INDIAN STATISTICAL INSTITUTE

B.Stat. III Year: 2025-26 (Semester I)

Parametric Inference: Mid-Semester Exam

Date: 11/09/2025 | Time: 2:30 PM - 4:30 PM | Full Marks: 30

Note: Show necessary steps to solve all of the following problems. There is no credit for a solution if the appropriate work is not shown even if the answer is correct. Here, $n > 2$ is a natural number. The total marks allotted in this question paper is 33. However, the final marks awarded shall be $\min\{x, 30\}$, where x denotes the marks actually obtained out of 33.

1. Suppose $(X_1, X_2) \sim N_2(\mu, \mu, 1, 4, 0.5)$ where the bivariate normal density of $N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ is given by:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right\}.$$

Determine with rigorous argument whether the following statements are true or false.

(a) X_1 is not a sufficient statistic for μ. [3]

(b) $\operatorname{cov}(X_1, \sin X_1 \cos X_2) \neq \operatorname{cov}(X_1, \cos X_1 \sin X_2)$. [3]

2. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. random variables from a distribution having density

$$f(x|\theta) = \frac{1}{2}e^{-|x-\theta|}, \text{ where } x \in \mathbb{R} \text{ and } \theta \in \mathbb{R}.$$

(a) Find a minimal sufficient statistic for θ. [3]

(b) Is the statistic $S = (X_1/X_2,\ X_2/X_3,\ X_3)$ complete? Justify. [2]

3. Let $S_\theta = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, x+y < \theta\}$ where $\theta > 0$. Suppose $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are independent observations from the uniform distribution on S_θ.

(a) Derive a complete sufficient statistic T for θ. [4]

(b) What is the uniformly minimum variance unbiased estimator δ_n of $\psi(\theta) = \ln\theta$? [2]

(c) Show that

$$\mathbb{E}\left[(X_1 + Y_1) \,\Big|\, \max_{1\le i\le n}(X_i + Y_i)\right] = \frac{2n+1}{3n} \max_{1\le i\le n}(X_i + Y_i).$$

[3]

4. Let $X_1, X_2, \ldots, X_n$ be independent samples from a continuous distribution with density

$$f(x|\alpha,\beta) = \alpha\, \beta^\alpha\, x^{-(\alpha+1)} \exp\left[-\left(\frac{\beta}{x}\right)^\alpha\right], \text{ where } x > 0, \text{ and } \alpha > 0,\ \beta > 0.$$

(a) Show that $\{f(x|\alpha,1) : \alpha > 0\}$ does not belong to an one-parameter exponential family. [3]

(b) Prove or disprove whether $\sum_{i=1}^n X_i^{-2}$ is independent of $\dfrac{\log X_1/X_3}{\log X_1/X_2}$. [4]

(c) Assuming α is known in $f(x|\alpha,\beta)$, derive the maximum likelihood estimator $\widehat{\beta}$ of β. [2]

(d) Suppose $\alpha = 1$ in part (c) and $\psi(\beta) := \mathbb{E}_\beta[\widehat{\beta}]$. Is $\widehat{\beta}$ an *asymptotically most efficient* estimator of $\psi(\beta)$ with respect to the Crame-Rao lower bound for the unbiased estimators of $\psi(\beta)$? Justify your answer. [4]