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ON GENERALIZED CONSISTENCY IN A ONE SECTOR MODEL OF OPTIMAL GROWTH*

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1. INTRODUCTION

In the standard treatment of optimum growth theory, the planner (or the economy) is endowed with a preference structure which extends over the whole future (or, at least a finite number of time periods to follow). Moreover, the preference function is given once for all and assumed not to change with the passage of time. For example, at the beginning of time period 1, the planner's welfare function may take the form $\sum_{t=1}^{\infty} \alpha^{t-1}u(c_t)$, where c_t is the per capita consumption in period t, $u(\cdot)$ is a time invariant per period utility function, usually assumed to be strictly concave and increasing and $0 < \alpha < 1$ is a disconut parameter. For a given technology and initial resource restriction, maximisation of the above welfare function (when such a maximum exists) will result in the choice of an optimum sequence of consumption labour ratios $(c_1^*, c_2^*, \ldots)$. It follows from an elementary application of Bellman's principle of optimality that while $(c_1^*, c_2^*, \ldots)$ maximises $\sum_{t=1}^{\infty} \alpha^{t-1}u(c_t)$, $(c_2^*, c_3^*, \ldots)$ maximises $\sum_{t=2}^{\infty} \alpha^{t-1}u(c_t)$ and in general, $(c_s^*, c_{s+1}^*, \ldots)$ maximises $\sum_{t=s}^{\infty} \alpha^{t-1}u(c_t)$. Thus, if the planner's welfare function does not change over time and at some period i, $1 < i$, the planner decides to reevaluate his plan for the remaining periods in his horizon, he would come out with a choice of $(c_i^*, c_{i+1}^*, \ldots)$ which is identical with the corresponding portion of his original plan, provided in the beginning of period i, he has the same capital stock as the i-th period optimal capital stock for the original plan. This was first observed by Strotz (1956) who showed that so long as the planner employs a geometrically declining sequence of discount parameters, all optimal plans corresponding to a summable welfare function satisfy this property. Strotz described the

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planner's behaviour under these conditions as *consistent*, since successive reevaluations do not change the planner's original decisions.

Clearly, the Strotz kind of consistency will not be observed when the planner's preference structure changes over time. A simple example of such changing preferences is obtained when in period i, the planner has a welfare function $\sum_{t=i}^{\infty} \alpha^{t-i} u_i(c_t)$ and $u_i(\cdot)$ varies with i. Under such a changing preference structure, in general the planner's decisions at any moment of time on the basis of the current preference function will be discarded when reevaluated on the basis of a future preference function. The asymptotic behaviour of the economy, when plans change on successive reevaluations was studied by Goldman (1968) for a special class of preference functions.

It seems almost certain, however, that when plans do get discarded over time, the planner should become aware of his own apparently paradoxical behaviour, due to which he never carries out his own decisions. One might ask why the planner does not consider his future disobedience as a constraint in his original maximization problem. If such additional constraints are introduced into the problem, the planner would be facing a *second best* version of his original plan.

Thus, on the basis of his past behaviour, the planner may try to forecast a probable pattern of behaviour regarding the future. In general, the planner's decision at the present moment will be a function of his expectations about the future. For example, under the usual kind of planning the planner will lay down a vector of optimum consumption-labour ratios $(c_1^*, c_2^*, \ldots)$. However, if the welfare function is changing over time, the planner would be aware of the fact that $(c_2^*, c_3^*, \ldots)$ will not be followed during periods $(2, 3, \ldots)$. Instead, depending on his past experience, he may believe that the actual magnitudes to prevail will be $(c_2^{*'}, \ldots)$. If the planner considers this as an additional constraint, his consumption-labour ratio choice for the first period will be $c_1^{*'}$, which, in general, would not be the same as c_1^*.

There is no reason, of course, for the planner's expectations regarding the future to actually materialize. However, it is still worthwhile to specify the circumstances under which the planner's expectations will indeed turn out to be true. This is precisely the purpose of this paper. In particular, we are interested in demonstrating the existence of an infinite sequence $\{c_i^*\}_{i=1}^{\infty}$ of consumption labour ratios which satisfies the property that the planner at point of time θ, knowing that $(c_1^*, \ldots, c_{\theta-1}^*)$ prevailed in the past and expecting that $(c_{\theta+1}^*, c_{\theta+2}^*, \ldots)$ will prevail in the future, actually chooses c_θ^* as his optimum consumption-labour ratio. If this is true for every θ, then the sequence $\{c_i^*\}_{i=1}^{\infty}$ may be said to satisfy a very general consistency requirement. The situation may be described as one involving consistency, because the planner, faced with the same pattern of expectations in every period and acting as a conscious maximizer, chooses a plan which is identically the same as his expected plan.

In the next section the problem is posed in its general form and Section 3 proves the main results. Finally, in Section 4 we make certain concluding remarks.

2. The general problem

In this section we pose the problem in its full generality and make a few remarks regarding the literature existing on the subject. We first turn to the existing literature.

The solitary reference on this problem seems to be an article by Phelps and Pollak (1968). According to their view, the process of intertemporal planning is not undertaken by a single planner. In general, one may visualize a sequence of planners each situated at a different point of time. For future reference, this sequence of planners may be denoted by $\{P_t\}_{t=1}^{\infty}$, where P_t is the planner situated at point of time t. Phelps and Pollak assume each planner P_θ to be endowed with a welfare function of the form

$$u(c_\theta)+\delta \sum_{t=\theta+1}^{\infty} \alpha^{t-\theta}u(c_t). \qquad \ldots \quad (2.1)$$

It may be noted that in (2.1), any two adjacent periods t, $t+1$, $t \geqslant \theta+1$, satisfy the property that the utility from consumption in period $t+1$ receives α times the weight received by the utility from consumption in period t. This tells us that although the planner is myopic to a certain degree, this does not result in any preferential treatment of two adjacent periods in the future. However, when it comes to evaluating the utility in the first period, which is presumably the period about which the planner is most concerned, there emerges a preferential treatment of today against tomorrow. This is reflected in the appearance of the parameter δ in the welfare function. In the case where $\delta < 1$, this implies that the planner puts a disproportionately high weight on the utility in the first period.

Since each planner is assumed to have the same kind of welfare function, we have, once again, the case of changing preferences over time. The planner P_θ will thus arrive at a consumption plan for all future on the basis of (2.1) which would be immediately discarded by $P_{\theta+1}$, because, on the basis of the preference function of $P_{\theta+1}$, the utility in period $\theta+1$ received too low a weight in the plan proposed by P_θ. If, however, P_θ, is aware of such future disobedience, he may try to formulate a second best plan of the nature proposed in the last section.

In the Phelps and Pollak model, an optimal path for each planner is described in terms of a sequence of savings ratios. They deal with a model for which the production side involves a linear technology and the preference side has a utility function which displays constancy of elasticity of marginal utility. If the planner's prescriptions are not likely to be followed in the future, he may try to formulate a pattern of expectation regarding a rate of savings to be followed in all future. Given such a savings rate, the planner will work out a second best savings rate for the current period, which in general will be different from the first best rate of savings. For any expectation s regarding the future savings behaviour, one can then define a function $\phi(s)$, which yield the savings ratio of the first period.

The authors then go on to show that under the assumptions made on the model, $\phi(s)$ will have a fixed point. Thus, if s^* is a fixed point of $\phi(s)$, then it will have the property that when each planner expects s^* to be followed in the future, he finds it his best strategy to follow s^* also. The savings ratio s^* then provides an equilibrium solution for the model in the sense that it involves generalized consistency and as Phelps and Pollak rightly point out, it resembles the solution to a non-cooperative game as visualised by Nash (1951).

A non-cooperative game usually deals with a given number of players, each with well-defined pay-off functions. Each player also has a strategy set from which he chooses his decision variable. Moreover, the pay-off function of each player is affected by the choice of strategies made by each other player in the game. An equilibrium solution to such a game then refers to a choice of strategies, one by each player, which has the property that the strategy chosen by each player is the best possible one against the best possible strategy of every other player. In the Phelps and Pollak model, there is an infinite number of players $\{P_t\}_{t=1}^{\infty}$, each with pay-off functions given by (2.1). The strategy of P_θ, viz. the savings ratio s_θ chosen in period θ, is affected by what was chosen by $(P_1, P_2, ..., P_{\theta-1})$ and what he expects to be chosen by $(P_{\theta+1}, P_{\theta+2}, ...)$. The players $(P_1, ..., P_{\theta-1})$ enter the scene because their choices affect the initial capital-labour ratio for P_θ. Thus, a sequence of savings ratios satisfying consistency of the generalized kind structurally resembles the solution to a non-cooperative game involving an infinite number of players.

In what follows, we try to generalize the results obtained by Phelps and Pollak. In the Phelps and Pollak treatment, the welfare function of planner $P_{\theta+1}$ is simply the welfare function of planner P_θ shifted one period into the future. Clearly, under the circumstance the $\phi(\cdot)$ function defined above will remain stationary with respect to time. In a more general situation the welfare functions of planners may keep changing in a quite arbitrary manner. In such a situation, $\phi(\cdot)$ would change over time and the simple Phelps and Pollak treatment of the problem in terms of establishing a fixed point for the $\phi(\cdot)$ function will not be applicable. However, this is precisely the problem we intend to sudy in this paper. The situation we have in mind involes viewing each P_t as endowed with some welfare function $w_t(c_t, c_{t+1}, ...)$. The particular assumptions we make regarding $w_t(\cdot)$ will be discussed later. On the production side we assume a neo-classical technology. This is a departure from the Phelps and Pollak set-up of a linear technology. In general, then, given any expected sequence $(s_{\theta+1}, s_{\theta+2}, ...)$ for the future and knowing that $(s_1, ..., s_{\theta-1})$ was followed in the past, P_θ will try to find his best choice s_θ as a function of these variables. It may be noted, that we are not necessarily restricting attention to a constant savings ratio as was assumed by Phelps and Pollak. Our task is to postulate proper assumptions on the model which will allow us to show the existence of a sequence $(s_1^*, s_2^*, ..., s_\theta^*, ...)$ which has the property that if P_θ expects $(s^*_{\theta+1}, s^*_{\theta+2}, \ldots)$ to be followed in the future and knows that $(s^*_1, s^*_2, \ldots, s^*_{\theta-1})$ was followed in the past, then his best choice would be to follow s^*_θ.

Each P_t may be assumed to have a strategy set S_t for his savings ratio. Moreover, on the basis of $w_t(\cdot)$ and the technology, each would have a choice function $\phi_t(s_1, \ldots, s_{t-1}, s_{t+1}, \ldots)$ such that $s_t = \phi_t(s_1, \ldots, s_{t-1}, s_{t+1} \ldots)$. In general, since $w_t(\cdot) \neq w_{t'}(.)$, $t \neq t'$, we can say that $\phi_t(\cdot) \neq \phi_{t'}(\cdot)$. Consider the function

$$Q(s_1, s_2, \ldots) = \phi_1(s_2, s_3, \ldots) \times \phi_2(s_1, s_3, \ldots) \times \ldots \times \phi_t(s_1, s_2, \ldots, s_{t-1}, s_{t+1}, \ldots) \times \ldots$$

Obviously then, $Q(.)$ maps $S_1 \times S_2 \times \ldots$ into itself. Our purpose is to show that $Q(\cdot)$ has a fixed point $(s^*_1, s^*_2, \ldots)$. Such a fixed point will form a non-cooperative equilibrium for the model in question.

Since $S_1 \times S_2 \times \ldots$ is an infinite dimensional space, the usual theorems available for finite dimensional spaces will not be useful for our purpose. There are, however, several fixed point theorems available for infinite dimensional spaces also and the particular one we choose is the Schauder-Tychonoff fixed point theorem (Dunford and Schwartz, 1958). The statement of this theorem goes as follows :

A continuous mapping of a convex compact subset of a locally convex linear topological space into itself has a fixed point.

In the next section we describe a specific model which will satisfy the conditions of the Schauder-Tychonoff fixed point theorem.

3. The existence theorem

For the purpose of this section, the welfare function of P_θ will be assumed to take the form

$$w_\theta(c_\theta, c_{\theta+1}, \ldots) = \sum_{t=\theta}^{\infty} \alpha_{\theta t} u_{\theta t}(c_t) \qquad \ldots \quad (3.1)$$

where, the $u_{\theta t}(c_t)$'s have the same interpretation as the $u_\theta(c_t)$'s dealt with so far. The assumptions made with respect to $u_{\theta t}(c_t)$ for each θ are stated below.

Assumption I :

(a) $u'_{\theta t}(c_t) > 0, \quad u''_{\theta t}(c_t) < 0;$

(b) For each θ there exists $b_\theta > -\infty$ such that $u_{\theta t}(c_t) > b_\theta$ for all t;

(c) There exists $u_\theta(c_t)$ satisfying properties (a) and (b) such that $u_{\theta t}(c_t) < u_\theta(c_t)$ for all t.

The sequence of discount parameters $\{\alpha_{\theta t}\}_{t=\theta}^{\infty}$ employed by P_θ are assumed to satisfy.

Assumption II :

(a) $\alpha_{\theta\theta} = 1 \quad \forall \theta$

(b) $0 < \frac{\alpha_{\theta, t+1}}{\alpha_{\theta t}} \leqslant \alpha < 1$ for some α and all $t \geqslant \theta$.

Assumption II(b), in conjunction with II(a), implies $(\alpha_{\theta t})_{t=1}^{\infty}$ is a monotone decreasing sequence with respect to t for each θ and that it decreases at the geometric rate, or, possibly even faster.

Production is assumed to be constrained by a one sector neoclassical technology

$$C_t+(K_{t+1}-K_t) = Y_t = F_t(K_t, L_t) \qquad \dots \quad (3.2)$$

where, $F_t(\cdot)$ is a linear homogeneous function of its arguments; Y_t is the level of output in period t; C_t is the level of consumption in period t; K_t is the level of capital and L_t the labour force in existence in period t. With labour growing at a constant exogenously given rate, i.e.

$$\frac{L_{t+1}-L_t}{L_t} = n$$

and under the assumption of full employment, (3.2) reduces to

$$c_t = f_t(k_t)+k_t-(1+n)k_{t+1} \qquad \dots \quad (3.3)$$

where $f_t(k_t) = F_t\left(\frac{K_t}{L_t}, 1\right)$ and the small case letters denote the per capita magnitudes. The function $f_t(\cdot)$ is assumed to satisfy :

Assumption III :

(a) $f_t'(\cdot) > 0, f_t''(\cdot) < 0$;

(b) $f_t(0) = 0$;

(c) $f_t'(k_t) \to +\infty$ as $k_t \to 0$; $f_t'(k_t) \to 0$ as $k_t \to +\infty$.

Thus, we are assuming technological progress of a fairly general nature. However, the following assumption will be made. Let $\bar{k}_t$ be the uniform bound imposed on the capital-labour ratio by $f_t(\cdot)$. (Under neoclassical assumptions and constant rate of growth of the labour force, such a $\bar{k}_t < +\infty$ will always exist for each $f_t(\cdot)$).

Assumption IV :

There exists $\bar{k}$ such that $\bar{k}_t < \bar{k} \;\forall\; t$, where $\bar{k} < +\infty$.

Clearly, Assumption IV along with Assumption I(b) and I(c) and Assumption II guarantees the boundedness of the welfare function (3.1).

If s_t is the savings ratio chosen by P_t, then (3.3) may be equivalently expressed by the following pair of equations.

$$c_t = (1-s_t)f_t(k_t) \qquad \dots \quad (3.4)$$

$$(1+n)k_{t+1}-k_t = s_tf_t(k_t). \qquad \dots \quad (3.5)$$

For any P_t, s_t is obviously bounded above by 1. As far as a lower bound on s_t is concerned several possibilities may be suggested. First of all, one may postulate $s_t \geqslant 0$. In this case, as is quite obvious from (3.5), investment would be irreversible. However, if we want to consider the possibility of reversible investment,[1] then the lower bound on s_t will in general depend on k_t. This can be seen as follows. The maximum amount of consumption possible corresponding to any K_t and L_t is

$$C_t = F_t(K_t, L_t)+K_t.$$

However, $$C_t = (1-s_t)F_t(K_t, L_t).$$

Therefore, $$F_t(K_t, L_t)+K_t = (1-s_t)F_t(K_t, L_t).$$

or, $$K_t = -s_tF_t(K_t, L_t)$$

or, $$s_t = -\frac{K_t}{F_t(K_t, L_t)} = -\frac{k_t}{f_t(k_t)}. \quad \dots \quad (3.6)$$

From (3.6), the lower bound on s_t is clearly a function of k_t. If we have to introduce reversible investment, we would then allow s_t to be negative but arbitrarily restrict it to be greater than or equal to some negative number (smaller than unity in absolute value). In any event, we shall be assuming that $s_t \in S_t$ where, S_t is a closed and bounded interval. If s_t is bounded below by r_t for planner P_t, then, we shall assume, $S_t = [r_t, 1]$ for each t, with $|r_t| < 1$.

Since $(c_t, c_{t+1}, \dots)$ is completely determined by $(k_t; s_t, s_{t+1} \dots)$ and k_t is completely determined by $(k_1; s_1, s_2, \dots, s_{t-1})$, $w_t(\cdot)$ may be looked upon as a function of $(k_1; s_1, s_2, \dots)$. To take this into account, we shall from now on write $w_t(k_1; s_1, s_2, \dots)$. (To be more precise one should write $w_t^*(k_1; s_1, s_2, \dots)$, where $w_t(c_t, c_{t+1}, \dots) = w_t^*(k_1; s_1, s_2, \dots)$.

As was pointed out earlier, the task of P_t is to choose s_t, given $(s_1, s_2, \dots, s_{t-1}, s_{t+1}, s_{t+2}, \dots)$. Thus, s_t^* is optimal for P_t if it maximises (3.1) subject to a given k_t, (which he inherits from the past, and cannot influence), the sequence $(s_{t+1}, s_{t+2}, \dots)$ and the production relation (3.3). Let the function $G_t(.)$ relate s_t^* to $(k_t; s_{t+1}, s_{t+2}, \dots)$. Since, k_t is completely determined by $(k_1; s_1, s_2, \dots)$, from (3.4), we may write

$$\begin{aligned} s_t^* &= G_t(k_t; s_{t+1}, s_{t+2} \dots) \\ &= \phi_t(k_1; s_1, s_2, \dots, s_{t-1}, s_{t+1}, s_{t+2}, \dots) \\ &= \phi_t(k_1; \bar{s}_t) \end{aligned} \quad \dots \quad (3.7)$$

where $\bar{s}_t = (s_1, s_2, \dots, s_{t-1}, s_{t+1}, \dots)$, and $k_1(>0)$, the initial capital-labour ratio at $t = 1$, is a parameter occurring in each $\phi_t(\cdot)$. $\phi_t(\cdot)$ in (3.7) is the same as the $\phi_t(\cdot)$'s defined in Section 2, where, in order not to complicate the discussion, we did not mention the role played by the parameter k_1.

[1]The referee has pointed out that instead of dealing with reversible investment, one could think in terms of arbitrary rates of capital depreciation.

The savings ratio s_t lies in the interval $[r_t, 1]$ and as such the strategy set of each P_t is convex and compact.

It is clear now that given a value of k_1, each $\phi_t(\cdot)$ maps the infinite Cartesian product space $S_1 \times S_2 \times \ldots \times S_{t-1} \times S_{t+1} \ldots$ into S_t. Define

$$Q(k_1; s_1, s_2, \ldots) = \phi_1(k_1; \bar{s}_1) \times \phi_2(k_1; \bar{s}_2) \times \ldots .$$

Then, $Q(\cdot)$ maps $S = \prod_{t=1}^{\infty} S_t$ into itself.

S is subset of the infinite Cartesian product of the real line and the latter is a locally convex topological space under the product topology. Moreover, S is compact in the same topology by Tychonoff's theorem, and it is obviously convex. Hence, if it can be shown that $Q(k_1; s_1, s_2, \ldots)$ is a single valued function which is continuous in the product topology, the existence of a fixed point will be established. At first, it will be shown that $Q(\cdot)$ is single valued and continuous. A further argument will show that $s^* = (s_1^*, s_2^*, \ldots) \neq (0, 0, 0, \ldots)$, where $s^* = Q(s^*)$. For this last result, however, we shall be needing stronger assumptions than the ones made so far.

Lemma 1 : *$w_t(k_1; s_1, s_2, \ldots)$ is a strictly concave function of s_t, given $(k_1; \bar{s}_t)$.*

Proof : It is sufficient to prove the lemma for $w_1(\cdot)$. Consider the corresponding T-period problem.

$$w_1^T(\cdot) = u_{11}((1-s_1)f_1(k_1)) + \alpha_{12}u_{12}(c_2) + \ldots + \alpha_{1T}u_{1T}(c_T)$$

where, $(k_1; s_2, s_3, \ldots)$ is given. We shall show that each component in the above sum is strictly concave with respect to s_1, which will prove that $w_1^T(\cdot)$ is strictly concave in s_1. A final limiting argument will show concavity for the infinite horizon model.

Let $s_1^1, s_1^2 \in S_1$, $s_1^1 \neq s_1^2$. Then for p such that $0 < p < 1$,

$$\begin{aligned} u_{11}((1-(p\,s_1^1+(1-p)s_1^2))f_1(k_1)) &= u_{11}(p(1-s_1^1)f_1(k_1)+(1-p)(1-s_1^2)f_1(k_1)) \\ &> p u_{11}((1-s_1^1)f_1(k_1))+(1-p)u_{11}((1-s_1^2)f_1(k_1)), \end{aligned}$$

by the strict concavity of $u_{11}(\cdot)$. Hence, $u_{11}(c_1)$ is strictly concave with respect to s_1. Consider now any $u_{1t}(c_t)$, $2 \leqslant t \leqslant T$. From, (3.4) and (3.5),

$$c_t = (1-s_t)f_t(k_t) = h_t(k_t) \qquad \text{(say)},$$

$$k_t = \frac{s_{t-1}f_{t-1}(k_{t-1})+k_{t-1}}{1+n} = g_{t-1}(k_{t-1}) \qquad \text{(say)}.$$

Therefore,

$$\begin{aligned} u_{1t}(c_t) &= u_{1t} \circ h_t(k_t) \\ &= u_{1t} \circ h_t \circ g_{t-1} \circ g_{t-2} \circ \ldots \circ g_2(k_2) \end{aligned}$$

and

$$k_2 = \frac{s_1 f_1(k_1)+k_1}{1+n} = i(s_1) \qquad \text{(say)}.$$

Since $f_t(\cdot)$'s are strictly concave, $h_t, g_{t-1}, \ldots, g_2$ are strictly concave functions of their arguments. Also $i(s_1)$ is a linear function of s_1. Hence, $u_{1t}(c_t)$ is a strictly concave function of s_1. The next step is to prove the strict concavity of the infinite horizon welfare function. Let $(k_1; p\,s_1^1+(1-p)s_1^2,\ s_2, s_3, \ldots, s_T)$, $0 < p < 1$, give rise to $(c_2, c_3, \ldots, c_T)$. Similarly, let $(k_1; s_1^1, s_2, s_3, \ldots, s_T)$ give rise to $(c_2^1, c_3^1, \ldots, c_T^1)$ and $(k_1; s_1^2, s_2, s_3, \ldots, s_T)$ give rise to $(c_2^2, c_3^2, \ldots, c_T^2)$. Then, from what we have shown above,

$$u_{11}((1-(p\,s_1^1+(1-p)s_1^2)f_1(k_1))+\alpha_{12}u_{12}(c_2)+\ldots+\alpha_{1T}u_{1T}(c_T)$$

$$> p(u_{11}((1-s_1^1)f_1(k_1))+\alpha_{12}u_{12}(c_2^1)+\ \ldots\ +\alpha_{1T}u_{1T}(c_T^1))$$

$$+(1-p)(u_{11}((1-s_1^2)f_1(k_1))+\alpha_{12}u_{12}(c_2^2)+\ldots+\alpha_{1T}u_{1T}(c_T^2))+\varepsilon$$

for some $\varepsilon > 0$.

Let $T \to \infty$. Then, keeping in mind Assumptions 1, 2 and 4, we have

$$w_1(k_1 : (ps_1^1+(1-p)s_1^2),\ s_2, s_3, \ldots) > pw_1(k_1; s_1^1, s_2, s_3, \ldots)+(1-p)w_1(k_1; s_1^2, s_2, s_3, \ldots)$$

Hence, $w_1(\cdot)$ is a strictly concave function of s_1.

Q.E.D.

Since, $w_t(\cdot)$ is concave with respect to s_t, $w_t(\cdot)$ is continuous over the interior of S_t with respect to s_t, given $(k_1;\ \bar{s}_t)$. We shall show that $w_t(\cdot)$ is continuous at the boundaries of S_t also. Consider $w_1(\cdot)$ once again. Let $s_1 = r_1$. Also, let $(c_1^0, c_2^0, \ldots)$ be the consumption-labour ratio sequence generated by $(k_1; s_1 = r_1, s_2, s_3, \ldots)$. From Assumption 1(b), (c) and 4, it follows that there exists B, $0 < B < \infty$ such that $B > |u_{1t}(c_t)-u_{1t}(c_t^0)|\ \forall c_t$. Also, $\sum_{t=T}^{\infty} \alpha^{t-1}B \to 0$ as $T \to \infty$. Hence, there exists N, $0 < N < \infty$ such that $T > N$ implies $\sum_{t=T}^{\infty} \alpha^{t-1}B < \frac{\varepsilon}{2}$, for any $\varepsilon > 0$. Now let

$$\sum_{t=1}^{N} \alpha^{t-1}|u_{1t}(c_t)-u_{1t}(c_t^0)\ | < \frac{\varepsilon}{2}.$$

Then,

$$\varepsilon > \sum_{t=1}^{N} \alpha^{t-1}|u_{1t}(c_t)-u_{1t}(c_t^0)|+\sum_{t=N+1}^{\infty} \alpha^{t-1}B > \sum_{t=1}^{\infty} \alpha^{t-1}|u_{1t}(c_t)-u_{1t}(c_t^0)|$$

$$> \sum_{t=1}^{\infty} \alpha_{1t}|u_{1t}(c_t)-u_{1t}(c_t^0)| \qquad \ldots \quad (3.8)$$

by Assumption 2. By the continuity of $u_{1t}(\cdot)$ and $f_t(\cdot)$, equations (3.4), (3.5) and the finiteness of N, it follows then that there exists $\delta > 0$ such that (3.8) holds whenever, $|s_1-r_1| < \delta$, given $(k_1; \bar{s}_1)$. Hence, $w_1(\cdot)$ is continuous at $s_1 = r_1$. In a similar fashion, continuity of $w_1(\cdot)$ at $s_1 = 1$ may be established. Similar arguments show the continuity of $w_t(\cdot)$ at the boundaries of S_t for any t.

It follows that the s_t which maximises $w_t(\cdot)$ given $(k_1; \bar{s}_t)$ is unique and that such a maximum always exists, S_t being a closed and bounded interval and $w_t(\cdot)$ being continuous over S_t. Thus, each $\phi_t(k_1; s_t)$, and therefore, $Q(k_1; s_1, s_2, \ldots)$ is single-valued.

The next step is to show that $\phi_t(k_1; \bar{s}_t)$ is continuous with respect to $\bar{s}_t$. Since $\phi_t(\cdot)$ is defined over an infinite dimensional space and the latter is compact in the product topology, the topology to be chosen for the domain set of $\phi_t(\cdot)$ in our proof of continuity will be the product topology. Thus, the open sets we shall be concerned with will be the infinite products of open neighbourhoods of the form $0_1 \times 0_2 \times 0_3 \times \ldots$ where, $0_1 \subset S_1$, $0_2 \subset S_2$, ... and only a finite number of 0_t's would be proper neighbourhoods. Thus, in the above definition of an open set, $0_t = S_t$ for all but finitely many t.

One way of showing the continuity would be to demonstrate that inverse images of open neighbourhoods of $\phi_t(\cdot)$ are open in the product topology. Instead of doing that, however, we try to define a metric for the space $\prod_{t=1}^{\infty} S_t$. Choose the norm $\sum_{t=1}^{\infty} \alpha^{t-1}|s_t|$. For any two sequences $\{s_t\}_{t=1}^{\infty}$ and $\{s_t'\}_{t=1}^{\infty}$, the metric induced by this norm is $\sum_{t=1}^{\infty} \alpha^{t-1}|s_t-s'_t|$. Since S_t is uniformly bounded for all t, the open sets generated by this metric are open in the product topology.

Lemma 2 : *$\phi_t(k_1; \bar{s}_t)$ is continuous in $\bar{s}_t$.*

Proof : It will be shown, first of all, that $w_\theta(k_1; s_1, s_2, \ldots)$ is continuous with respect to $(s_1, s_2, s_3, \ldots)$ for all θ. Choose any $(s_1^0, s_2^0, s_3^0, \ldots, s_\theta^0, \ldots)$. Also, let $(k_1; s_1^0, s_2^0, \ldots)$ generate $(k_\theta^0; c_\theta^0, c_{\theta+1}, \ldots)$. Then,

$$w_\theta(k_1; s_1^0, s_2^0, \ldots) = \sum_{t=\theta}^{\infty} \alpha_{\theta t} u_{\theta t}(c_t^0).$$

Choose $\epsilon > 0$. Then, by arguments following Lemma 1, there exist T, B, $0 < T < \infty$, $0 < B < \infty$ such that

$$\epsilon > \sum_{t=T}^{\infty} \alpha^{t-\theta} B > \sum_{t=T}^{\infty} \alpha^{t-\theta} |u_{\theta t}(c_t) - u_{\theta t}(c_t^0)|, \quad \text{for any } c_t.$$

Choose $m > 0$ such that

$$\epsilon > \sum_{t=\theta}^{T-1} \alpha^{t-\theta} m + \sum_{t=T}^{\infty} \alpha^{t-\theta} B.$$

Let
$$\sum_{t=\theta}^{T-1} \alpha^{t-\theta} |u_{\theta t}(c_t) - u_{\theta t}(c_t^0)| < \sum_{t=\theta}^{T-1} \alpha^{t-\theta} m. \qquad \ldots (3.0)$$

Then,

$$\varepsilon > \sum_{t=\theta}^{T-1} \alpha^{t-\theta} m + \sum_{t=T}^{\infty} \alpha^{t-\theta} B$$

$$> \sum_{t=\theta}^{\infty} \alpha^{t-\theta} |u_{\theta t}(c_t) - u_{\theta t}(c_t^0)|$$

$$> \sum_{t=\theta}^{\infty} \alpha_{\theta t} |u_{\theta t}(c_t) - u_{\theta t}(c_t^0)|.$$

Thus, $\sum_{t=\theta}^{\infty} \alpha_{\theta t} u_{\theta t}(c_t)$ lies in an ε-neighbourhood of $\sum_{t=\theta}^{\infty} \alpha_{\theta t} u_{\theta t}(c_t^0)$. It follows from (3.9), however, $|u_{\theta t}(c_t) - u_{\theta t}(c_t^0)| < \bar{m}$, $t = \theta, \theta+1, \ldots, T-1$, for some $\bar{m}$. But $u_{\theta t}(\cdot)$ is continuous. Hence, the last observation implies $|c_t - c_t^0| < b_t$, $t = \theta, \theta+1, \ldots, T-1$, where b_t can be made as small as one likes, depending on the choice of m. Clearly, from the continuity of $f_t(\cdot)$ and the fact that k_1 is fixed, one can choose δ, $0 < \delta < |1-r_t|$ $t = 1, \ldots, T-1$, such that $|s_t - s_t^0| < \delta$, $t = 1, 2, \ldots, T-1$ implies $|c_t - c_t^0| < \bar{m}$, $t = \theta, \theta+1, \ldots, T-1$, where $(c_\theta, c_{\theta+1}, \ldots, c_{T-1})$ is generated by $(k_1; s_1, s_2, \ldots, s_{T-1})$. Now choose $\varepsilon(\delta) > 0$ such that $\sum_{t=1}^{\infty} \alpha^{t-1} |s_t - s_t^0| < \varepsilon(\delta)$ implies $|s_t - s_t^0| < \delta$, $t = 1, 2, \ldots, T-1$. Since, S_t is bounded for all t, the choice of $\varepsilon(\delta)$ is always possible.

Thus, given any $\varepsilon > 0$, there exists $\varepsilon(\delta) > 0$ such that

$$\sum_{t=\theta}^{\infty} \alpha_{\theta t} |u_{\theta t}(c_t) - u_{\theta t}(c_t^0)| < \varepsilon$$

whenever,
$$\sum_{t=1}^{\infty} \alpha^{t-1} |s_t - s_t^0| < \varepsilon(\delta).$$

Since, $\delta < |1-r_t|$, $t = 1, 2, \ldots, T-1$, $\sum_{t=1}^{\infty} \alpha^{t-1} |s_t - s_t^0| < \varepsilon(\delta)$ defines an open neighbourhood in $\prod_{t=1}^{\infty} S_t$. Hence, $w_\theta(k_1; s_1, s_2, \ldots)$ is continuous at $(k_1; s_1^0, s_2^0, \ldots)$ with respect to $(s_1, s_2, \ldots)$. However, $(s_1^0, s_2^0, \ldots)$ was chosen arbitrarily. Therefore, $w_\theta(\cdot)$ is continuous with respect to $(s_1, s_2, \ldots)$.

Assume now $\phi_t(k_1; \bar{s}_t)$ is not continuous at $\bar{\sigma}_t = (s_1, s_2, \ldots, s_{t-1}, s_{t+1}, \ldots)$. This implies that $\bar{\sigma}_t^n = (s_1^n, s_2^n, \ldots, s_{t-1}^n, s_{t+1}^n, \ldots) \to \bar{\sigma}_t$, $s_t^{n*} \not\to s_t^*$, where, $s_t^{n*} = \phi_t(k_1; \bar{\sigma}_t^n)$ and $s_t^* = \phi_t(k_1; \bar{\sigma}_t)$. Since, S_t is bounded, there exist subsequences $\bar{\sigma}_t^{nj}$ and s_t^{nj*} such that

$$\bar{\sigma}_t^{nj} \to \bar{\sigma}_t, \; s_t^{nj*} \to \bar{s} \neq s_t^*.$$

But,
$$w_t(k_1; s_1^{nj}, \ldots, s_{t-1}^{nj}, s_t^{nj*}, s_{t+1}^{nj}, \ldots) \geqslant w_t(k_1; s_1^{nj}, \ldots, s_{t-1}^{nj}, s_t^*, s_{t+1}^{nj}, \ldots).$$

Taking limits and using the continuity of $w_t(\cdot)$, the left hand side goes to

$$w_t(k_1; s_1, \ldots, s_{t-1}, \bar{s}, s_{t+1}, \ldots)$$

and the right hand side goes to

$$w_t(k_1; s_1, \ldots, s_{t-1}, s_t^*, s_{t+1}, \ldots).$$

However, from the strict concavity of $w_t(\cdot)$ with respect to s_t,

$$w_t(k_1; s_1, \ldots, s_{t-1}, \bar{s}, s_{t+1}, \ldots) < w_t(k_1; s_1, \ldots, s_{t-1}, s_t^*, s_{t+1}, \ldots)$$

which violates the continuity of $w_t(\cdot)$.

Hence, $\phi_t(.)$ is continuous with respect to $\bar{s}_t$.

Q.E.D.

Lemma 3 : *$Q : S \to S$ is continuous in $(s_1, s_2, \ldots)$.*

Proof : Let $\bar{\sigma}^n = (s_1^n, s_2^n, \ldots) \to \sigma = (s_1, s_2, \ldots)$. Then $s_t^{n*} \to s_t^* \;\forall\; t$ from Lemma 2. Hence, by the choice of our metric $\sum\limits_{t=1}^{\infty} \alpha^{t-1} |s_t^{n*} - s_t^*| \to 0$. That is to say, $\sigma^{n*} \to \sigma^*$, where $\sigma^{n*} = (s^{n*}, s_2^{n*}, \ldots)$ and $\sigma^* = (s_1^*, s_2^*, \ldots)$. Therefore, $Q(\cdot)$ is continuous.

Q.E.D.

Theorem 1 : *$Q : S \to S$ has a fixed point.*

Proof : Apply the Schauder-Tychonoff fixed point theorem.

Q.E.D.

The assumptions we have made on the model so far are too general for any fruitful discussion of the properties of the fixed point except that it exists. Under stronger assumptions, however, it is possible to characterize the fixed point in greater detail. For the purposes of the following discussion, we shall assume $u_{\theta t}(\cdot) = u_\theta(\cdot)$ $f_t(\cdot) = f(\cdot) \forall t$. $u_\theta(\cdot)$ will be assumed to satisfy.

Assumption 5 :

(a) $u'_\theta(\cdot) > 0$, $u''_\theta(\cdot) < 0 \forall \theta$;

(b) There exists $b_\theta > -\infty$ such that $u_\theta(\cdot) > b_\theta$ for each θ.

As far as the production side is concerned we merely retain Assumption 3 and discard Assumption 4. The sequence $\{\alpha_{\theta t}\}_{t=\theta}^{\infty}$ is still assumed to satisfy Assumption 2.

The planner's welfare function now becomes

$$w_\theta(.) = \sum_{t=\theta}^{\infty} \alpha_{\theta t} u_\theta(c_t)$$

where, production is constrained by (3.3).

From what has been shown so far, existence of a fixed point in this model follows from Assumptions 5(a), (b), 2 and 3. (Assumption 1(c) is not needed for the existence theorem in this case).

In Lemma 1, we proved the strict concavity of $w_t(k_1; s_1, s_2, \ldots)$ with respect to s_t given k_1 and $\bar{s}_t$. It follows that $w_t(k_1; s_1, s_2, \ldots)$ is twice differentiable with respect to s_t (given k_1 and $\bar{s}_t$) almost everywhere (in the sense of Lebesgue measure). Consider now the point $s_1 = s_2 = \ldots = s_{t-1} = s_{t+1} = \ldots = 0$. Let 0_t be an open neighbourhood contained in $S_t = [r_t, 1]$ such that 0_t contains 0.

Assumption 6:

Given $s_1 = s_2 = \ldots = s_{t-1} = s_{t+1} = \ldots = 0$, $w_t(k_1; s_1, s_2, \ldots)$ is differentiable with respect to s_t for $s_t \in 0_t$.

If Assumption 6 holds for evey t, then along with the other assumptions, Theorem 2 follows.

Theorem 2: *Suppose Assumptions 2, 3 and 5 hold. Also let Assumption 6 hold for all t. Then, there exists σ^* such that $\sigma^* = \phi(\sigma^*)$. Moreover, given any β, where $1 > \beta > 0$, there exists a $\bar{\theta}$ such that if $\alpha_{\bar{\theta}, \bar{\theta}+1} \geqslant \beta$ then $\sigma^* \neq (0, 0, 0, \ldots)$.*

Proof: The existence of σ^* follows from Theorem 1. Suppose $\sigma^* = (s_1^*, s_2^*, s_3^*, \ldots) = (0, 0, 0, \ldots)$. We shall achieve a contradiction by showing the existence of a $\theta = \bar{\theta}$ such that $s_1^* = s_2^* = s_3^* = \ldots = s_{\theta-1}^* = 0$ implies $s_\theta^* = 0$ cannot be an optimum choice.

Consider any sequence of savings ratios $(s_1, s_2, s_3, \ldots)$ with $s_t = 0$ for all $t \neq \theta$. Let $(c_1, c_2, c_3, \ldots)$ and $(k_1, k_2, k_3, \ldots)$ be the corresponding sequence of consumption-labour ratios and capital-labour ratios respectvely. With these values, planner P_θ's welfare function becomes

$$w_\theta(c_\theta, c_{\theta+1}, \ldots) = u_\theta(c_\theta) + \alpha_{\theta, \theta+1}\, u_\theta(c_{\theta+1}) + \alpha_{\theta, \theta+2}\, u_\theta(c_{\theta+2}) + \ldots \qquad \ldots \quad (3.10)$$

Note that $u_{\theta t}(\cdot) = u_\theta(\cdot)$ for all t, by assumption. From (3.4) and the assumption that $f_t(\cdot) = f(\cdot)$ for all t, we find,

$$c_\theta = (1 - s_\theta) f(k_\theta).$$

Similarly, from (3.4), (3.5) and the fact that $s_{\theta+1} = 0$, we have

$$c_{\theta+1} = f(k_{\theta+1}) = f\left(\frac{s_\theta f(k_\theta) + k_\theta}{1+n} \right).$$

Thus, (3.10) can be written

$$w_\theta(c_\theta, c_{\theta+1}, \ldots) = u_\theta((1 - s_\theta) f(k_\theta)) + \alpha_{\theta, \theta+1} u_\theta \left(f \left(\frac{s_\theta f(k_\theta) + k_\theta}{1+n} \right) \right) + \alpha_{\theta, \theta+2} u_\theta(c_{\theta+2}) + \ldots$$

$$\ldots \quad (3.11)$$

For an infinitesimal increase in s_θ (all other s_t's remaining zero), the total change in welfare in periods θ and $\theta+1$ in (3.11) is then given by

$$-u_\theta'((1-s_\theta)f(k_\theta))f(k_\theta)+\alpha_{\theta,\,\theta+1}u_\theta'\left(f\left(\frac{s_\theta f(k_\theta)+k_\theta}{1+n}\right)\right)f'\left(\frac{s_\theta f(k_\theta)+k_\theta}{1+n}\right)\frac{f(k_\theta)}{1+n} \quad \ldots \quad (3.12)$$

For the remaining periods, denote the total change in welfare by δ_θ. *Note that δ_θ must be strictly positive.* Now let $s_\theta = 0$. Then, (3.12) becomes

$$-u_\theta'(f(k_\theta))f(k_\theta)+\alpha_{\theta,\,\theta+1}u_\theta'\left(f\left(\frac{k_\theta}{1+n}\right)\right)f'\left(\frac{k_\theta}{1+n}\right).\frac{f(k_\theta)}{1+n}. \quad \ldots \quad (3.13)$$

Thus, if in period θ, s_θ is increased infinitesimally from zero (with $s_t = 0$, all $t \neq \theta$), then the total change in welfare is

$$-u_\theta'(f(k_\theta))f(k_\theta)+\alpha_{\theta,\,\theta+1}u_\theta'\left(f\left(\frac{k_\theta}{1+n}\right)\right)f'\left(\frac{k_\theta}{1+n}\right)\left(\frac{f(k_\theta)}{1+n}\right)+\delta_\theta. \quad \ldots \quad (3.14)$$

Our proof is complete if (3.14) is demonstrated to be strictly positive for some $\theta = \bar{\theta}$.

If $s_t = 0$ for all t, then from (3.4) and (3.5), $k_{t+1} < k_t$ and $c_{t+1} < c_t$ for all t. Hence, noting that $c_\theta = f(k_\theta)$, $c_{\theta+1} = f\left(\frac{k_\theta}{1+n}\right)$ and $k_{\theta+1} = \frac{k_\theta}{1+n}$, it follows from the strict concavity of the utility function that,

$$u_\theta'\left(f\left(\frac{k_\theta}{1+n}\right)\right)f(k_\theta) > u_\theta'(f(k_\theta))f(k_\theta). \quad \ldots \quad (3.15)$$

(3.15) holds for any θ, provided $s_1 = s_2 = \ldots = s_\theta = 0$. On the other hand, since $k_{t+1} < k_t$ for all t, it follows from assumption II(c) that there exists a $\bar{\theta}$ such that

$$\beta \,.\, \frac{f'(k_{\bar{\theta}+1})}{1+n} > 1,$$

since β is given.

Let $\alpha_{\bar{\theta},\,\bar{\theta}+1} \geqslant \beta$. Then, from (3.15),

$$\alpha_{\bar{\theta},\bar{\theta}+1}u_{\bar{\theta}}'\left(f\left(\frac{k_{\bar{\theta}}}{1+n}\right)\right)\frac{f'(k_{\bar{\theta}+1})}{1+n}\,.\,f(k_{\bar{\theta}}) > u_{\bar{\theta}}'(f(k_{\bar{\theta}}))f(k_{\bar{\theta}}). \quad \ldots \quad (3.16)$$

From (3.16), it follows then (3.14) must be strictly positive for $\theta = \bar{\theta}$, as was to be proved.

Q.E.D.

In a model of optimal growth, an equilibrium solution involving all planners saving at the rate zero is economically somewhat unacceptable. This would indicate a situation where no planner cares for the future at all. In the context of the game theoretic framework, however, this situation is not entirely unexpected. For example, in the Prisoner's Dilemma problem, the non-cooperative equilibrium solution is strictly dominated by other solutions which can never be achieved.

If, however, we still insist on altruistic motives on the part of planners, then Theorem 2 is the answer. Theorem 2 shows that the very fact that all the preceding saving rates are zero would lead some planner at some stage to start saving.

4. Directions for further research

Our apporach in this paper has been basically descriptive. We have dealt with a model where all planners have arbitrary utility functions and discount parameters. Given the fact that the utility functions and discount parameters can vary from planner to planner, we tried to establish an equilibrium sequence of savings ratios which is acceptable to all planners.

It would be interesting to see if our results hold in the case where discounting is absent. Brock (1971) has shown the insensitivity of the optimum consumption path in a finite horizon model with respect to the terminal capital stock when discounting is absent. His results may turn out to be useful for our purpose.

Again, given the existence of a fixed point, it would be important to see whether it is unique, or at least under what conditions it is. The uniqueness is easily established if the mapping $Q(\cdot)$ turns out to be a contraction.

One would also like to know if a given equilibrium of savings ratios implies convergence of the capital-labour ratio to a steady state, and if it does, how the steady state compares with the Cass-Koopmans (Cass, 1965 ; Koopmans, 1967) steady state.

No attempt has been made here to show how the equilibrium may be attained. Clearly, no planner may have any idea about the preferences of future generations. It seems necessary, therefore, to construct a process of expectations formation about future behaviour (based on past experience) which would lead the path of optimum savings ratios to the non-cooperative equilibrium.

Lastly, although the horizon has been assumed to be infinite for each planner, the result holds with equal force for finite horizon. The summable nature of the preference functions was necessary to prove the lemmas regarding continuity and concavity. However, if one is willing to assume these properties, the summable preference function can also be dispensed with.

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