

**MODULAR FORMS**  
**MMATH II YEAR**

26 NOVEMBER 2025  
DURATION 3 HOURS  
FINAL EXAM 2025  
MAX MARKS: 90  
ANSWER ALL QUESTIONS

**You are allowed one page (both sides) cheat sheet**

1. Let  $\eta(z)$  denote the Dedekind eta function. For  $q = 2, 3, 5, 7, 13$  let  $r = 24/(q - 1)$  and define

$$\Phi(z) = \left( \frac{\eta(qz)}{\eta(z)} \right)^r.$$

(a) State Dedekind's functional equation.

(b) Assuming Dedekind's functional equation and required properties of the Dedekind sum, show that  $\Phi(z)$  is a modular function for  $\Gamma_0(q)$ .

(c) Show that  $\Phi$  has a simple zero at infinity.

(5+ 18 + 7 = 30 marks)

2. Let  $\Gamma = SL_2(\mathbb{Z})$ . Let  $f \in M_{2k}(\Gamma)$  and  $T(m)$  be the  $m$ -th Hecke operator. Suppose the Fourier expansions are given by

$$f(z) = \sum_{n=0}^{\infty} a(n)e^{2\pi inz}, \quad \text{and} \quad (T(m)f)(z) = \sum_{n=0}^{\infty} A(n)e^{2\pi inz}$$

(a) Show that

$$A(n) = \sum_{\substack{d|(m,n) \\ d \geq 1}} d^{2k-1} a\left(\frac{mn}{d^2}\right).$$

(b) Let  $\tau(n)$  denote the Ramanujan tau function. Show that for any prime  $p$ , either  $\tau(p) \neq 0$  or  $\tau(p^2) \neq 0$ .

(10+10=20 marks)

3. Let  $T_k(m)$  be the  $m$ -th Hecke operator and for any prime  $q|N$  with  $Q = q^\ell || N$ , let  $w_Q$  be the Atkin-Lehner operator on  $S_k(\Gamma_0(N))$ .

(a) For  $(m, N) = 1$ , show that

$$T_k(m)w_Q = w_Q T_k(m).$$

(b) For  $(m, N) \neq 1$ , show that  $T_k(m)$  may not be self adjoint.

(c) Suppose  $q^2|N$ , then show that  $T_k(q)f \in S_k^{\text{old}}(\Gamma_0(N))$  for any  $f \in S_k(\Gamma_0(N))$ .

(10+10+10=30 marks)

4. Let

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz) \in S_k(\Gamma)$$

be a Hecke form for the full modular group.

(a) Show that

$$g_1(z) := \sum_{n=1}^{\infty} a_f(6n)e(nz)$$

is a modular form of some level.

(b) Show that

$$g_2(z) := \sum_{\substack{n=1 \\ (n,6)=1}}^{\infty} a_f(n)e(nz)$$

is a modular form of level 36.

(10+10=20 marks)