and various areas of optimization problems. This highlights the important attention that SDLCPs have received in optimization theory in recent years. The SDLCP was introduced in a slightly different form by Kojima et al. It is well known that the linear complementarity problem can be reformulated as an SDLCP. In recent years, much emphasis has been put on the study of interior-point algorithms for SDLCP. Interior-point methods (IPMs) are now the most effective methods for solving optimization problems. One may distinguish between IPMs according to whether they are feasible or infeasible. A feasible IPM can be started only if a strictly feasible point is known. Otherwise, an infeasible IPM is recommended.

Conclusion and Future Work

The concept of complementarity is a crucial issue for optimization theory. It is expected that efficient numerical methods will be developed based on the complementarity condition to solve a wide range of optimization problems. In the future, the theory of complementarity will help develop additional theoretical frameworks and solution methods for both the absolute value equation and the variational inequality problems. **For further details of the research publications, visit** https://sites.google.com/view/abhilashaeshaarupdas/home.

Brief Biographical Notes: Dr. Arup Kumar Das is a faculty member of the Indian Statistical Institute and has held the post of professor grade since 2009. He did his PhD in Mathematics from the Indian Statistical Institute. His PhD thesis was widely acclaimed and considered one of the best by Richard W. Cottle, Professor Emeritus at Stanford University. His research interests include Matrix Theory and Analysis, Set-valued mapping, Optimization Theory, Game Theory, Stochastic Process, and Theory of Algorithms. His teaching interests include Optimization Theory, Game Theory, Linear Algebra, Matrix Theory and Analysis. He has more than 30 years of teaching experience at the Indian Statistical Institute, where he has taught full-time courses in the institute's regular degree programs. Dr. Arup Kumar Das teaches Mathematics, Statistics, and Optimization Theory, apart from other courses. Many of his research works are published in several reputed peer-reviewed journals, namely Linear Algebra and its Applications, Linear and Multilinear Algebra, SIAM Journal on Matrix Analysis and Applications, Mathematical Programming series A, Journal of Mathematical Modeling and Algorithms, Journal of Applied Statistics, Journal of Optimization Theory and Applications, Journal of Computational and Applied Mathematics, and Annals of Operations Research.

He is the author of several research monographs. He is a reviewer for the American Mathematical Society**.** He served as guest editor for several book volumes published by Springer, World Scientific, and Elsevier. He was the convener of the PhD & DSc committee in QROR at the institute from 2014 to 2020. He has several academic achievements. Dr. Arup Kumar Das visited Informatics and Mathematical Modeling at the Technical University of Denmark in 2003 and Lichthof, TU Berlin, in 2012. Apart from many keynote addresses and invited talks, he delivered talks at programs held at Dibrugarh University and St. Anthony's College, Shillong. He is a registered member of CSIR as a PhD Supervisor and of SERB and N-PDF as a Post-Doctoral Supervisor. He has been supervising students with mathematics degrees who have INSPIRE, DST/CSIR NET/GATE fellowships for their PhD degrees, as well as post-doctoral students under the N-PDF fellowship. For further details about the author, visit https://sites.google.com/view/abhilashaeshaarupdas/home.





Visit of Professor Robert J. Aumann, Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Israel, at the Indian Statistical Institute, Kolkata, during the Platinum Jubilee Celebration in 2007. Professor Aumann received the Nobel Memorial Prize in Economic Sciences in 2005 for his work on conflict and cooperation through game theory analysis. He shared the prize with Thomas Schelling.



With Professor Richard W. Cottle of Stanford University during the International Symposium on Mathematical Programming in Copenhagen, Denmark, in 2003.



