*Signature of the invigilator*________________________

INDIAN STATISTICAL INSTITUTE

(203 B. T. Road, Kolkata 700 108)

Bachelor of Statistics (B.Stat.) IIIrd Year
Academic Year 2024 - 2025: Semester II

Optional Course on Random Graphs

Final Examination

Instructor: Antar Bandyopadhyay

Date: April 30, 2025 Total Points: 50
Time: 10:30 AM - 01:30 PM Duration: 180 minutes

Note: There are five problems with a total of 60 points. Solve as many as you can. Maximum you can score is 50. Show all your works. Use only the pages provided. The rough works must be done on the pages provided for the same. This is an open note examination. You are allowed to use your own hand written notes (such as class notes, your homework solutions, list of theorems, formulas etc). No printed materials or photo copies are allowed, in particular you are not allowed to use books, photocopied class notes etc. Do not forget to write your roll number. Good luck!

Roll No.: ________________________

Please DO NOT write any thing below
For Instructor's use only

Problems	1	2	3	4	5	Total
Points						

*Signature of the grader/instructor*________________________

[Please Only Turn at 10:30 AM]

[Please DO NOT write anything on this page!]

1. Consider the $\mathcal{G}(n, p_n)$ model with standard notations. Let P_n be the property that a graph on n vertices "*is not a tree*". Show that P_n is an *increasing property*. Show that $\hat{p}_n = 1/n$ is a *threshold* for the property P_n? [2 + 10 = 12]

2. Suppose G_n is random *multi-graph* which follows a configuration model, say, $\text{CM}(n; \{d_i\})$, where $d_1, d_2, \cdots, d_n$ are positive integers, such that, $\sum_{i=1}^{n} d_i \in 2\mathbb{N}$. Let S_n be the number of *self-loops* in G_n. Find $\mathbf{E}[S_n]$ and $\mathbf{Var}(S_n)$ in terms of n and $d_1, d_2, \cdots, d_n$. [4+8 = 12]

3. Let $X_1, X_2, \cdots, X_n$ be i.i.d. Poisson(λ) variables, for some $\lambda > 0$. Let $D_i = 2(X_i + 1)$, $1 \leq i \leq n$. Given $(D_1, D_2, \cdots, D_n) = (d_1, d_2, \cdots, d_n)$, $G_n \sim \text{CM}(n; \{d_i\})$. Let S_n be the number of *self-loops* in G_n. Find $\lim_{n\to\infty} \mathbf{E}[S_n]$ and $\lim_{n\to\infty} \mathbf{Var}(S_n)$. [4 + 8 = 12]

4. Let $(\mathcal{T}_n)_{n\geq 1}$ be a random graph process, such that, vertex set of $\mathcal{T}_n$ is $[n] := \{1, 2, \cdots, n\}$. Starting with one *isolated* vertex as $\mathcal{T}_1$, the sequence $(\mathcal{T}_n)_{n\geq 1}$, is inductively constructed as follows:

   Having constructed $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_n$, $\mathcal{T}_{n+1}$ is the graph $\mathcal{T}_n$ along with a (new) vertex $(n+1)$, which joins to any of the vertices of $\mathcal{T}_n$ uniformly at random.

   Show that for every n, $\mathcal{T}_n$ is a tree. Let τ_n be the distance of the vertex n from the vertex 1 in $\mathcal{T}_n$. Show that

$$\frac{\tau_n - \log n}{\sqrt{\log n}} \xrightarrow{d} \text{Normal}(0, 1).$$

   [2 + 10 = 12]

5. Consider the $\mathcal{G}\left(n, \frac{\lambda}{n}\right)$ model in standard notations. If $\lambda < 1$ then show that

$$\frac{|\mathcal{C}_{max}|}{\log n} \xrightarrow{\mathbf{P}} \frac{1}{I_\lambda},$$

   where $I_\lambda = \lambda - 1 - \log \lambda$ and $|\mathcal{C}_{max}|$ is the size of the largest component. [You may use any result done in the class without writing a proof, but by stating it clearly in our context.] [12]

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