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7 # Monochromatic Empty Triangles in Two-Colored Point Sets

1 **Indian Statistical Institute, Kolkata**

Dissertation submitted in partial fulfilment of the of the requirements for the degree of

Master of Technology

in

Computer Science

by

Adrish Paik

M.tech CS2302

Under the guidance of

5 **Professor Sandip Das**

Dept. Advanced Computing and Microelectronics Unit(ACMU)

Indian Statistical Institute, Kolkata

June 2025

4

ABSTRACT

A key problem in combinatorial geometry is the identification of properties of a subset of a point set which has properties like convexity, monochromaticity, and emptiness. This thesis actual works on this sides of combinatorial geometry: efficiently finding empty monochromatic triangles in two-colored point sets on the plane. While earlier research focused on counting empty triangles in point set without any color, our work takes an optimal algorithm to explicitly detect them. By using geometric insights and visibility-based techniques. We then address a relaxed but equally compelling variant: monochromatic triangles containing at most one point of the opposite color. For this problem, we derive a quadratic lower bound, revealing how even slight relaxations in geometric constraints dramatically influence lower bound of empty monochromatic triangly in two colored point set.

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Declaration

I, **Adrish Paik**, hereby declare that the dissertation titled **Monochromatic Empty Triangles in Two-Colored Point Sets**, submitted to the Indian Statistical Institute, Kolkata, is a record of original work completed by me as a part of the academic requirements for the **Master of Technology in Computer Science** degree.

This work was carried out under supervision of **Professor Sandip Das**, and has not been submitted elsewhere for the award of any other degree or diploma. I have ensured academic integrity and followed ethical guidelines during this research. Any references to the work of others have been appropriately cited and acknowledged.

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June 11, 2025

Acknowledgements

1 I would like to express my heartfelt gratitude to my research supervisor, **Prof. Sandip Das**, at ISI Kolkata, for offering me the opportunity to pursue this project under his guidance. His insightful feedback, constructive suggestions, and continuous encouragement were invaluable throughout the research process. The conversations and brainstorming sessions with him significantly shaped the outcome of this work.

I sincerely acknowledge the support of **Mr. Saumya Sen** SRF under **Prof. Sandip Das** , who helped me by giving valuable insights and
9 timely help during the course of this project.

Finally, I am equally thankful to the **Indian Statistical Institute, Kolkata**, for facilitating this work by providing the necessary resources and an environment conducive to research.

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CERTIFICATE

I certify that I have read the thesis titled *"Monochromatic Empty Triangles in Two-Colored Point Sets"*, prepared under my guidance by Adrish Paik, and in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Master of Technology in Computer Science of the Indian Statistical Institute.

Prof. Sandip Das

ACMU, ISI Kolkata

June 11, 2025

1 Introduction

A key problem in combinatorial geometry is the identification of properties of a subset of a point set. Here, we discuss properties like convexity, monochromaticity and emptiness for a point set. The main work of Erdős and Szekeres[5] presented that for any integer $k > 3$, there exists a threshold $n(k)$ where any planar point set P in general position with at least $n(k)$ points must contain a k-point convex subset. For decades, mathematicians believed that sufficiently large point sets would necessarily contain an empty convex k-gon—a k-point convex configuration with no other points from P in its interior. While this holds true for $k \leq 5$, in 1983, the construction of the Horton set showed that there exist arbitrarily large point sets with no empty convex heptagons. In general, if a point set has n points, then the number of triangles that can be formed is $\binom{n}{3}$. Thus, in a brute-force approach, an algorithm would run in $O(n^3)$ time, which is quite inefficient for larger values of n.

Consider $MC(n, m, k)$ be the largest number of compatible m-holes that can be found in every k-colored n-set. Where n-set defines a point set V with n points such that no 3-points are collinear.In here we mainly focus on $MC(n, 3, 2)$, in section 5 we find value of $MC(n, 3, 2)$[6].

In section 3 we gave an optimal algorithm of finding all monochromatic empty triangle in two-colored pont set.That actually modification of Dobkin et al.[3] finding empty triangle algorithm.

Lastly in section 6 and 7 we gave an analysis that lower bound of monochromatic triangle with at most one other color point in it's interior,i.e. $AMC_1(n, 3, 2)$ is quadratic in total number points in point set V.

2 Preliminaries

- **Two-colored Point set:** A two-colored point set defines a set of points in a geometric space(in generally in $2D$ or a plane) where each point is assigned one of two colors(red(R) or blue(B)).

  A two-colored point set V is a finite set of points partitioned into two disjoint subsets:

$$S = R \cup B, \quad R \cap B = \emptyset,$$

- **Monochromatic Polygon:** A polygon is said to be monochromatic polygon if and only if all of its vertices are assigned or belongs to a same color. A polygon is a closed geometric shape made with finite number of straight line segments connected end to end.

- **Convex Polygon:** A convex polygon is a polygon where all interior angles are less than $180°$ and every line segment between any two points lies entirely inside the polygon and edge intersection happens only at the endpoints of line segments.

  Smallest convex polygon is a triangle. A triangle is always a convex.

- **Empty Polygon:** A polygon is said to be empty if no other points from point set belong to interior of that polygon.

- **Visibility graph:** Considering a star-shaped polygon P (in general it should not be any particular shape) of n vertices with one vertex p considering as kernel. The vertices of the polygon P are ordered anticlockwise around p numbered as $p_1, p_2, ..., p_{n-1}$ and $p = p_n$. For a pair of vertices of P is said to be visible from the kernel p, if the entire line segment that joins that pair of vertices completely lies inside the polygon P.

All such possible edges and polygon P together known as visibility graph for kernel point p.

3 An Optimal Algorithm For Finding Monochromatic Empty Triangle

Before entering the main algorithm, we have to going through some definitions and some notation which will be useful for describing the algorithm more appropriately.

- **Notations:**

  I. ADD(ij): it creates an edge from i to j. This edge will be stored at both p_i and p_j for later use.

  II. TURN(ij, jk): it returns left or right depending on whether p_k lies to the left or to the right of the directed line passing through p_i and p_j in this order. (Note that it cannot lie on the line.)

  III. FRONT(Q): it returns the index of the first point in queue Q.

  IV. DEQUEUE(Q): it removes the first point from queue Q.

  V. ENQUEUE(k, Q): it adds the point p_k to queue Q.

Now the algorithm works as:

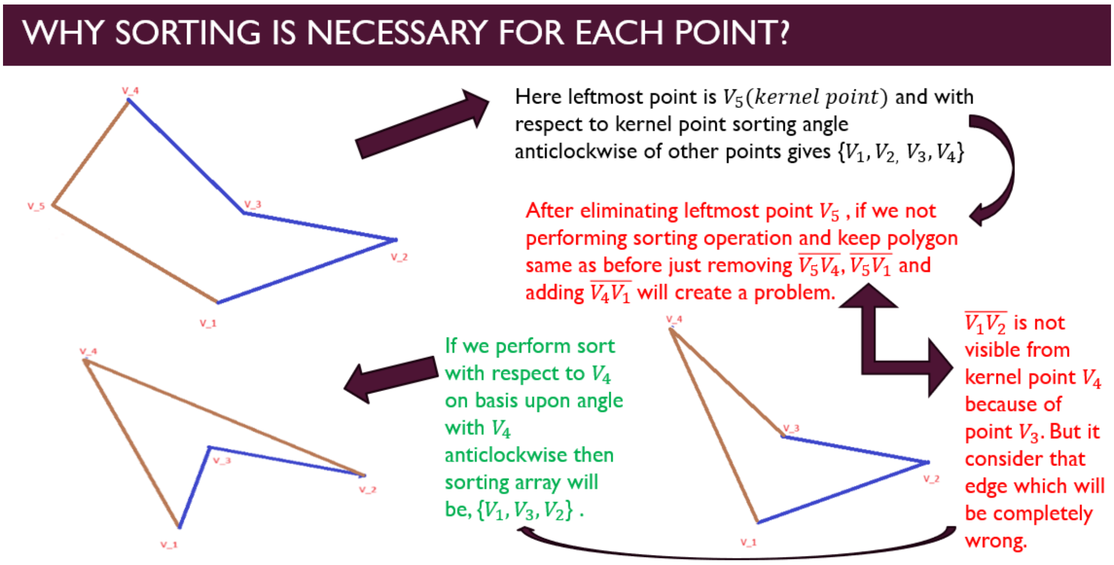
1. For each $p \in V$, sort all other points by angle around p anticlockwise, resulting in an ordered set V_p. From V_p remove all the points to the left of p and add p instead. This results in a star-shaped polygon P_p with p in the kernel.

2. For each $p \in V$, compute the visibility graph VG_p inside P_p, including the edges of P_p, not including the visibility edges involving p.

3. For each $p \in V$, compute all edges in VG_p. Each of these forms, together with p, an empty monochromatic triangle.

For running above algorithm our assumption is no three points are collinear and no two points are on same vertical line. If two points are same vertical line then we can rotate coordinate system in such a way that new coordinate system vertical lines don't have any two points from point set.

Here step-2 is the more important part of the algorithm finding visibility of point p i.e. VG_p. Below we discuss about procedure of visibility graph of any point p.



Algorithm 1 Visibility Graph Construction

Input : Polygon P with n points (p_1 to p_n),
Color assignments for each point,
last_point_color (color of p_n)

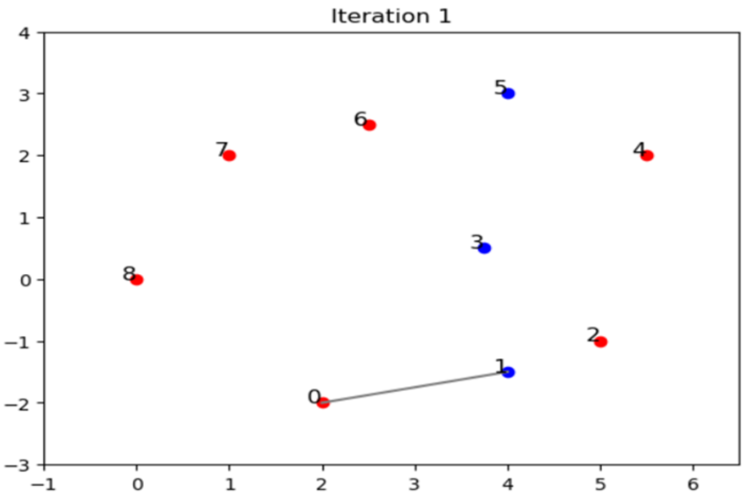
Output: Visibility graph edge set E_P

```
Initialize (n − 1) empty queues Q_1 to Q_{n−1} for i ← 1 to n − 1 do
    Q_i ← ∅ // Initialize queues for each point
end

for i ← 1 to n − 2 do
    Proceed-kernel-colour(i, i + 1) // Process adjacent points
end

return E_P // Return computed visibility edges

Procedure Proceed-kernel-colour(i, j)
    while Q_i ≠ ∅ and TURN(overline(FRONT(Q_i) i), overline(ij)) = left do
        Proceed-kernel-colour(FRONT(Q_i), j)
            tcpRecursive call Dequeue(Q_i) // Remove front element
    end

    if Color(i) = last_point_color or Q_i ≠ ∅ then
        Enqueue(i, Q_j) // Add i to j's queue
        if Color(i) = Color(j) then
            Add edge (i, j) to E_P // Visible points of same color
        end
    end
```

4 Visual representation of Visibility Graph Construction

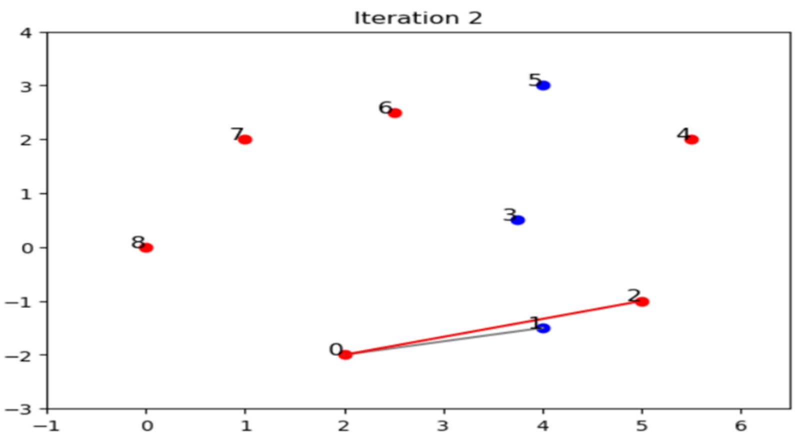
1. First iteration:

```
Iteration 1:
Count of edges 1
Edge Set: [(0, 1)]
Queues: [[], [0], [], [], [], [], [], [], []]
```



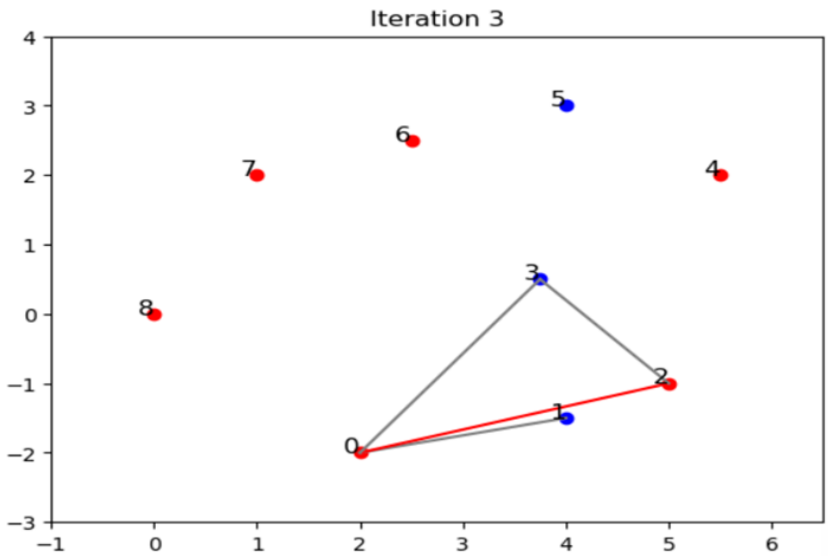
2. Second iteration:

```
Iteration 2:
Count of edges 2
Edge Set: [(0, 1), (0, 2)]
Queues: [[], [], [0], [], [], [], [], [], []]
```



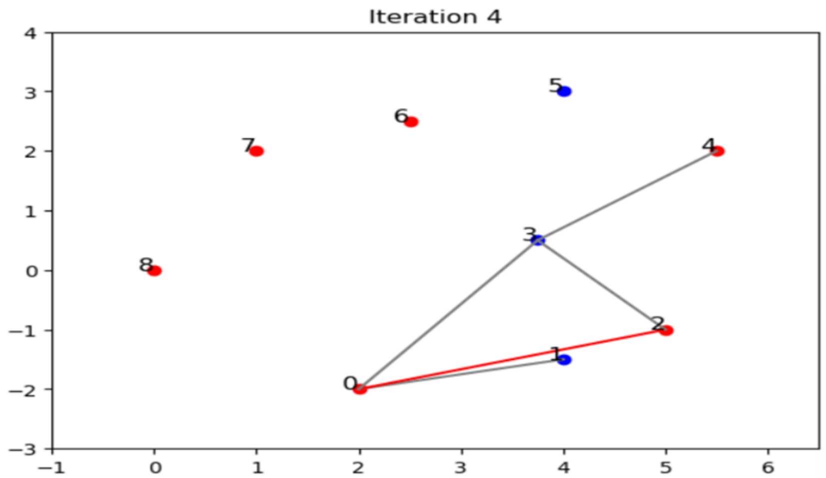
3. Third iteration:

```
Iteration 3:
Count of edges 4
Edge Set: [(0, 1), (0, 2), (0, 3), (2, 3)]
Queues: [[], [], [], [0, 2], [], [], [], [], []]
```



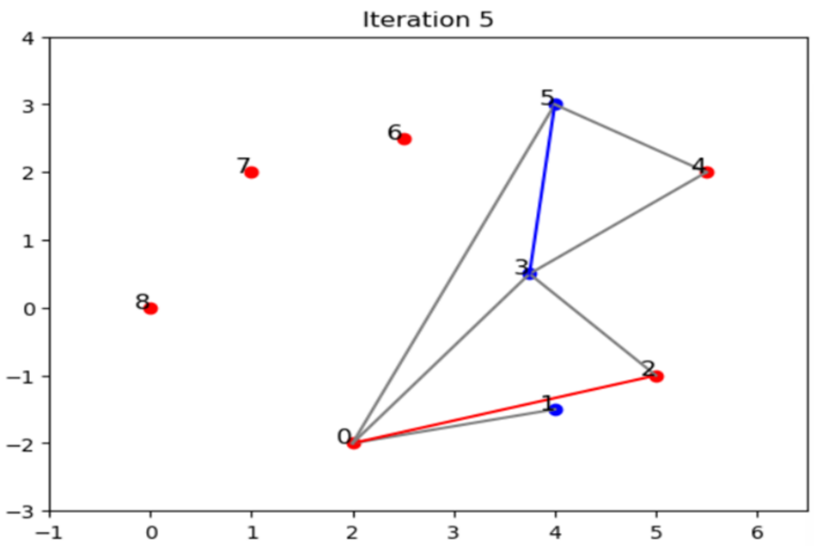
4. Fourth iteration:

```
Iteration 4:
Count of edges 5
Edge Set: [(0, 1), (0, 2), (0, 3), (2, 3), (3, 4)]
Queues: [[], [], [], [0, 2], [3], [], [], [], []]
```



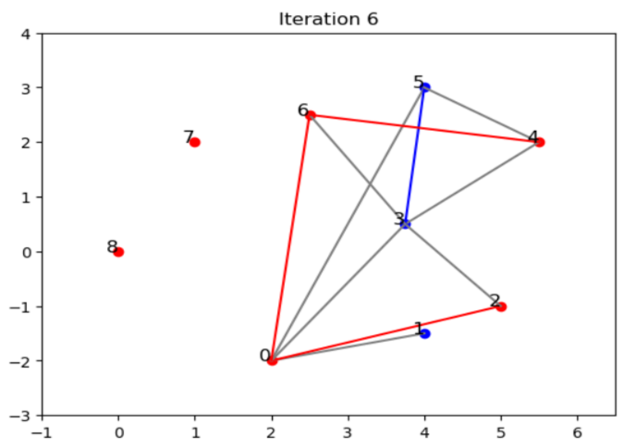
5. Fifth iteration:

```
Iteration 5:
Count of edges 8
Edge Set: [(0, 1), (0, 2), (0, 3), (2, 3), (3, 4), (0, 5), (3, 5), (4, 5)]
Queues: [[], [], [], [2], [], [0, 3, 4], [], [], []]
```

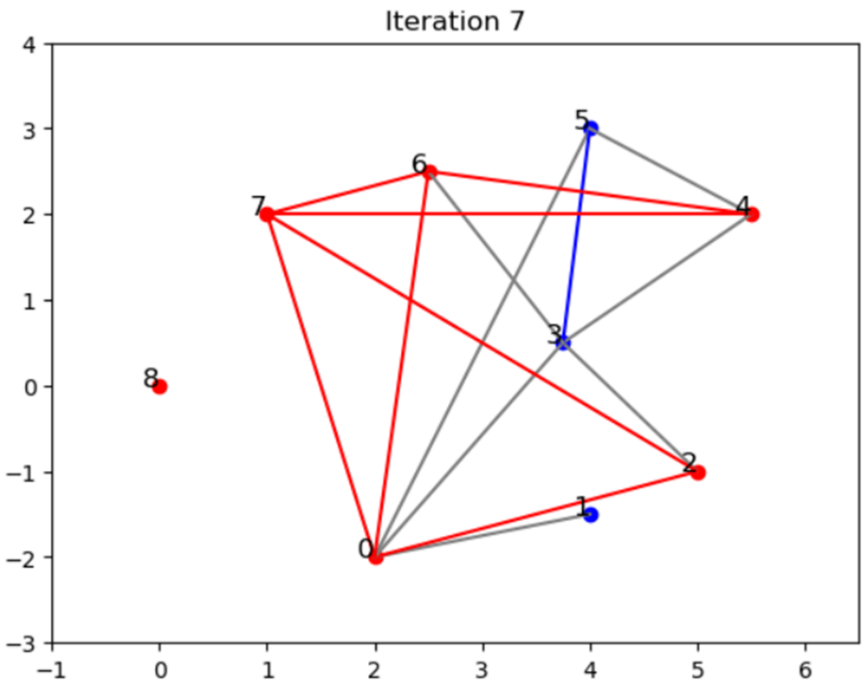


6. Sixth iteration:

```
Iteration 6:
Count of edges 11
Edge Set: [(0, 1), (0, 2), (0, 3), (2, 3), (3, 4), (0, 5), (3, 5), (4, 5), (0, 6), (3, 6), (4, 6)]
Queues: [[], [], [], [2], [], [], [0, 3, 4], [], []]
```



7. **Seventh iteration:**




```
Count of edges 15
Final Edge Set: [(0, 1), (0, 2), (0, 3), (2, 3), (3, 4),
 (0, 5), (3, 5), (4, 5), (0, 6), (3, 6), (4, 6), (0, 7),
 (2, 7), (4, 7), (6, 7)]

Final Queues: [[], [], [], [], [], [], [], [0, 2, 4, 6], []]
Monochromatic edges of color red : [(0, 2), (0, 6), (4, 6),
  (0, 7), (2, 7), (4, 7), (6, 7)]

Count of Monochromatic edges 7
```

Monochromatic means here red
ones only as colour of 8-th vertices
is red.

Final result of the visibility algorithm ends here.

5 Finding value of $MC(n, 3, 2)$

This lower bound actually discussed in János Pach & Géza Tóth[1] paper. Here we try to elaborate on it.

Theorem 1 (Dilworth's Theorem - Chain/Antichain Lower Bound)**.** *Let $(P, \leq)$ be a finite poset with $|P| = m$. Then P contains either:*

(i) a chain of size at least $\lceil \sqrt{m} \rceil$, or

(ii) an antichain of size at least $\lceil \sqrt{m} \rceil$.

Proof. We proceed with a combinatorial argument:

1. **Chain decomposition:** Let C be a maximal chain in P. If $|C| \geq \lceil \sqrt{m} \rceil$, then (i) holds and we are done.

2. **Assume no large chain exists:** Suppose instead that all chains in P have size $< \lceil \sqrt{m} \rceil$. By Dilworth's original theorem, P can be partitioned into k chains, where k equals the size of the largest antichain. Since each chain has size $\leq \lceil \sqrt{m} \rceil - 1$, we have:

$$m \leq k \left(\lceil \sqrt{m} \rceil - 1 \right).$$

3. **Lower bound on antichain size:** Rearranging the inequality gives:

$$k \geq \frac{m}{\lceil \sqrt{m} \rceil - 1} > \sqrt{m}.$$

(The last inequality holds because $\lceil \sqrt{m} \rceil - 1 < \sqrt{m}$.)

Thus, the largest antichain has size $k > \sqrt{m}$, implying $k \geq \lceil \sqrt{m} \rceil$. Hence, (ii) holds.

In either case, the theorem follows. □

Lemma 1 (Order Lemma). *Let $P_1P_2P_3$ be a triangle containing interior points $Q_1, Q_2, \dots, Q_m$. The set $\{P_1, P_2, P_3, Q_1, \dots, Q_m\}$ can be triangulated such that at least $m + \lceil\sqrt{m}\rceil + 2$ triangles include P_1, P_2, or P_3 as a vertex.*

Proof. We define a partial order $\prec$ on $\{Q_1, \dots, Q_m\}$:

$$Q_i \prec Q_j \iff \text{triangle } Q_iP_1P_2 \text{ contains } Q_j.$$

By Dilworth's Theorem, there exists either:

1. A chain $Q_1 \prec \cdots \prec Q_{m'}$ of size $m' = \lceil\sqrt{m}\rceil$, or
2. An antichain $\{Q_1, \dots, Q_{m'}\}$ of size m'.

Case 1: Chain exists. Add edges Q_iQ_{i+1} $(1 \le i < m')$ and Q_iP_1, Q_iP_2 $(1 \le i \le m')$. Triangulate the region near P_1P_2. Connect remaining points $Q_{m'+1}, \dots, Q_m$ to P_1 or P_2 without edge crossings. Include edges P_1P_3 and P_2P_3.

The degrees of P_1 and P_2 sum to $m+m'+4$. The resulting triangulation has at least $m + m' + 2$ triangles incident to P_1 or P_2.

Case 2: Antichain exists. No line Q_iQ_j $(1 \le i < j \le m')$ intersects P_1P_2, so all such lines cross P_1P_3 and P_2P_3. For any $i < j$, either $Q_j \in \triangle P_1P_3Q_i$ or $Q_i \in \triangle P_1P_3Q_j$. Proceed as in Case 1, swapping roles of P_2 and P_3.

In both cases, the lemma holds. □

Lemma 2 (Discrepancy Lemma). *Given a set of n blue points and $n+k$ red points in general position in the plane, there exist at least $\frac{(n+k)(k-2)}{3}$ monochromatic empty triangles.*

Proof. Let P be an arbitrary red point. Label the remaining red points as $P_1, P_2, \ldots, P_{n+k-1} = P_0$, ordered by their visibility from P.

- **Angle Analysis:** For each i, the angle $\angle P_i P P_{i+1}$ is less than π, except possibly for $\angle P_0 P P_1$. This ensures that the interiors of the triangles $P_i P P_{i+1}$ are pairwise disjoint for $i = 1, \ldots, n+k-2$.
- **Empty Triangles:** Since there are only n blue points, at most n of these triangles can contain a blue point. Thus, at least $(n+k-2)-n = k-2$ triangles must be empty and red.
- **Repetition for All Red Points:** Applying this argument to each of the $n+k$ red points yields at least $(n+k)(k-2)$ empty red triangles. Each triangle is counted at most three times (once for each of its vertices), so the total number of distinct empty red triangles is at least $\frac{(n+k)(k-2)}{3}$.

□

Theorem 2. *Any two-colored set of n points in general position in the plane spans at least $cn^{4/3}$ monochromatic empty triangles, where $c > 0$ is an absolute constant.*

Proof. Given a set S of $r(S)$ red and $b(S)$ blue points, define the **discrepancy** of S as:

$$d(S) := |r(S) - b(S)|.$$

Let S be a two-colored set of n points in general position, with $n \geq$

1000. A point $P \in S$ is called **rich** if it is adjacent to at least $\sqrt[3]{n}$ empty monochromatic triangles.

Algorithm FIND-RICH-POINTS (S)

Step 0: Initial Setup
if $d(S) \geq \sqrt[3]{n}/100$ **then**
  Apply Discrepancy Lemma to find $\Omega(n^{4/3})$ triangles. **Stop.**
else
  Assume $d(S) < \sqrt[3]{n}/100$. Then:

$$b(S) > \frac{n}{2} - \frac{\sqrt[3]{n}}{200}, \quad r(S) > \frac{n}{2} - \frac{\sqrt[3]{n}}{200}.$$

  Set $i = 1$ and $S_1 = S$.
end if
Step i: Iterative Process
Induction Hypothesis: For $i > 1$, $b(S_i) = b(S_{i-1}) - 1$, so:

$$b(S_i) > \frac{n}{2} - \frac{\sqrt[3]{n}}{200} - i, \qquad i \leq n/5$$

1. **Convex Hull Filtering:**
  Compute convex hull of S_i.
  Remove red points from hull boundary until only blue remain. Call this S'.
  No blue points removed: $b(S') = b(S_i)$.
2. **Discrepancy Check:**
if $d(S') \geq \sqrt[3]{n}/100$ **then**
  Apply Discrepancy Lemma. **Stop.**
else

$r(S') \geq b(S') - d(S') > \frac{n}{4}$.
end if
3. **Hull Size Check:**
if hull of S' has $m \geq \sqrt[3]{n}/50$ blue points **then**
Remove them to form S''. Then:
$d(S'') \geq \sqrt[3]{n}/100$ and $|S''| \geq r(S') > n/4$.
Apply Discrepancy Lemma. **Stop.**
else
Assume $m \leq \sqrt[3]{n}/50$ blue points on hull.
end if
4. **Triangulation and Subregion Analysis:**
Let $P_1, \ldots, P_m$ be hull points (clockwise).
Triangulate hull with diagonals P_iP_j, forming triangles $T_1, \ldots, T_{m-2}$.

For each T_j, let b_j (blue) and r_j (red) be interior points.
if $\exists j$ with $|b_j - r_j| > \sqrt[3]{n}/50$ **then**
Case 1: Apply Discrepancy Lemma to subregion with $\geq n/6$ points. **Stop.**
else
Case 2: $\exists T_j$ with $b_j \geq \frac{n}{4m} \geq \frac{50n^{2/3}}{4}$.
By Order Lemma, triangulate blue points in T_j:At least $\sqrt[3]{n}$ empty triangles share a vertex P. Mark P as **rich**.
end if
5. **Termination or Iteration:**
if $i \geq n/5$ **then**
Stop.
else
Set $S_{i+1} = S_i \setminus \{P\}$, $i := i + 1$, repeat Step i.
end if

Conclusion

- **Early Stop:** Discrepancy Lemma yields $\Omega(n^{4/3})$ triangles.
- **Final Stop:** More than $n/5$ rich points generate $\geq \frac{n}{5} \cdot \sqrt[3]{n} = \Omega(n^{4/3})$ empty monochromatic triangles.

Thus we can conclude that $MC(n, 3, 2) = cn^{\frac{4}{3}}$, more specifically the constant value will be $c = \frac{1}{15}$.

Thus we can implies that $MC(n, 3, 2) = cn^{\frac{4}{3}}$, more specifically constant value will be $c = \frac{1}{15}$.

$\square$

Let A and B be two disjoint color classes of a point set S in the plane such that $|A| = |B| = \frac{n}{2}$, assuming n is even. The total number of triangles that can be formed from n points is $\binom{n}{3}$. Thus the minimum number of monochromatic triangles occurs will be,

$$2 \cdot \binom{n/2}{3} = \frac{n(n-2)(n-4)}{24} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}.$$

To find bound the number of empty triangle now independent of color,we have to find upper bound of non-monochromatic triangle. If such example or point distribution find that satisfy the claim that it gives maximum number of non-empty triangle than subtracting it from $\frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}$ wil gave perfect lower bound of monochromatic empty triangle in two-colored point set.[7]

6 Construction of $AMC_1(n, 3, 2)$

We analyze the distribution of blue points across disjoint wedges formed by red points(see Figure 6.1). Let n be the total number of points, with $\frac{n}{2}$ red and $\frac{n}{2}$ blue (for even n). For odd n, consider red points are more than blue points if not we can swap the color of points.In worst case points are equally divide,so in this portion we are more focus on worst case scenario.

- **(A)** Consider a red point(p),with respect to the point p we can draw wedges to the remaining $\frac{n}{2} - 1$ red points.
- **(B)** Number of blue points can be maximum $\frac{n}{2}$.
- Let us assume,
  - α: Number of empty red wedges.
  - β: Number of red wedges exactly one blue point inside.
  - γ: Number of wedges with two or more blue points inside.

Now if number of red points are more than or equal to blue point, then we can derive following inequality,

1. The total number of red wedges (considering $|V| = n$ and $|R| \geq |B|$) :
$$\alpha + \beta + \gamma \geq \frac{n}{2} - 1 \quad \text{(from A)}$$
2. The total number of blue points distributed across triangles is at most $\frac{n}{2}$:
$$0 \cdot \alpha + 1 \cdot \beta + 2 \cdot \gamma \leq \frac{n}{2} \quad \text{(from B)}$$

$$\implies \beta + 2 \cdot \gamma \leq \frac{n}{2}$$

3. By comparing the number of triangles with the number of blue points they contain:

$$\beta + 2\gamma \leq \alpha + \beta + \gamma + 1 \implies \gamma \leq \alpha + 1$$

From the first inequality,substituting $\gamma \leq \alpha + 1$ we get:

$$\alpha + \beta + \alpha + 1 \geq \frac{n}{2} - 1 \implies 2\alpha + \beta \geq \frac{n}{2} - 2$$

Adding both sides by $\beta \geq 0$ we get,

$$\text{implies, } 2\alpha + 2\beta \geq \frac{n}{2} - 2$$

$$\text{implies, } \alpha + \beta \geq \frac{n}{4} - 1$$

For odd n, let $n = 2k + 1$. The number of red points becomes $k + 1$ means, wedges becomes k, and the number of blue points is at most k. The same logic applies:

$$\alpha + \beta \geq \frac{k}{2} \approx \frac{n-1}{4} \geq \frac{n}{4} - 1$$

Conclusion:

In all cases (odd or even n), the lower bound holds:

$$\boxed{\alpha + \beta \geq \frac{n}{4} - 1 \quad \text{(Inequality 1)}}$$

This means that at least $\frac{n}{4} - 1$ triangles(red one) contain at most one point(blue) with respect to a kernel point p. In other words, we can also say that the number of monochromatic (more specific red) at most one point(blue) inside its interior with respect to any point(red) p is at least $\frac{R}{2} - 1$ where $|R| \geq |B|$ in any two colored point set. □

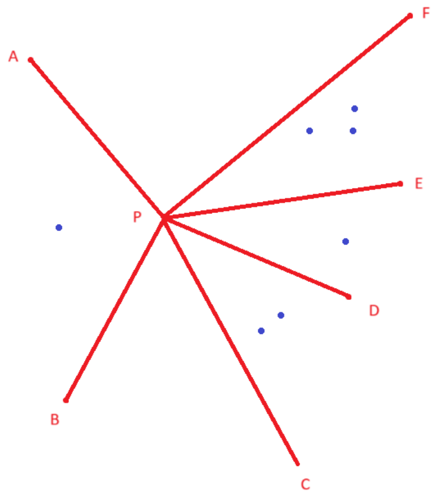


Figure 6.1: Visual representation of wedges with kernel point P

Here red points are **A,B,C,D,E,F** and kernel point is **P**.

Total number of wedges are **APB, APC, APD, APE, APF, BPC, BPD, BPE, BPF, CPD, CPE, CPF, DPE, DPF, EPF** and disjoint wedges are **APB, BPC, CPD, DPE, EPF, FPA**.

1. **BPC, FPA** are empty red wedges.
2. **APB, DPE** are red wedges with exactly one **blue point** in it's interior.
3. **CPD, EPF** are red wedges with more than one **blue point** in it's interior.

All wedges (ones side, i.e. less than 180 degree) are form a triangle as no three points are colinear.Total number of points are **7** + **7** = 14 and number of red wedges are 14/2 - 1 = 7.

7 Finding value of $AMC_1(n, 3, 2)$

Theorem 3. *Any two-colored point set with n points in general position in the plane spans at least cn^2 monochromatic empty triangles with at most one points in it's interrior, where $c > 0$ is an absolute constant.*

Proof. Consider total number of points be $2n + k$.

Input: Consider $|R| = n + k$ (number of red points)

$|B| = n$ (number of blue points) and $i = 0$.

Step 1: Draw convex hull of point set V.

Step 2: Check which colour has maximum points in point set. WLOG consider $|R| \geq |B|$. (If not, swap the points color.)

Step 3: While all points on convex hull are blue:

Remove all blue points on $CH(V)$ from V,where $CH(V)$

denotes the convex hull points of V.

make $V = V - CH(V)$

Step 4: Take any red point p from $CH(V)$ and set $i = i + 1$. From point p, from point p, draw wedges to the remaining $(n+k-i)$ red points.

Step 5: If $i \leq n + k - 2$, go to **Step 1**.

Otherwise, terminate (**STOP**).

Few observance of above algorithm

- Algorithm always runs upon on red points,if anyhow blue points are more than red points than we swap the colors of points. This actually doesn't gave any impact on counting.As in monochromacity we are focused on all points in polygon should to be same. Which colour it represents in general is not important.
- We stop on $i \leq n + k - 2$ as for a kernel point like p we need atleast two other points to form a triangle.
- From the last conclusion we can easily said that on each iteration we got $\frac{n+k-i}{2}$ number of monochromatic triangles.

Therefore total number of monochromatic triangle will be,

$$\alpha + \beta = \sum_{i=0}^{(n+k-2)} (\alpha_i + \beta_i)$$

$$= \sum_{i=0}^{(n+k-2)} \frac{(n + k - i)}{2} \quad \text{(From Inequality 1)}$$

$$= \frac{1}{2}(n + k - 2)(n + k) +$$

$$\frac{(n + k - 2)(n + k - 1)}{4}$$

$$= \frac{n^2}{4} + O(n)$$

This algorithm completes our proof of the lower bound of the monochromatic triangle with at most one different color point in its interrior is $O(n^2)$, i.e. $AMC_1(2n + k, 3, 2) = c_1 n^2$.Thus, $AMC_1(n, 3, 2) = cn^2$. □

The time complexity for drawing the convex hull is $O(n \log n)$. In worst case we should drawing this convex hull for $n + k - 2$ times. Thus overall time for above algorithm will be $O(n^2 \log n)$.

8 Future Direction

1. Betterment of the lower bound of the empty monochromatic triangle, that is, $O(n^{\frac{4}{3}})$ in the two-colored point set.

2. Giving an output sensitive algorithm for finding monochromatic triangle in two-colored point set.

3. Finding lower bound monochromatic empty triangle in three colored point set with atmost one point inside.

4. Finding lower bound of monochromatic quadrilateral in two colored point set.

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9 REFERENCES

1. János Pach, Géza Tóth, Monochromatic empty triangles in two-colored point sets
2. O. Aichholzer, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, J. Urrutia, Empty monochromatic triangles.
3. David P. Dobkin, Herbert Edelsbrunner, and Mark H. Overmars, Searching For Empty Convex polygon.
4. Oswin Aichholzer, Ruy Fabila-Monroyb, David Flores-Peñaloza, Thomas Hackl,Clemens Huemer, Jorge Urrutiab,Empty monochromatic triangles.
5. P. Erdős, G. Szekeres, A combinatorial problem in geometry, Compos. Math.
6. Olivier Devillers, Ferran Hurtado, Gyula Károlyi, Carlos Seara , Chromatic variants of the Erdős–Szekeres theorem on points in convex position.
7. Robert Brady,William Gasarch, A Pedagogical Approach to Ramsey Multiplicity(page 5)
8. Bhaswar B.Bhattacharyaa, Sandip Das, Sk Samim Islam, Saumya Sen, Growth rates of the number of empty triangles and simplices.