Dynamics of random graphs and interacting particle systems

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Dynamics of random graphs and interacting particle systems

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Introduction

In this thesis we broadly have two objectives. In the first part we construct a class of Markov process on the space of sparse graphs that are governed by an unbounded integrable graphon from a sequence of dynamically evolving sparse graphs. This construction is inspired in part by dense graph dynamics constructed in Athreya et al. [1]. In the second part of the thesis we understand trajectorial representations of solution $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}_+$ of nonlinear equation given by

$$\partial_t u = \frac{1}{2}\Delta u - u^{1+\alpha}, u(0, \cdot) = u_0$$

where $u_0 : \mathbb{R}^d \to \mathbb{R}_+$ and $0 < \alpha < 1$, in terms of a α-stable collision tree generated by annihilating Brownian spheres. This compliments the work done in Sznitman [38] where α was an positive integer.

Dynamics of random graphs: One of the early papers on the study of the dynamics of real-world graphs, particularly in the context of evolving social networks, is the work of Holland and Leinhardt [18]. They proposed time-evolving random graphs as a model for the evolution of social networks. This foundational work has been further developed by Snijders [34] and Snijders et al. [35]. They provided detailed accounts of these developments and described statistical procedures to monitor the effects of network dynamics. We will briefly discuss about compartmental models and agent based models at the end of this chapter.

Despite various efforts, the mathematical exploration of this topic is still in its nascent stages. Many models often involve highly structured networks, typically lattices, which differ greatly from the networks discussed here. Additionally, much of the existing research focuses on sparse networks where node degrees remain bounded. Typical metrics of interest include the mixing times and cover times of random walks on these networks, across different dynamics. These metrics are crucial for understanding how information propagates through the network (see Avena et al. [4], Leskovec [22] and Levin and Peres [23]).

Graph sequences can be classified based on the relationship between the number of edges E and the number of vertices V. Broadly, we will characterize them into three classes: extremely sparse graphs, sparse graphs, and dense graphs. An extremely sparse graph sequence is one with $o(n)$ edges. A sparse graph sequence on n vertices has a number of edges between $\omega(n)$ and $o(n^2)$. A dense graph sequence on n vertices has $\Theta(n^2)$ edges, for example the complete graph.

We will focus our attention on sparse graph sequences where there is a natural "limit object" of a convergent sparse graph sequence under the cut metric(see [7, 8, 9, 10]), namely a symmetric measurable function $W : [0, 1]^2 \to \mathbb{R}_+$. These measuarable functions are called graphons and they offer a comprehensive and adaptable framework to address the question of graph limits as the number of vertices tend to infinity.

Borgs et al. [7] showed that every graphon produces a discrete sparse graph sequence converging to the graphon under a suitable metric. Given a graphon, they constructed a graph on n vertices by connecting two vertices with a probability which depends on a thinning parameter ρ_n and the graphon. As $n \to \infty$, the connection probability between vertices goes to 0.

In Athreya et al. [1], they first construct random dynamics for discrete dense graph sequences and establish the existence of limiting dynamics on the space of graphons as the number of vertices goes to infinity. In this thesis we try to understand the same in the sparse graph setting. We construct random dynamics using interacting particle systems for discrete sparse graph sequence and using L^p theory we study the limiting dynamics on the space of unbounded graphons as the number of vertices goes to infinity. We will follow [1] by assigning types to the vertices of the sparse graph sequence. The types will evolve according to the Moran model (a population genetics model) and the edge connection probabilities will depend on the types. In our main result we characterize the limiting dynamics of the system. We illustrate this in the next example.

Example: Consider a population size of n and assign each individual in the population one of the two types, Type 0 or Type 1. Assume that each individual, independently at rate 1, randomly selects an individual from the population (possibly itself) and adopts its type. We denote $X^n(s)$ to be the number of individuals of type 0 at times s. For each $s \geq 0$ think of the n individuals as the vertices of a random graph $G^n(s)$. Then we connect individuals i and j by an edge with probability ρ_n if they are of the same type and remain disconnected if their types are different, where $\rho_n \to 0$ and $n\rho_n \to \infty$. Let $Y^n(s) = \frac{1}{n}X^n(ns)$ represent the fraction of individuals of Type 1 at time s on time scale n. It is well known that if $Y^n(0)$ converges weakly to $Y(0)$, then $Y^n = (Y^n(s))_{s\geq 0}$ converges weekly to $Y = (Y(s))_{s\geq 0}$ in path space with respect to Skorohod topology, where the limiting process is the Wright-Fisher diffusion on $[0, 1]$, given by SDE

$$Y(s) = Y(0) + \int_0^s \sqrt{Y(u)(1 - Y(u)}dW(u)$$

with initial condition $Y(0)$ and with $W = (W(s))_{s\geq 0}$ being standard Brownian motion.

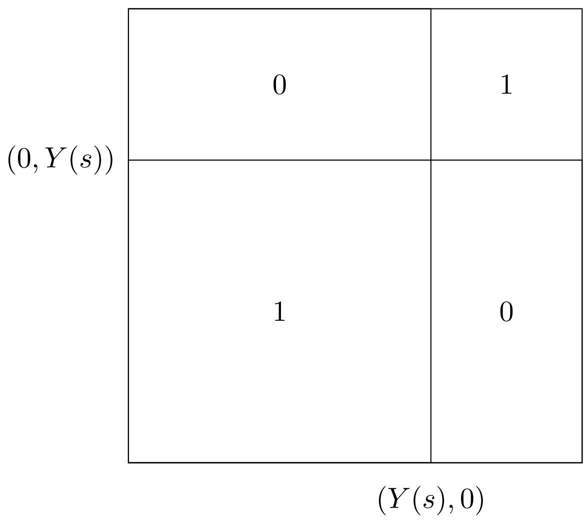

FIGURE 1: Graphical representaion of $\tilde{h}(s)$, with $Y(s)$ the fraction of Type 0 in the population at time s.

We will show in Theorem 1.2.1 of Chapter 1 that $\frac{G^n}{\rho_n}$ converges weakly on path space of graphons valued process to $\tilde{h}$ where

$$\tilde{h}(s;x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0, Y_0(s)) \times [0, Y_0(s)) \\ 1 & \text{if } (x,y) \in [Y_0(s), 1) \times [Y_0(s), 1) \\ 0 & \text{if } (x,y) \in [Y_0(s), 1) \times [0, Y_0(s)) \\ 0 & \text{if } (x,y) \in [Y_0(s), 1) \times [0, Y_0(s)) \end{cases}$$

(see Figure 1). The precise definitions of the model and metric spaces are provided in chapter 1 and Theorem 1.2.1 applies to a larger class of examples.

We will now briefly review the literature where dynamically involving discrete graphs and their limits have been considered. We shall begin with results from dense graphs.

As mentioned before a dense graph sequence is a graph sequence whose number of edges is of order $\Theta(n^2)$. The space of dense graphs can be endowed with the subgraph count metric (or cut metric). The limit theory (established by Lovász and Szegedy [24]) for dense graph sequences under this metric is well understood. Under this metric, the limits are symmetric measurable functions $\kappa : [0.1]^2 \to [0,1]$, called graphons as mentioned above.

Crane [14] conducted the first mathematically rigorous investigation into the limiting dynamics of time-varying graphs as the number of vertices approaches infinity in the context of dense graphs. While it is Crane [14] conducted the first mathematically rigorous investigation into the limiting dynamics of time-varying graphs as the number of vertices approaches infinity in the context of dense graphs. While it is conceptually and mathematically straightforward to construct random dynamics for graphs with n vertices for any fixed n, ensuring that the limit is a diffusion as n approaches infinity is more challenging. Crane's starting point is the Aldous-Hoover theory for infinitely exchangeable arrays. Let $\mathcal{G}_\infty$ be the space of infinite exchangeable arrays equipped with the product topology. Define modulus map $|\cdot|$ that takes an array $\Gamma \in \mathcal{G}_\infty$ to a graphon $|\Gamma|$ (which is well-defined with probability 1 when the array is exchangeable). One of Crane's main findings is that an $\mathcal{G}_\infty$-valued exchangeable Markov process $((\Gamma(s))_{s\geq 0}$ induces a graphon valued Markov process $(|\Gamma(s)|)_{s\geq 0}$, results in a graphon-valued Markov process with locally bounded variation (Refer to Definition 5.7 in [14] for the definition of locally bounded variation.). However, Crane showed that this method only leads to jump processes and deterministic flows in the space of graphons.

Basu and Sly [6] studied a variant of the voter model on a dynamically changing network within the context of dense graphs. In this network, the vertices represent agents, each of whom can hold one of two opinions. When an edge connects two agents with differing opinions, the connection occurs with a probability of $\frac{\beta}{n}$. In such cases, one agent performs a voter model step, changing its opinion to match the other agent's. Whereas with a probability of $1 - \frac{\beta}{n}$, the existing edge between two agents is broken and reconnected to a new agent, chosen either randomly from the entire network (rewire-to-random model) or from among agents holding the same opinion (rewire-to-same model). They showed that in both models, the time for the dynamics to terminate exhibits a phase transition based on the model parameter β. When β is sufficiently small, the network is highly likely to quickly split into two disconnected communities with opposing opinions. Conversely, when β is large enough, the dynamics persist longer, and the density of opinion changes significantly before the dynamics terminates.

Avena et al. [4] considered a discrete-time dynamic version of the configuration model. It is known that the mixing time of a random walk on a random graph constructed according to the configuration model on n vertices is of the order $\log n$. They stuided the effects of making the random graph dynamic, where at each time unit, a fraction α_n of the edges is randomly rewired, while preserving the degrees of the vertices. As a result, when analyzing a random walk on this dynamic configuration model, the stationary distribution remains constant over time, making the study of its mixing time a well-posed question. Under mild assumptions on the prescribed degree sequence, they proved that the random walk on the dynamic model mixes faster than the $\log n$ order known for the static model.

Athreya et al. [1] considered random dynamics for discrete dense graphs with n vertices for each fixed n and investigated the limit as n approaches infinity. To generate random dynamics on a graph, they utilized a model from population genetics. Specifically, they considered a finite population (represented by the vertices) where individuals change type based on resampling governed by a Moran model. One treats each individual as a vertex of a graph. At any time, each pair of individuals are connected by an edge with a probability that is given by a type-connection matrix, whose entries depend on the current empirical type distribution of the entire population via a fitness function. They showed that under appropriate scaling of time, as the population-size goes to infinity, the evolution of the associated adjacency matrix converges to a random process in the space of graphons, driven by Wright-Fisher diffusion and the type-connection matrix as $n \to \infty$.

Recently, Braunsteins et al. [11] developed a dynamic Erdős-Rényi random graph (ERRG) with n vertices, where each edge switches on at rate λ and switches off at rate μ, independently of other edges. They studied the evolution of the associated empirical graphon in the limit as $n \to \infty$. They extended the Large Deviation Principle (LDP) from [13] to a sample-path LDP for a dynamic random graph with edges switching on and off randomly. The equilibrium of these dynamics aligns with the setup in [13], making the sample-path LDP a dynamic version of the static LDP derived in that work. Furthermore, they developed a sample-path LDP for a class of processes where edges evolve dependently (see [12]). In this class, each vertex is assigned a type that changes randomly over time, and fluctuations in vertex types influence how the edges interact while switching on and off.

In Chapter 1 we explore random dynamics of discrete sparse graphs on n vertices for each fixed n and study the limit as $n \to \infty$. We follow Borgs et al. [7, Section 2.7] to construct the sparse graph sequence. We introduce dynamics by assigning types to each vertex which evolve according to the Moran model. We connect two vertices with probability that depends on the types of the vertices. The connection probability of two vertices does not depend on the rest of the population. In chapter 2 we constructed the sparse graph sequence using a Markov Chain and a clumping procedure as considered in Athreya and Röllin [3]. We introduce a dynamics by imposing types on the vertices and probability of connecting an edge depend on them. In this set up there is dependencies across edges unlike in the model in Chapter 1. We established a criteria for convergence of finite dimensional distribution and tightness of processes in Lemma 1.3.1 and Lemma 1.3.2 respectively. Under appropriate scaling of time, as number of vertices goes to infinity, the evolution of the dynamic sprase graph sequence converges to a random process in the space of graphons, driven by Wright-Fisher diffusion and the type-connection matrix. The main results are recorded in Theorem 1.2.1 and Theorem 2.2.3.

Trajectorial representation: We shall now discuss the second objective of the thesis. In [37] Sznitman described a system of N Brownian spheres of radius $N^{-\frac{1}{d-2}}$ in $\mathbb{R}^d$, where $d \geq 3$. The spheres are destroyed as soon as a collision occurs between any two of them. These spheres are called Brownian spheres because the center of these spheres evolve according to independent Brownian motions, initially distributed as $u_0(dx) = V(x)dx$. We can refer the centers as particles and the spheres as balls of radius

$N^{-\frac{1}{d-2}}$ around the particles. Particles which does not get destroyed in the system are called alive. Sznitman showed that limiting (under $N \to \infty$) probability of each particle being alive is given by the solution, $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}_+$, of the nonlinear equation

$$\partial_t u = \frac{1}{2}\Delta u - c_d u^2, u(0, \cdot) = u_0$$

where c_d is a constant depending on dimension. The initial motivation was to try to investigate the connection between a detailed and a reduced description of particle evolution.

Further in [38] Sznitman used concept of wild trees from [40] and constructions in the proof of uniqueness of the Boltzmann process in [36] to construct a trajectorial representation for solution $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}_+$ of nonlinear equation given by

$$\partial_t u = \frac{1}{2}\Delta u - u^{1+k}, u(0, \cdot) = u_0$$

where $u_0 : \mathbb{R}^d \to \mathbb{R}_+$ and $k \in \mathbb{N}$. Here the trajectorial representation means expressing the solutions of the nonlinear system as a set of finite paths in $\mathbb{R}^d$. Each path is treated as a vertex of a tree. This trees are called marked trees. Paths meet their immediate ancestor path in a bundle of k paths. Later in [39] Sznitman studied a system of a N independent diffusions undergoing collisions and derived a propagation of chaos result when N goes to infinity. This a step in understanding why such a structure for the limit collision tree occurs between nondestructing independent particles should arise naturally in a constant mean free travel time regime.

In Chapter 3 we will generalise the above trajectorial representation for solution $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}_+$ of nonlinear equation given by

$$\partial_t u = \frac{1}{2}\Delta u - u^{1+\alpha}, u(0, \cdot) = u_0$$

where $u_0 : \mathbb{R}^d \to \mathbb{R}_+$ and $0 < \alpha < 1$. Since α is not an integer we modify the methods used in [39] by employing an α-stable subordinator. We first construct the space of marked trees by attaching finite $\mathbb{R}^d$ valued paths and a α-stable subordinator to each node of trees (finite or infinite). We place a suitable measure on the space of marked trees such that the trees are finite almost surely (see Lemma 3.2.4 and Lemma 3.4.5). We then define an interaction on the trees which decides whether a root path (finite path attached to the root of a tree) survives or gets destroyed under the interaction. Under the above measure we prove our main results, the surviving probability of the root path satisfies the above equation. The main results are recorded in Theorem 3.2.6 and Theorem 3.4.7.

Future work: Our focus in the results on dynamics of random graphs in this thesis are derived from models in population genetics. An interesting future direction of work would be to consider epidemiological models on co-evolvoing sparse graphs. We conclude this chapter with a brief discussion on epidemiological models, (such as Compartmental and agent based models), on networks that are of interest.

In epidemiology, modeling the spread of pathogens through social contact networks has a long-established tradition. Particularly for sexually transmitted infections, it is crucial to consider that partnership networks evolve at a similar pace to the infectious spread. Most mathematical analyses in this field employ compartmental models, originating from the work of Kermack and McKendrick [21]. These models generate systems of ordinary differential equations, which inherently overlook stochastic effects. Agent based models are also frequently used, but these are typically intractable mathematically (see Morris and Kretzschmar [26]). Holme and Saramäki [19] provide a comprehensive survey of simulation results from physics and computer science, highlighting numerous real-world applications and efforts to unify the various emerging sub-disciplines.

Compartmental models are a mathematical framework used to study the dynamics of infectious diseases by categorizing a population into compartments that represent different disease states. One of the most fundamental compartmental models is the SIR model, which categorizes individuals into three states: susceptible (S), those who can contract the disease; infectious (I), those who have the disease and can spread it; and recovered (R), those who have either gained immunity or died. The transition occurs when susceptible individuals come into contact with infectious individuals, leading to infection, and over time, infectious individuals move to the recovered compartment. The model is particularly useful for diseases where recovery confers lasting immunity. The SIR model serves as a foundation for more complex models like SEIR (which includes an exposed phase) or SIRS (which accounts for waning immunity) (see Ross, Ross & Hudson [31, 32], Kermack & McKendrick [21], and Kendall [20] and references there in.)

Agent-based models (ABMs) simulate disease spread by modeling individual interactions, capturing heterogeneity in immunity, contact patterns, and risk factors. Unlike aggregate models, which assume uniform mixing, ABMs incorporate network structures and emergent behaviors, providing greater accuracy in representing real-world disease dynamics. However, they are often intractable mathematically (see Morris and Kretzschmar [26]) and come with higher computational costs. Using these models to construct dynamically evolving sparse random graphs are interesting open questions.

Chapter 1

Random sparse graph dynamics: independent edges

In this chapter we construct a class of graphon-valued Markov processes from a sequence of dynamically evolving discrete sparse random graphs. We will consider graph on n-vertices and each vertex assigned a type which evolve dynamically according to Moran model from population genetics. An edge is placed between two vertices with probability depending on the type of those two vertices, independently of all other edges. We let n go to infinity and obtain a dynamics on space of sparse graphs which are governed by a limiting graphon. A similar approach in dense graph regime was carried out in the recent work of [1, Athreya et al.]. We will use the sparse graph limit theory from [7, Section 2.7].

1.1 Preliminaries

In this Section we will discuss the necessary preliminaries required to state and prove our main result. First in Section 1.1.1, we shall provide necessary background required on sparse graph sequences on n vertices. Second we shall understand the limit of sparse graph sequences, as $n \to \infty$, via symmetric integrable functions called graphons (see Section 1.1.1). We impose dynamics on the vertices by assigning types to each vertex which evolve according to a Moran model which we discuss in the Section 1.1.2. We conclude this section by discussing the fundamental concepts of weak convergence such as tightness on the space of graph valued processes.

Now we fix some notations which we will need. Let $f, g : \mathbb{N} \to \mathbb{R}_+$.

Small-O-notation: We write $f(n) = o(g(n))$ iff $\forall \epsilon > 0$ $\exists n_0$ such that $\forall n > n_0$ we have $f(n) < \epsilon g(n)$.

Big-O-notation: We write $f(n) = O(g(n))$ iff $\exists k > 0$ and $\exists n_0$ such that $\forall n > n_0$ we have $f(n) \leq kg(n)$.

Big-Θ-notation: We write $f(n) = \Theta(g(n))$ iff $\exists k_1 > 0$,$\exists k_2 > 0$ and $\exists n_0$ such that $\forall n > n_0$ we have $k_1 g(n) \leq f(n) \leq k_2 g(n)$.

Small-ω-notation: We write $f(n) = \omega(g(n))$ iff $\forall \epsilon > 0$, $\exists n_0$ such that $\forall n > n_0$ we have $f(n) > \epsilon g(n)$.

1.1.1 Sparse graph sequences

In this section, we will discuss sparse graph sequences and weighted graphs. We will establish a metric for measuring the distances between any two weighted graphs. We will only consider simple graphs, that is graphs without loops and multiple edges.

For $n \geq 1$, a graph $G^n (\equiv G)$ on a vertex set $V(G) := \{1, 2, \ldots, n\}$, the set of edges $E(G)$ is a subset of $\mathcal{P}_2(V(G))$ where $\mathcal{P}_2(V(G))$ is collection of two element subsets of $V(G)$.

Definition 1.1.1 (Sparse Graph). *A sequence of graphs $(G^n)_{n\geq 1}$, where each graph G^n is defined on n vertices $\{1, 2, \ldots, n\}$, is called a sparse graph sequence if the number of edges $E(G^n)$ satisfies the following two conditions:*

- *$E(G^n) = O(n^{1+\alpha})$, where $\alpha \in (0, 1)$. (The number of edges grows at most as $n^{1+\alpha}$, which is faster than linear but slower than quadratic in n.)*
- *$E(G^n) = \omega(n)$, (The number of edges grows faster than n, ensuring the graph is not too sparse.)*

A Dense graph sequence is a sequence of graphs on n vertices where the number of edges is $\Theta(n^2)$ while in a extremely sparse sequence of graphs with n vertices the number of edges is $o(n)$ edges.

We will need to consider weighted graphs to understand the sparse graph limits. Weighted graphs are graphs in which each edge is assigned a weight which is a non-negative real number. These weights may be used to represent the concepts like distances, costs, or strength of connection between vertices in a real world network.

Definition 1.1.2 (Weighted Graph). *A graph G on a finite vertex set $V(G)$ and with an edge set $E(G)$ is called a weighted graph if there exists a weight function $\beta : V(G) \times V(G) \to [0, \infty)$ which satisfies:*

(i) $\beta_{ij}(G) = \beta_{ji}(G)$ *for all* $i, j \in V$,

(ii) $\beta_{ij}(G) \geq 0$ *for all* $i, j \in V$ *and* $\beta_{ii}(G) = 0$ *for all* $i \in V$,

(iii) If $i \neq j$ *then* $\beta_{ij}(G) > 0$ *iff* $\{i, j\} \in E(G)$.

Here, we interpret the value $\beta_{ij}(G)$ as the weight of the edge $\{i, j\}$. Also we write $\beta(G) = (\beta_{ij}(G))_{i,j\in V(G)}$. Note that, $E(G) = \{\ \{i, j\} : i \neq j, \beta_{ij}(G) > 0\}$.

For any weighted graph G and constant $c > 0$, we shall define cG to be weighted graph on the same set of vertices and edge weights $\beta_{ij}(cG) = c\beta_{ij}(G)$. Moreover any unweighted graph can be interpreted as a weighted graph by setting $\beta_{ij}(G) = 1$ whenever $\{i, j\} \in E(G)$.

Borges et al. addressed the issue of convergence of sparse graph sequences in [7] and [8]. They showed that under a suitable metric a Cauchy sequence of graphs converges to functions called graphon which we define next.

Definition 1.1.3 (Graphon). *A graphon is any symmetric function $\kappa : [0, 1]^2 \to \mathbb{R}_+$ which is integrable under Lebesgue measure. Let $\mathcal{W}$ be the space of all graphons.*

Any weighted graph can be thought of as a graphon. More precisely let G^n be a weighted graph with vertices $V(G_n) = \{1, 2, \ldots, n\}$ and edge set $E(G^n)$. First divide the interval $[0, 1]$ into intervals

$I_k = [\frac{k-1}{n}, \frac{k}{n})$ for $k = 1, \ldots, n-1$ and $I_n = [\frac{n-1}{n}, 1]$ of length $\frac{1}{n}$. We define the function $\kappa_{G^n} : [0,1]^2 \to \mathbb{R}_+$ by assigning it the constant value $\beta_{ij}(G^n)$ on $I_i \times I_j$ for every $i, j \in V(G^n)$. So κ_{G^n} is symmetric, simple Lebesgue measurable function on $[0,1]^2$. The representation of a graph via a graphon is not unique because it is invariant with respect to vertex labels. Indeed, many graphs may have the same graphon representation. We consider equivalence classes of graphons, where the equivalence is determined by measure preserving bijections. More precisely, if $\kappa_1, \kappa_2 \in \mathcal{W}$ we say $\kappa_1 \equiv \kappa_2$ if $\exists\ \sigma : [0,1] \to [0,1]$, a measure preserving bijection such that

$$\kappa_1(x,y) = \kappa_2(\sigma x, \sigma y), \quad x, y \in [0,1].$$

Let $\widetilde{\mathcal{W}}$ be the quotient space under the above equivalence relation on the space $\mathcal{W}$. From here on we think of each weighted graph G as a graphon κ_G, which is the representation of G from $\widetilde{\mathcal{W}}$. Now we define the distance between two graphons, distance between two graphs or distance between a graph and a graphon.

Definition 1.1.4 (Cut norm)**.** *For any measurable function $\kappa : [0,1]^2 \to \mathbb{R}$, the cut norm of κ is defined as:*

$$\|\kappa\|_\square = \sup_{S,T \subseteq [0,1]} \left| \int_{S \times T} \kappa(x,y)\,dxdy \right|$$

where S, T are Lebesgue measurable subsets of [0,1].

For any two graphons κ_1 and κ_2 we let,

$$d_\square(\kappa_1, \kappa_2) = \|\kappa_1 - \kappa_2\|_\square. \tag{1.1.1}$$

Now if two weighted graphs G^n and G_1^n have the same set of vertices $V(G^n)$, then it is clear that we can express their cut-distance as

$$d_\square(G^n, G_1^n) = \max_{S,T \subset V(G^n)} \left| \sum_{i \in S, j \in T} (\beta_{ij}(G^n) - \beta_{ij}(G_1^n)) \right|.$$

We define the cut metric.

Definition 1.1.5 (Cut metric)**.** *Define the cut metric*

$$\delta_\square(\kappa_1, \kappa_2) := \inf_\sigma d_\square({\kappa_1}^\sigma, \kappa_2)$$

where infimum ranges over all measure preserving bijections $\sigma : [0,1] \to [0,1]$ and where the graphon κ^σ is defined as $\kappa^\sigma(x,y) = \kappa(\sigma(x), \sigma(y))$.

The L^1- norm of κ is given by

$$\|\kappa\|_1 = \int_{[0,1]\times[0,1]} |\kappa(x,y)|\,dxdy.$$

The distance $\delta_\square$ between two graphons is a pseudometric. We have a equivalence relation $\kappa_1 \equiv \kappa_2$ iff $\kappa_1(x,y) = \kappa_2(\sigma(x), \sigma(y))$ for some measure preserving bijection $\sigma : [0,1] \to [0,1]$. Then define $\widetilde{\mathcal{W}}$ to be the collection of equivalence classes of graphons. If $\kappa \in \mathcal{W}$, then, without loss of generality, we may use κ to denote the equivalence class in $\widetilde{\mathcal{W}}$ that contains κ.

Now $\delta_\square$ will be a metric on $\widetilde{\mathcal{W}}$. Then L^1-norms of κ_1 and κ_2 are same iff $\kappa_1 \equiv \kappa_2$. Define $\widetilde{\mathcal{W}}_1 := \{\kappa \in \widetilde{\mathcal{W}} : \|\kappa\|_1 \leq 1\}$. Then, $(\widetilde{\mathcal{W}}_1, \delta_\square)$ is a compact metric space, see [7, Theorem 2.13].

For any two graphons κ_1 and κ_2 define

$$d_1(\kappa_1, \kappa_2) := \|\kappa_1 - \kappa_2\|_1 \tag{1.1.2}$$

and note that

$$\delta_\square(\kappa_1, \kappa_2) \leq d_\square(\kappa_1, \kappa_2) \leq d_1(\kappa_1, \kappa_2). \tag{1.1.3}$$

In graph context, even if the weighted graphs G and G_1 have different set of vertices, we can define their cut-distance through the cut-distance of their associated graphons, that is,

$$\delta_\square(G, G_1) := \delta_\square(\kappa_G, \kappa_{G_1}). \tag{1.1.4}$$

For a graphon κ and weighted graph G we define

$$d_\square(G, \kappa) := d_\square(\kappa_G, \kappa), \quad \delta_\square(G, \kappa) := \delta_\square(\kappa_G, \kappa).$$

Now we construct a sparse random graph sequence from a graphon κ as detailed in [7].

Sparse κ-random graph model. Let κ be a graphon. Let $\rho_n > 0$ satisfy $\rho_n \to 0$ and $n\rho_n \to \infty$ as $n \to \infty$. Let the vertex set be given by $[n] = \{1, 2, \ldots, n\}$. Let $U_1, \ldots, U_n$ be i.i.d. random variables chosen uniformly in [0,1]. We define $G^n \equiv G(n, \kappa, \rho_n)$ to be the graph defined by connecting i and j with probability $\min\{\rho_n \kappa(U_i, U_j), 1\}$. Next we state the result that under suitable scaling the sparse random graph sequence converges to the respective graphon.

Theorem 1.1.6. *Let κ be a graphon and $G^n \equiv G(n, \kappa, \rho_n)$ be a sparse graph sequence. If $\rho_n > 0$ satisfy $\rho_n \to 0$ and $n\rho_n \to \infty$ as $n \to \infty$. Then we have,*

$$d_\square\left(\frac{G^n}{\rho_n}, \kappa\right) \to 0 \ \textit{and} \ \delta_\square\left(\frac{G^n}{\|G^n\|_1}, \frac{\kappa}{\|\kappa\|_1}\right) \to 0$$

as $n \to \infty$ with probability 1.

Proof. See [7, Theorem 2.14 and Corollary 2.15]. □

1.1.2 Moran model

In this section we discuss Moran model from population genetics and it's limit. The Moran model introduced by P.A.P. Moran in 1958, (see [25]), is a mathematical model used to describe the evolution of allele frequencies within a fixed-size population. It is a foundational tool in population genetics and evolutionary theory. The Moran model was motivated by the need to study how genetic drift and natural selection of alleles(individuals) in finite populations over time. The process has been studied extensively in evolutionary biology, ecology, and mathematical biology to understand fixation probabilities, extinction events, and evolutionary dynamics. It also finds relevance in areas like game theory and evolutionary graph theory, where it is used to model competition or selection processes in structured populations.

The $m-$multitype Moran model has $m+1$ types, which are labelled as $0,...,m$. We consider n individuals with each individual having a type from $\{0,\ldots,m\}$. We assume at rate 1 each individual independently adopts the type of randomly chosen individual. Let, $X_l^{m,n}(s)$ denote the number of individuals of type l at time s, where $0\le l\le m-1$ and

$$X^{m,n}(s)=(X_0^{m,n}(s),\ldots,X_{m-1}^{m,n}(s))$$

be corresponding vector of type counts. We exclude $X_m^{m,n}(s)$ from $X^{m,n}(s)$ because it is fully determined by the constraint

$$X_m^{m,n}(s)=n-\sum_{l=0}^{m-1}X_l^{m,n}(s),$$

making its inclusion redundant. This simplifies the model by reducing its dimensionality without losing any information.

Consider the space time rescaling

$$Y^{m,n}(s)=\frac{1}{n}X^{m,n}(ns),s\ge 0$$

which consists m components

$$Y^{m,n}(s)=(Y_0^{m,n}(s),\ldots,Y_{m-1}^{m,n}(s)).$$

One may view $Y^{m,n}\equiv\{Y^{m,n}(s)\}_{s\ge 0}$ as processes in the Skorohod space $D([0,\infty),[0,1]^m)$. It is known (see Dawson [15, Section 2]) that if

$$Y^{m,n}(0)\Rightarrow Y^m(0)\quad(n\to\infty)\text{ then }Y^{m,n}\Rightarrow Y^m\quad(n\to\infty)\tag{1.1.5}$$

where " $\Rightarrow$ " denotes weak convergence at the process level under the Skorohod topology. The limiting process consists of m components

$$Y^m(s)=(Y_0^m(s),...,Y_{m-1}^m(s)),$$

taking values in the $m-$dimensional simplex

$$\mathcal{S}^m=\left\{y=(y_0,\ldots,y_{m-1})\in\mathbb{R}^m: y_l\ge 0\text{ for all }0\le l\le m-1,\sum_{l=0}^{m-1}y_l\le 1\right\},$$

and is referred to as the *Wright-Fisher diffusion.* The diffusion has generator,

$$(L^m\phi)(y)=\sum_{0\le k<l<m}y_ky_l\frac{\partial^2\phi(y)}{\partial y_k\partial y_l},\quad y\in\mathcal{S}^m,$$

for test functions $\phi:\mathcal{S}^m\to\mathbb{R}$ that are twice continuously differentiable in each coordinate.

We can write

$$Y_m^m(s)=1-\sum_{l=0}^{m-1}Y_l^m(s).$$

Now let us define the cumulative distribution function of the type distribution, defined as

$$F^m(s;x) := \sum_{l=0}^{\lfloor (m+1)x \rfloor} Y_l^m(s), \quad x \in [0,1), \tag{1.1.6}$$

Observe that for fixed $s > 0$, $x \to F^m(s;x)$ can have jumps at $x \in \{0, \frac{1}{m+1}, \dots, \frac{m}{m+1}\}$. Now we will be constructing sequences of graphon valued stochastic processes based on Moran model.

1.2 Sparse graph dynamics and Main result

In this section we will state our main result (Theorem 1.2.1). Before that we will discuss càdlàg paths, weak convergence of graphons on the metric space $(\widetilde{\mathcal{W}}_1, \delta_\square)$ and graphon dynamics following from Moran model.

Let X and Y be two metric spaces. A function $f : X \to Y$ is said to have càdlàg paths if f is right-continuous at every point and f has left-hand limits at every point. We will denote by $D(X,Y)$ the collection of functions $f : X \to Y$ that are càdlàg.

We will use " $\to$ " to denote convergence with respect to the metric space $(\widetilde{\mathcal{W}}_1, \delta_\square)$ and we will use " $\Rightarrow$ " to denote weak convergence for stochastic processes taking values in $D([0,\infty), \widetilde{\mathcal{W}}_1)$, the set of cádlág paths in $\widetilde{\mathcal{W}}_1$.

Let $\tilde{h}(s)$ be a stochastic process taking values in $D([0,\infty), \widetilde{\mathcal{W}}_1)$. We will write $\tilde{h}(s)$ to denote the value of the process at time $s \geq 0$, which is an equivalence class of graphons. If $h(s)$ is a representative graphon of the equivalence class $\tilde{h}(s)$, we write $h(s;x,y)$ to denote the value of that graphon evaluated at coordinates $(x,y) \in [0,1]^2$.

We will use the Moran model to construct a sequence of random graphs that evolve in time. Consider the interval $[0,1]$ with the Euclidean metric, and let $\mathcal{L} = C([0,1),[0,1))$, the space of continuous functions from $[0,1)$ to $[0,1)$ endowed with the uniform topology. We call $\mathcal{L}$ the *fitness landscape*. It was introduced by Athreya et al. in [1] to construct dynamics of discrete dense graphs. The fitness landscape plays a crucial role in modeling the effects of random selection of individuals in a finite population. It determines how likely type of an individual to be adopted compared to other types, influencing the evolutionary dynamics of the population. In the definition of $\mathcal{L}$, the first appearance of $[0,1]$ represents a space of types, and the second appearance of [0, 1] represents some sort of fitness associated with each type through a function from $\mathcal{L}$.

Let us assume the following:

(R1) Let $r : [0,1] \times [0,1] \to [0,\infty)$ be a function which is symmetric and integrable under Lebesgue measure. We further assume that r is continuous at (x,y) if $(x,y) \in [0,1) \times [0,1)$ and $r(x,y) = 0$ if $x = 1$ or $y = 1$. Note that r can be unbounded.

(H1) For $m,n \in \mathbb{N}$, we consider $H^{m,n}, H^m$ to be random elements in $D([0,\infty),\mathcal{L})$, which will be referred to as *fitness landscape processes*. We write $H^m(s)$ for the fitness landscape at time s, and $H^m(s;x)$ for the fitness of type x at time s. Assume $H^{m,n} \Rightarrow H^m$ as $n \to \infty$.

(Y1) $Y^{m,n}(0) \Rightarrow Y^m(0)$ as $n \to \infty$. We assume $Y^{m,n}, Y^m, H^{m,n}, H^m$ are all defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We denote by $\bar{F}$ to be the right-continuous generalised inverse of a distribution function F with support $[0,1]$, defined in the following way

$$\bar{F}(u) := \inf\{x \in [0,1] : F(x) > u\}, \quad u \in [0,1), \tag{1.2.1}$$

and, for convenience, we set $\bar{F}(1) := \lim_{x \uparrow 1} \bar{F}(x)$.

Now we proceed towards constructing discrete sparse graph sequence.

Construction of random sparse graphs. Consider n vertices which follow the $m-$multi type Moran model defined in Section 1.1.2. Let $\tau_i^{m,n}(s)$ denote the type of individual i divided by $m+1$, at time ns. At time ns if the type of individual i equals k, where $0 \leq k \leq m$, then $\tau_i^{m,n}(s) = \frac{k}{m+1}$. We are going to follow sparse κ-random graph model (recall Section 1.1) to construct time varying discrete random sparse graphs. We first fix m and n. We have vertex set $[n] = \{1, 2, \ldots, n\}$, these vertices follow the Moran model. We are going to construct graph $G^{m,n}(s)$ on n vertices at time s.

Let, $\{U_{ij}^n : n \in \mathbb{N}, 1 \leq i < j \leq n\}$ being a collection of independent uniform random variables on $[0,1]$. We assume $\{U_{ij}^n : n \in \mathbb{N}, 1 \leq i < j \leq n\}$ are all defined on common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Now, we can construct graph $G^{m,n}(s)$ at time $s \geq 0$ by connecting i and j if

$$U_{ij}^n < \min\left\{\rho_n r\left(H^{m,n}(s;\cdot)(\tau_i^{m,n}(s)),\ H^{m,n}(s;\cdot)(\tau_j^{m,n}(s))\right),\ 1\right\}$$

where $\{\rho_n\}_{n\geq 1}$ is a fixed sequence such that $\rho_n \to 0$ and $n\rho_n \to \infty$ as $n \to \infty$. Thus, $(G^{m,n})_{n\in\mathbb{N}}$ is a sequence of random sparse graph processes, which evolve in time due to induced dynamics of the Moran model of $m+1$ types. By using the same set of edge variables $\{U_{ij}^n : n \in \mathbb{N}, 1 \leq i < j \leq n\}$ for the entire period, the construction ensures that the edges themselves are not randomized at every time step and that the graph's evolution is entirely driven by the type dynamics from the Moran process or the fitness landscape.

Note that the structure of $G^{m,n}(s)$ depends on the stae of the Moran model at time ns. Also the types of the vertices are influencing the existence of edges between them through the function $\tau_i^{m,n}(s)$. Now we state our main result of this chapter. Recall $F^m, \bar{F}$ from (1.1.6) and (1.2.1).

Theorem 1.2.1. *Let $G^{m,n}$ be constructed as above. Assume* **(R1)**, **(H1)** *and* **(Y1)**. *Then we have*

$$\frac{G^{m,n}}{\|G^{m,n}\|_1} \Rightarrow \frac{\tilde{h}^m}{\|\tilde{h}^m\|_1} \quad (\text{as } n \to \infty)$$

where for $s \geq 0$ the equivalence class $\tilde{h}^m(s)$ has a representative $h^m(s)$ of the form

$$h^m(s;x,y) = \begin{cases} r(H^m(s;\bar{F}^m(s;x)), H^m(s;\bar{F}^m(s;x))) & \text{if } (x,y) \in [0,1) \times [0,1) \\ 0 & \text{otherwise} \end{cases}$$

In $(G^{m,n})_{n\in\mathbb{N}}$, the probability for two vertices to be connected, depends on the types associated with the vertices. The limiting graphon is actually a $(m+1)\times(m+1)$-block graphon whose block boundaries move according to the Wright-Fisher diffusion and block heights or values are controlled by H^m.

We will finish this section with an elementary example to understand our main result, Theorem 1.2.1. We will explain the result in the case of $m = 1$. The limiting graphon h^1 will be an 2×2-block graphon, whose block boundaries are influenced by Wright-Fisher diffusion. And the block heights will depend on H^1.

Example 1.2.2. *Let*

$$H^{1,n}(s,u) = H^1(s;u) = u$$

and $r : [0,1] \times [0,1] \to [0,\infty)$ *be a function which is symmetric and integrable under Lebesgue measure.* r *is continuous at* (x,y) *if* $(x,y) \in [0,1) \times [0,1)$ *and* $r(x,y) = 0$ *if* $x = 1$ *or* $y = 1$.

Then at time s, $G^{1,n}(s)$ *is the random graph with vertex set* $[n] = \{1,2,\dots,n\}$, *in which, vertices* i *and* j *are connected by an edge if*

$$U_{ij}^n < \min\{\rho_n r(\tau_i^{1,n}(s), \tau_j^{1,n}(s)), 1\}.$$

Then by Theorem 1.2.1, as $n \to \infty$, *the graph process* $\frac{G^{1,n}}{\|G^{1,n}\|_1}$ *converges to* $\frac{\tilde{h}^1}{\|\tilde{h}^1\|_1}$. *Here, the connection probability of individuals* i *and* j *does not depend on the rest of the population.*

Then r *is being evaluated at the points* $\{0, \frac{1}{2}\} \times \{0, \frac{1}{2}\}$. *Then without loss of generality, we can take* r *as a* 2×2 *matrix, namely,*

$$r = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

which gives the representation

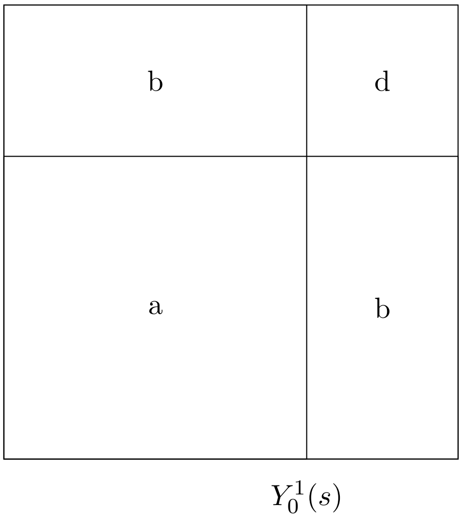

FIGURE 1.1: Graphical representaion of $h^1(s)$, with $Y_0^1(s)$ the fraction of Type 0 in the population at time s.

$$h^1(s;x,y) = \begin{cases} a & \text{if } (x,y) \in [0, Y_0^1(s)) \times [0, Y_0^1(s)) \\ d & \text{if } (x,y) \in [Y_0^1(s), 1) \times [Y_0^1(s), 1) \\ b & \text{if } (x,y) \in [Y_0^1(s), 1) \times [0, Y_0^1(s)) \\ b & \text{if } (x,y) \in [Y_0^1(s), 1) \times [0, Y_0^1(s)) \end{cases}$$

Similarly for general m, r *will be evaluated at the points* $\left\{0, \frac{1}{m+1}, \dots, \frac{m}{m+1}\right\} \times \left\{0, \frac{1}{m+1}, \dots, \frac{m}{m+1}\right\}$, *then we can view* r *as a matrix* $(r_{ij})_{0 \le i,j \le m}$.

1.3 Proof of Theorem 1.2.1

In this section we first state and prove necessary lemmas related to convergence of finite dimensional distribution and tightness of graphon valued stochastic processes.

1.3.1 Weak convergence of graphon valued stochastic processes

Let $(\tilde{h}^n)_{n\in\mathbb{N}}$ be a sequence of $D([0,\infty),\widetilde{\mathcal{W}}_1)$-valued stochastic processes. In order to prove weak convergence in the graphon space, that is, $\tilde{h}^n \Rightarrow \tilde{h}$ in $D([0,\infty),\widetilde{\mathcal{W}}_1)$, we need to establish (see [17, Theorem 7.8])

(*i*) For any $d \geq 1$, convergence of finite dimensional distributions $((\tilde{h}^n(s_i))_{1\leq i\leq d} \Rightarrow ((\tilde{h}(s_i))_{1\leq i\leq d}$ for all points $s_1,\ldots,s_d \geq 0$ at which $\tilde{h}$ is continuous almost surely and

(*ii*) Tightness of the sequence $(\tilde{h}^n)_{n\in\mathbb{N}}$.

Lemma 1.3.1. *Let $\tilde{h}^n, \tilde{h}$ in $D([0,\infty),\widetilde{\mathcal{W}}_1)$. If $\tilde{h}^n(s) \to \tilde{h}(s)$ almost surely as $n\to\infty$ whenever s is a point at which $\tilde{h}$ is continuous. Then we have $((\tilde{h}^n(s_i))_{1\leq i\leq d} \Rightarrow ((\tilde{h}(s_i))_{1\leq i\leq d}$ for all points $s_1,\ldots,s_d \geq 0$ at which $\tilde{h}$ is continuous almost surely.*

Proof. Let $\tilde{h}^n(s) \to \tilde{h}(s)$ almost surely as $n\to\infty$ whenever s is a point at which $\tilde{h}$ is continuous. Let $s_1,\ldots,s_d \geq 0$ be points at which $\tilde{h}$ is continuous almost surely and $f_1,\ldots,f_d$ be continuous functions on $(\tilde{\mathcal{W}}_1,\delta_\square)$. Then we have,

$$\lim_{n\to\infty} f_1(\tilde{h}^n(s_1))\ldots f_d(\tilde{h}^n(s_d)) = f_1(\tilde{h}(s_1))\ldots f_d(\tilde{h}(s_d))$$

almost surely.

Since, $(\tilde{\mathcal{W}}_1,\delta_\square)$ is compact we have $f_1,\ldots,f_d$ are bounded. Then using bounded convergence theorem, we have,

$$\lim_{n\to\infty} E[f_1(\tilde{h}^n(s_1))\ldots f_d(\tilde{h}^n(s_d))] = E[f_1(\tilde{h}(s_1))\ldots f_d(\tilde{h}(s_d))].$$

This implies, $((\tilde{h}^n(s_i))_{1\leq i\leq d} \Rightarrow ((\tilde{h}(s_i))_{1\leq i\leq d}$ for all points $s_1,\ldots,s_d \geq 0$ at which $\tilde{h}$ is continuous almost surely.

$\square$

Remark 1.3.1. *Assuming the Lemma 1.3.1 and using [17, Theorem 7.8] we can conclude that $\tilde{h}^n \Rightarrow \tilde{h}$ in $D([0,\infty),\widetilde{\mathcal{W}}_1)$ iff the following hold*

(*i*) *$\tilde{h}^n(s) \to \tilde{h}(s)$ almost surely as $n\to\infty$ whenever s is a point at which $\tilde{h}$ is continuous.*

(*ii*) *Tightness of the sequence $(\tilde{h}^n)_{n\in\mathbb{N}}$.*

Tightness of processes: Let (E,r) be a compact metric space. This implies (E,r) is also complete and separable. Let, $D([0,\infty),E)$ be the set of cádlág paths in E. Cádlág paths are right continuous and have left limits everywhere. Then, under suitable metric d (see Ethier and Kurtz [17]), we will have, $D([0,\infty),E)$ is complete and separable since (E,r) is also complete and separable (see Ethier and Kurtz [17, Theorem 5.6]).

Lemma 1.3.2. *(E, r) be a compact metric space. Let $\{X_n\}_{n\geq 1}$ be a sequence of processes taking values in $D([0,\infty), E)$. Then, we have, the sequence $\{X_n\}_{n\geq 1}$ is tight iff for every $\eta > 0$ and $T > 0$ $\exists$ $\epsilon > 0$ such that*

$$\sup_n \mathbb{P}(w'(X_n, \epsilon, T) > \eta) \leq \eta$$

where $w'(X_n, \epsilon, T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} r(X_n(s), X_n(t))$ and $\{t_i\}$ ranges over all partition of form $0 \leq t_0 < t_1 < \ldots < t_{n-1} < T = t_n$ with $\min_{1\leq i\leq n}(t_i - t_{i-1}) > \epsilon$.

Proof of Lemma 1.3.2. The proof of the Lemma is standard but we present it here for completeness sake. Suppose (E, r) be a compact metric space, then (E, r) is also complete and separable. Let $\{X_n\}_{n\geq 1}$ be a sequence of processes taking values in $D([0,\infty), E)$. Then using Ethier and Kurtz [17, Corollary 7.4], and compactness of (E, r) we can easily conclude that the sequence $\{X_n\}_{n\geq 1}$ is relatively compact iff for every $\eta > 0$ and $T > 0$ $\exists$ $\epsilon > 0$ such that

$$\sup_n \mathbb{P}(w'(X_n, \epsilon, T) > \eta) \leq \eta$$

where $w'(X_n, \epsilon, T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} r(X_n(s), X_n(t))$ and $\{t_i\}$ ranges over all partition of form $0 \leq t_0 < t_1 < \ldots < t_{n-1} < T = t_n$ with $\min_{1\leq i\leq n}(t_i - t_{i-1}) > \epsilon$.

Next, $D([0,\infty), E)$ is complete and separable since (E, r) is also complete and separable (see Ethier and Kurtz [17, Theorem 5.6]).

Then, Ethier and Kurtz [17, Theorem 2.2], implies that the sequence $\{X_n\}_{n\geq 1}$ is relatively compact iff the sequence $\{X_n\}_{n\geq 1}$ is tight. Since $\{X_n\}_{n\geq 1}$ takes values in $D([0,\infty), E)$ which is complete and separable.

Finally we can conclude, the sequence $\{X_n\}_{n\geq 1}$ is tight iff for every $\eta > 0$ and $T > 0$ $\exists$ $\epsilon > 0$ such that

$$\sup_n \quad \mathbb{P}(w'(X_n, \epsilon, T) > \eta) \leq \eta$$

where $w'(X_n, \epsilon, T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} r(X_n(s), X_n(t))$ and $\{t_i\}$ ranges over all partition of form $0 \leq t_0 < t_1 < \ldots < t_{n-1} < T = t_n$ with $\min_{1\leq i\leq n}(t_i - t_{i-1}) > \epsilon$.

$\square$

1.3.2 Proof of Theorem 1.2.1

Recall from 1.2, $H^{m,n}, H^m$ as in **(H1)** and $Y^{m,n}, Y^m$ as in **(Y1)** as stated in Section 1.2. Then (1.1.5) implies $Y^{m,n} \Rightarrow Y^m$ as $n \to \infty$ and from **(H1)** we have $H^{m,n} \Rightarrow H^m$ as $n \to \infty$. Using Skorohod representation, we may assume there exists a probability space on which we can define distributionally equivalent $Y^{m,n}, Y^m, H^{m,n}, H^m$ in such a way that

$$(Y^{m,n}, H^{m,n}) \to (Y^m, H^m) \text{ as } n \to \infty \quad \mathbb{P} - a.s. \tag{1.3.1}$$

Define the graph $h^{m,n}(s)$ on vertices $[n]$ at time s, whose edge weights given by

$$\beta_{ij}(h^{m,n}(s)) := r(H^{m,n}(s; \tau_i^{m,n}(s)), H^{m,n}(s; \tau_j^{m,n}(s))).$$

Recall $\{\rho_n\}_{n\geq 1}$ is a fixed sequence such that $\rho_n \to 0$ and $n\rho_n \to \infty$ as $n \to \infty$. Define the graph $K^{m,n}(s)$ on vertices $[n]$ at time s, whose edge weights are

$$\beta_{ij}(K^{m,n}(s)) := \min\{\rho_n r(H^{m,n}(s;\tau_i^{m,n}(s)), H^{m,n}(s;\tau_j^{m,n}(s))), 1\}. \tag{1.3.2}$$

From Remark 1.3.1 it is immediate that Theorem 1.2.1 will follow if

(S1) Whenever s is a point at which $\tilde{h}^m$ is continuous, we have $\mathbb{P}$-a.s.

$$\delta_\Box\left(\frac{G^{m,n}(s)}{\|G^{m,n}(s)\|_1}, \frac{\tilde{h}^m(s)}{\|\tilde{h}^m(s)\|_1}\right) \to 0 \text{ as } n \to \infty \tag{1.3.3}$$

and

(S2)

$$\left\{\frac{G^{m,n}}{\|G^{m,n}\|_1}\right\}_{n\geq 1} \text{ is tight in } \widetilde{W}_1 \tag{1.3.4}$$

hold. We will now prove **(S1)** and **(S2)**.

Proof of (S1). Recall for $s \geq 0$ the equivalence class $\tilde{h}^m(s)$ has a representative $h^m(s)$ of the form

$$h^m(s;x,y) = \begin{cases} r(H^m(s;\bar{F}^m(s;x)), H^m(s;\bar{F}^m(s;x))) & \text{if } (x,y) \in [0,1) \times [0,1) \\ 0 & \textit{otherwise} \end{cases}$$

We make the following claim.

Claim 1: For $s \geq 0$, whenever s is a point at which $\tilde{h}^m$ is continuous, we have, $\mathbb{P}$-a.s.

$$\delta_\Box\left(\frac{G^{m,n}(s)}{\rho_n}, \tilde{h}^m(s)\right) \to 0 \quad as \quad n \to \infty. \tag{1.3.5}$$

We will complete the proof of this theorem assuming the above claim. We defer its proof to Section 1.3.3. This implies, whenever s is a point at which $\tilde{h}^m$ is continuous, we have, $\mathbb{P}$-a.s.

$$\left|1 - \frac{\|G^{m,n}(s)\|_1}{\rho_n\|\tilde{h}^m(s)\|_1}\right| \to 0 \quad as \quad n \to \infty \tag{1.3.6}$$

and $\mathbb{P}$-a.s. we have that the sequence $\left\{\frac{\rho_n}{\|G^{m,n}(s)\|_1}\right\}_{n\geq 1}$ is bounded. Further, whenever s is a point at which $\tilde{h}^m$ is continuous, we have (see [7, Proof of Corollary 2.15]), $\mathbb{P}$-a.s.

$$\begin{aligned} &\delta_\Box\left(\frac{G^{m,n}(s)}{\|G^{m,n}(s)\|_1}, \frac{h^m(s)}{\|h^m(s)\|_1}\right) \\ \leq &\frac{\rho_n}{\|G^{m,n}(s)\|_1}\left(\delta_\Box(\rho_n^{-1}G^{m,n}(s), h^m(s)) + \|h^m(s)\|_1\left|1 - \frac{\|G^{m,n}(s)\|_1}{\rho_n\|h^m(s)\|_1}\right|\right). \end{aligned} \tag{1.3.7}$$

Then from (1.3.5), (1.3.6) and the above we get, $\mathbb{P}$-a.s.

$$\delta_\Box\left(\frac{G^{m,n}(s)}{\|G^{m,n}(s)\|_1}, \frac{h^m(s)}{\|h^m(s)\|_1}\right) \to 0 \quad as \quad n \to \infty.$$

This implies **(S1)**. □

Proof of (S2). Let $s \geq 0$ be given. We have

$$\delta_{\square}\left(\frac{G^{m,n}(s)}{\|G^{m,n}(s)\|_1}, \frac{K^{m,n}(s)}{\|K^{m,n}(s)\|_1}\right) \leq \frac{\rho_n}{\|K^{m,n}(s)\|_1}\left\{\frac{\delta_{\square}(G^{m,n}(s), K^{m,n}(s)) + \Big|\|K^{m,n}(s)\|_1 - \|G^{m,n}(s)\|_1\Big|}{\rho_n}\right\}. \tag{1.3.8}$$

It can be easily seen that for $s \geq 0$ and $\eta > 0$

$$\left\{\frac{\delta_{\square}(G^{m,n}(s), K^{m,n}(s)) + \Big|\|K^{m,n}(s)\|_1 - \|G^{m,n}(s)\|_1\Big|}{\rho_n} > \frac{\|K^{m,n}(s)\|_1}{\rho_n}\eta\right\} \subset \left\{\rho_n^{-1} d_{\square}(G^{m,n}(s), K^{m,n}(s)) > \frac{\|K^{m,n}(s)\|_1}{\rho_n}\frac{\eta}{2}\right\}. \tag{1.3.9}$$

In the $m-$multitype Moran model note that the number of resampling events in the interval $[0, T]$ follows a Poisson distribution of mean nT. Here $T > 0$ is fixed. Let $A_{n,m}(T)$ be the event that the number of resampling events is not more than n^2 in the interval $[0, T]$. So let $Z \sim Poisson(nT)$ then we have,

$$\mathbb{P}[(A_{n,m}(T))^c] = \mathbb{P}[Z > n^2] \leq \frac{\mathbb{E}[Z^2]}{n^4} \leq \frac{2T^2}{n^2}. \tag{1.3.10}$$

We shall now condition on $H^{m,n}, Y^{m,n}, H^m, Y^m \; \forall n \geq 1$ and define the conditional probability measure $\mathbb{P}'(\cdot)$ as

$$\mathbb{P}'(\cdot) = \mathbb{P}(\cdot | H^{m,n}, Y^{m,n}, H^m, Y^m; n \geq 1). \tag{1.3.11}$$

Then using (1.3.8) on the event $A_{n,m}$ we get for $\eta > 0$,

$$\mathbb{P}'\left[\sup_{s\in[0,T]} \delta_{\square}\left(\frac{G^{m,n}(s)}{\|G^{m,n}(s)\|_1}, \frac{K^{m,n}(s)}{\|K^{m,n}(s)\|_1}\right) > \eta \,\Big|\, A_{n,m}(T)\right] \leq \mathbb{P}'\left[\sup_{s\in[0,T]} \frac{\delta_{\square}(G^{m,n}(s), K^{m,n}(s)) + \Big|\|K^{m,n}(s)\|_1 - \|G^{m,n}(s)\|_1\Big|}{\rho_n} > \frac{\|K^{m,n}(s)\|_1}{\rho_n}\eta \,\Big|\, A_{n,m}(T)\right]. \tag{1.3.12}$$

Then using (1.3.9) and the above we get,

$$\mathbb{P}'\left[\sup_{s\in[0,T]} \delta_{\square}\left(\frac{G^{m,n}(s)}{\|G^{m,n}(s)\|_1}, \frac{K^{m,n}(s)}{\|K^{m,n}(s)\|_1}\right) > \eta \,\Big|\, A_{n,m}(T)\right] \leq \mathbb{P}'\left[\sup_{s\in[0,T]} \rho_n^{-1} d_{\square}(G^{m,n}(s), K^{m,n}(s)) > \frac{\|K^{m,n}(s)\|_1}{\rho_n}\frac{\eta}{2} \,\Big|\, A_{n,m}(T)\right]. \tag{1.3.13}$$

In the above equation, on the event $A_{n,m}$ we apply a union bound over all resampling events and using from Borgs et al. [7, Lemma 7.3], we get

$$\mathbb{P}'\left[\sup_{s\in[0,T]} \delta_{\square}\left(\frac{G^{m,n}(s)}{\|G^{m,n}(s)\|_1}, \frac{K^{m,n}(s)}{\|K^{m,n}(s)\|_1}\right) > \eta \,\Big|\, A_{n,m}(T)\right] \leq \frac{n^3}{2^{\omega(n)}}.$$

Claim 2: $\{\frac{K^{m,n}}{\|K^{m,n}\|_1}\}_{n\geq 1}$ is tight in $\widetilde{W}_1$.

We will prove the claim in Section 1.3.3. At present we assume the claim. Then using (1.3.10) and the above, we can conclude $\left\{\frac{G^{m,n}}{\|G^{m,n}\|_1}\right\}_{n\geq 1}$ is tight. (See condition (b) of Ethier and Kurtz[17, Corollary 7.4] and Ethier and Kurtz[17, Inequality (6.3)]. This concludes the proof of Theorem 1.2.1. □

1.3.3 Proof of Claim 1 and Claim 2

Recall $Y^{m,n}, Y^m$ as in (1.1.5) and $H^{m,n}, H^m$ as in **(H1)** in Section 1.2. Using Skorohod representation, as in (1.3.1) we can construct $Y^{m,n}, Y^m, H^{m,n}, H^m$ in such a way that

$$(Y^{m,n}, H^{m,n}) \to (Y^m, H^m) \text{ as } n\to\infty \quad \mathbb{P}-a.s. \tag{1.3.14}$$

*Proof of **Claim 1**.* Fix $s\geq 0$ such that, s is a continuity point of (Y^m, H^m). We have

$$\begin{aligned}\delta_\Box\left(\frac{G^{m,n}(s)}{\rho_n}, \tilde{h}^m(s)\right) \leq \delta_\Box\left(\frac{G^{m,n}(s)}{\rho_n}, \frac{K^{m,n}(s)}{\rho_n}\right) + \delta_\Box\left(\frac{K^{m,n}(s)}{\rho_n}, h^{m,n}(s)\right)\\ + \delta_\Box(h^{m,n}(s), \tilde{h}^m(s))\end{aligned} \tag{1.3.15}$$

from triangle inequality. Then to prove the claim it is enough to show the following

(i) $\mathbb{P}-$almost surely
$$\lim_{n\to\infty} \delta_\Box(h^{m,n}(s), \tilde{h}^m(s)) = 0. \tag{1.3.16}$$

(ii) $\mathbb{P}-$almost surely
$$\lim_{n\to\infty} \delta_\Box\left(\frac{K^{m,n}(s)}{\rho_n}, h^{m,n}(s)\right) = 0. \tag{1.3.17}$$

(iii) $\mathbb{P}-$almost surely
$$\lim_{n\to\infty} \delta_\Box\left(\frac{K^{m,n}(s)}{\rho_n}, \frac{G^{m,n}(s)}{\rho_n}\right) = 0. \tag{1.3.18}$$

We first prove (i). Define,

$$F^{m,n}(s;x) := \sum_{l=0}^{\lfloor (m+1)x\rfloor} Y_l^{m,n}(s), \quad x\in[0,1),$$

then using (1.3.14) and above we get, for all $x\in[0,1)$

$$F^{m,n}(s;x) \to F^m(s;x) \quad \mathbb{P}-\text{almost surely as } n\to\infty.$$

Recall from (1.2.1), for $x\in[0,1)$ we have,

$$\bar{F}^{m,n}(s;x) = \begin{cases} \frac{a}{m+1} & \text{if } \sum_{l=0}^{a-1} Y_l^{m,n}(s) \leq x < \sum_{l=0}^{a} Y_l^{m,n}(s),\\ 0 & \text{otherwise,}\end{cases} \tag{1.3.19}$$

where $a\in\{1,\ldots,m\}$.

Then from the above and (1.3.19) we have, for all $x\in[0,1)$, $\mathbb{P}$-a.s.

$$\bar{F}^{m,n}(s;x) \to \bar{F}^m(s;x) \text{ as } n\to\infty. \tag{1.3.20}$$

As $H^{m,n}(s;\cdot)$ is continuous on $[0,1]$ for all $n \geq 1$ (see **(H1)**) then using (1.3.14) and (1.3.20) we have for all $x \in [0,1)$, $\mathbb{P}$-a.s.

$$H^{m,n}(s;\bar{F}^{m,n}(s;x)) \to H^{m}(s;\bar{F}^{m}(s;x)) \text{ as } n \to \infty. \tag{1.3.21}$$

As r is continuous (see **(R1)**) then using (1.3.21) we have for all $(x,y) \in [0,1)^2$, $\mathbb{P}$-a.s.

$$r(H^{m,n}(s;\bar{F}^{m,n}(s;x)), H^{m,n}(s;\bar{F}^{m,n}(s;y))) \to h^{m}(s;x,y) \text{ as } n \to \infty. \tag{1.3.22}$$

Recall $h^{m,n}(s)$ is the graph on vertices $[n]$, whose edge weights are

$$\beta_{ij}(h^{m,n}(s)) = r(H^{m,n}(s;\tau_i^{m,n}(s)), H^{m,n}(s;\tau_j^{m,n}(s))).$$

Now,

$$\tau_i^{m,n}(s) = \begin{cases} \frac{a}{m+1} & \text{if } \sum_{l=0}^{a-1} Y_l^{m,n}(s) \leq x < \sum_{l=0}^{a} Y_l^{m,n}(s), \\ 0 & \text{otherwise}, \end{cases} \tag{1.3.23}$$

where $a \in \{1,\dots,m\}$.

Then, for all $(x,y) \in [0,1)^2$ we have,

$$\kappa_{h^{m,n}(s)}(x,y) = r(H^{m,n}(s;\bar{F}^{m,n}(s;x)), H^{m,n}(s;\bar{F}^{m,n}(s;y))) \tag{1.3.24}$$

and (1.3.22) implies for all $(x,y) \in [0,1)^2$, $\mathbb{P}$-a.s.

$$\kappa_{h^{m,n}(s)}(x,y) \to h^{m}(s;x,y) \text{ as } n \to \infty. \tag{1.3.25}$$

Integrating $\kappa_{h^{m,n}(s)}(x,y)$ over $[0,1)^2$ we get

$$\begin{aligned} &\int_{[0,1)^2} \kappa_{h^{m,n}(s)}(x,y)dxdy \\ &= \sum_{a,b=0}^{m} r\left(H^{m,n}\left(s;\frac{a}{m+1}\right), H^{m,n}\left(s;\frac{b}{m+1}\right)\right) Y_a^{m,n}(s)Y_b^{m,n}(s). \end{aligned} \tag{1.3.26}$$

By **(H1)** we have, $\mathbb{P}$-a.s.

$$\begin{aligned} &\lim_{n\to\infty} \int_{[0,1)^2} \kappa_{h^{m,n}(s)}(x,y)dxdy \\ &= \lim_{n\to\infty} \sum_{a,b=0}^{m} r\left(H^{m,n}\left(s;\frac{a}{m+1}\right), H^{m,n}\left(s;\frac{b}{m+1}\right)\right) Y_a^{m,n}(s)Y_b^{m,n}(s) \\ &= \sum_{a,b=0}^{m} r\left(H^{m}\left(s;\frac{a}{m+1}\right), H^{m}\left(s;\frac{b}{m+1}\right)\right) Y_a^{m}(s)Y_b^{m}(s) \\ &= \int_{[0,1)^2} h^{m}(s;x,y)dxdy. \end{aligned}$$

So we have,

$$\lim_{n\to\infty} \int_{[0,1)^2} \kappa_{h^{m,n}(s)}(x,y)dxdy = \int_{[0,1)^2} h^{m}(s;x,y)dxdy \quad \mathbb{P}-a.s. \tag{1.3.27}$$

Then using Scheffe's lemma (see [33]) we have

$$\delta_{\square}(h^{m,n}(s), \tilde{h}^{m}(s)) \to 0 \text{ as } n \to \infty \quad \mathbb{P} - a.s.,$$

where h^m is a representative of $\tilde{h}^m$. So we have proved (i).

Now we prove (ii). From (i) we have

$$d_1(h^{m,n}(s), \tilde{h^m}(s)) \to 0 \text{ as } n \to \infty \quad \mathbb{P} - a.s.$$

Recall the graph $K^{m,n}(s)$

$$\beta_{ij}(K^{m,n}(s)) = \min\{\rho_n r(H^{m,n}(s;\tau_i^{m,n}(s)), H^{m,n}(s;\tau_j^{m,n}(s))), 1\}.$$

Note that

$$\begin{aligned} d_1\left(\frac{K^{m,n}(s)}{\rho_n}, h^{m,n}(s)\right) &= \frac{1}{n^2}\sum_{i,j=1}^{n} \left|\rho_n^{-1}\beta_{ij}(K^{m,n}(s)) - \beta_{ij}(h^{m,n}(s)\right| \\ &= \frac{1}{n^2}\sum_{i,j=1}^{n} \max\{\beta_{ij}(h^{m,n}(s)) - \rho_n^{-1}, 0\}. \end{aligned} \tag{1.3.28}$$

Since for $i, j = 1, \ldots, n$ we have, $\beta_{ij}(h^{m,n}(s))$ are uniformly bounded $\mathbb{P}$-a.s. then from above we get, $\mathbb{P}$-a.s.

$$\frac{1}{n^2}\sum_{i,j=1}^{n} \max\{\beta_{ij}(h^{m,n}(s)) - \rho_n^{-1}, 0\} \to 0 \quad \text{as} \quad n \to \infty. \tag{1.3.29}$$

Then, from (1.3.28) and above we have $\mathbb{P}$-a.s.,

$$d_1\left(\frac{K^{m,n}(s)}{\rho_n}, h^{m,n}(s)\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore

$$\lim_{n\to\infty} \delta_{\square}\left(\frac{K^{m,n}(s)}{\rho_n}, h^{m,n}(s)\right) = 0$$

$\mathbb{P}$-a.s.. So we have proved (ii).

It remains to prove (iii). Note that $\|K^{m,n}(s)\|_1 \le 1$. Then from (1.3.29) and Borgs et al. [7, Lemma 7.3] we have $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \delta_{\square}\left(\frac{G^{m,n}(s)}{\rho_n}, \frac{K^{m,n}(s)}{\rho_n}\right) = 0.$$

This concludes the proof of **Claim 1**. □

Now we prove **Claim 2**.

*Proof of **Claim 2**.* Recall $(\widetilde{\mathcal{W}}_1, \delta_{\square})$ which is a compact space (see Borges et al.[7, Theorem 2.13]), so $(\widetilde{\mathcal{W}}_1, \delta_{\square})$ is also complete and separable. Then under suitable metric the space $D([0,\infty), \widetilde{\mathcal{W}}_1)$ is complete and separable, because $(\widetilde{\mathcal{W}}_1, \delta_{\square})$ is also complete and separable. Recall $K^{m,n} : \Omega \to D([0,\infty), \widetilde{\mathcal{W}}_1)$ $\forall n \ge 1$ from (1.3.2). Then compactness of $(\widetilde{\mathcal{W}}_1, \delta_{\square})$ and Lemma 1.3.2 implies that to prove $\{K^{m,n}\}_{n\ge1}$ is tight, we need to show for every $\eta > 0$ and $T > 0$ $\exists$ $\epsilon > 0$ such that

$$\sup_n \mathbb{P}(w'(K^{m,n}, \epsilon, T) > \eta) \leq \eta \tag{1.3.30}$$

where

$$w'(K^{m,n}, \epsilon, T) = \inf_{\{t_i\}} \max_i \sup_{s,t \in [t_{i-1}, t_i)} \delta_\square(K^{m,n}(s), K^{m,n}(t))$$

where $\{t_i\}$ ranges over all partition of form $0 \leq t_0 < t_1 < \ldots < t_{n-1} < T = t_n$ with $\min_{1 \leq i \leq n}(t_i - t_{i-1}) > \epsilon$.

Fix $s \geq 0$. Note that

$$\beta_{ij}(K^{m,n}(s)) = \min\left\{\rho_n r\left(H^{m,n}\left(s; \frac{a}{m+1}\right), H^{m,n}\left(s; \frac{b}{m+1}\right)\right), 1\right\}.$$

Fix $s, t \geq 0$. Observe that,

$$\begin{aligned}&|\beta_{ij}\left(K^{m,n}(s)\right) - \beta_{ij}\left(K^{m,n}(t)\right)| \\ \leq &\rho_n \left| r\left(H^{m,n}\left(s; \frac{a}{m+1}\right), H^{m,n}\left(s; \frac{b}{m+1}\right)\right) - r\left(H^{m,n}\left(t; \frac{a}{m+1}\right), H^{m,n}\left(t; \frac{b}{m+1}\right)\right)\right|\end{aligned} \tag{1.3.31}$$

and $0 \leq Y_l^{m,n} \leq 1$ for $0 \leq l \leq m$. We will use (1.3.31) and the block structure derived from the definition of $K^{m,n}(s)$ and $K^{m,n}(t)$ to derive an upper bound for $d_\square(K^{2,n}(s), K^{2,n}(t))$. We will illustrate the bound for $m = 2$ precisely and the general case follows similarly. For $m = 2$ the block structure is described in Figure 1.2 where $c_{s,t}^{a,b}$ given by

$$c_{s,t}^{a,b} = \rho_n \left| r\left(H^{2,n}\left(s; \frac{a}{3}\right), H^{2,n}\left(s; \frac{b}{3}\right)\right) - r\left(H^{2,n}\left(t; \frac{a}{3}\right), H^{2,n}\left(t; \frac{b}{3}\right)\right)\right|$$

for $0 \leq a, b \leq 2$ which is the right hand side of (1.3.31).

Using the definition of $K^{2,n}(\cdot)$ and the above block structure it is easy to see

$$\begin{aligned}&d_\square(K^{2,n}(s), K^{2,n}(t)) \\ &\leq \sum_{a,b=0}^{2} \left| r\left(H^{2,n}\left(s; \frac{a}{3}\right), H^{2,n}\left(s; \frac{b}{3}\right)\right) - r\left(H^{2,n}\left(t; \frac{a}{3}\right), H^{2,n}\left(t; \frac{b}{3}\right)\right)\right| \\ &+ 2\sum_{k=0}^{1} \left|Y_k^{2,n}(s) - Y_k^{2,n}(t)\right|.\end{aligned}$$

For general m also we can similarly show

$$\begin{aligned}&d_\square(K^{m,n}(s), K^{m,n}(t)) \\ &\leq \sum_{a,b=0}^{m} \left| r\left(H^{m,n}\left(s; \frac{a}{m+1}\right), H^{m,n}\left(s; \frac{b}{m+1}\right)\right) - r\left(H^{m,n}\left(t; \frac{a}{m+1}\right), H^{m,n}\left(t; \frac{b}{m+1}\right)\right)\right| \\ &+ 2\sum_{k=0}^{m-1} \left|Y_k^{m,n}(s) - Y_k^{m,n}(t)\right|.\end{aligned} \tag{1.3.32}$$

We have $(Y^{m,n}, H^{m,n})_{n\geq 1}$ is tight from (1.3.1). This implies $\left\{\left(H^{m,n}\left(\cdot, \frac{a}{m+1}\right), H^{m,n}\left(\cdot, \frac{b}{m+1}\right)\right)\right\}_{n\geq 1}$ is

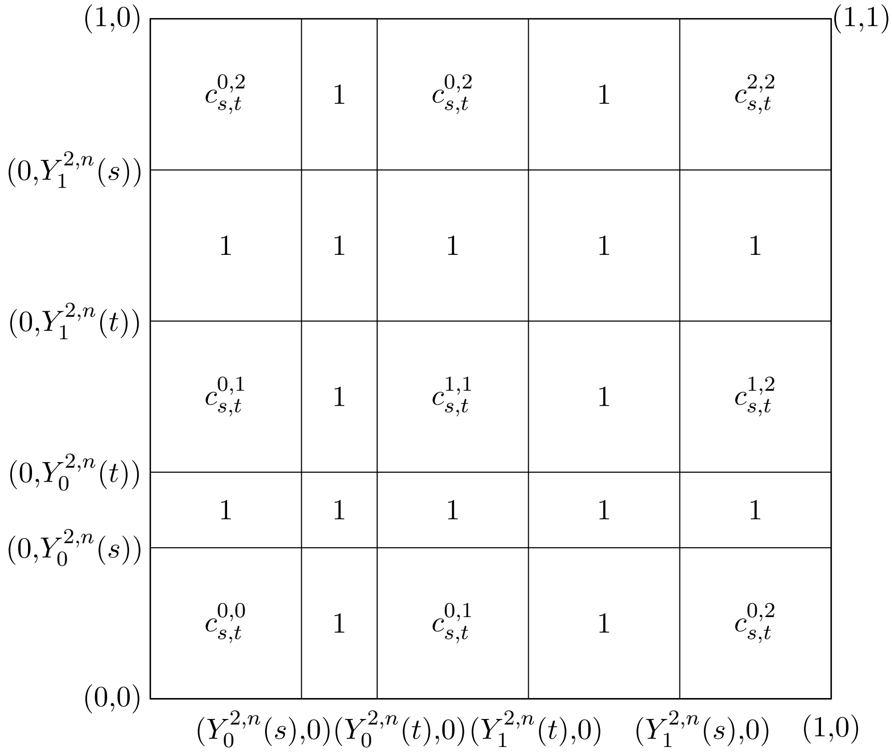

FIGURE 1.2: Graphical representaion of upper bounds of each block value (omitted n from the notation)

tight for $0 \leq a, b \leq m$. Then continuity of r on $[0,1) \times [0,1)$ implies $\left\{ r\left(H^{m,n}\left(\cdot; \frac{a}{m+1}\right), H^{m,n}\left(\cdot; \frac{b}{m+1}\right)\right)\right\}_{n \geq 1}$ is tight for $0 \leq a, b \leq m$ (See Ethier and Kurtz [17, Problem 13]). Then using (1.3.32) we get $\{K^{m,n}\}_{n\geq 1}$ is tight in $\widetilde{W}_1$.

It is easy to see that, $id : (\widetilde{\mathcal{W}}, \delta_{\square}) \to (\widetilde{\mathcal{W}}, \delta_{\square}))$ is a continuous function and $\frac{id}{\|\cdot\|_1}$ is a continuous function from $D([0,\infty), \widetilde{\mathcal{W}} \setminus \{\tilde{0}\})$ to $D([0,\infty), \widetilde{\mathcal{W}})$. Then, continuous mapping theorem implies $\left\{\frac{K^{m,n}}{\|K^{m,n}\|_1}\right\}_{n\geq 1}$ is tight in $\widetilde{W}_1$. So **Claim 2** is proved. $\square$

Chapter 2

Random sparse graph dynamics: dependent edges

In this chapter we construct graphon valued stochstic processes in the context of sparse graphs using the methods from Athreya and Röllin [3]. In the previous Chapter, the connection probability of individuals i and j in the graph $G^{m,n}$ were independent of all other pairs. Here the connection probabilities in $G^{m,n}$ will be assigned via a Markov chain $\{X_k\}_{k\geq 1}$. The specific steps we take to impose random dynamics on the vertices are the followings. We first sample N individuals, labelled $X_1, ..., X_N$ where $X_i \in [0,1]$. This sampling is influenced by an underlying Moran model. After sampling, the individuals are clumped into n equally spaced bins, which are represented by the intervals $A_{n,i} = [\frac{i-1}{n}, \frac{i}{n})$ for $1 \leq i \leq n-1$ and $A_{n,n} = [\frac{n-1}{n}, 1]$. We connect i, j if there exists $0 \leq p < N$ such that either $X_p \in A_{n,i}$ and $X_{p+1} \in A_{n,j}$ or $X_{p+1} \in A_{n,j}$ and $X_p \in A_{n,i}$. We choose N in such a way that the graph sequence become sparse. We let n go to infinity and obtain a dynamics on space of sparse graphs which are governed by a limiting graphon.

2.1 Preliminaries

Athreya and Röllin, studied dense and sparse graph sequences, where vertices are sampled via an ergodic process (allowing dependencies) and edges are sampled independently using a graphon that represented the network (See [3], [2]). Their motivation was to model realistic networks with probabilistic dependencies. They constructed the random sparse graph sequence G^n through a specific sampling procedure. Using a Markov chain whose transition probabilities are governed by a given graphon, they constructed a sparse random graph sequence using a "clumping procedure" and identified it's limit. In this section we present this construction.

Let $r : [0,1]^2 \to [0, \infty)$ be a graphon. We shall make the following assumptions on r.

Assumption 1: There is a constant $\delta > 0$ and an integrable function $\phi : [0,1] \to \mathbb{R}_+$ such that

$$0 < \delta \leq \frac{r(x,y)}{\int_0^1 r(x,z)dz} \leq \phi(y) \quad 0 \leq x, y \leq 1.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which we define a Markov chain $X = \{X_k\}_{k\geq 0}$ on $[0,1]$ with transition probabilities given by

$$\mathbb{P}[X_{l+1} \in dy | X_l = x] = \frac{r(x,y)}{\int_0^1 r(z,x)dz} dy \quad \text{for} \quad l \geq 0.$$

As κ is symmetric, the Markov chain is time-reversible with stationary distribution

$$\pi(dx) = \frac{\int_0^1 r(x,u)du}{\int_0^1 \int_0^1 r(u,v)dudv} dx.$$

We are going to assume that X_0 follows distribution π, which means the chain is stationary. Then the probability of a transition from dx to dy is given by joint distribution

$$\mathbb{P}[X_k \in dx, X_{k+1} \in dy] = \pi(dx) \frac{r(x,y)}{\int_0^1 r(x,z)dz} dy = \frac{r(x,y)}{\int_0^1 \int_0^1 r(u,v)dudv} dxdy$$

Let $n \geq 1$ and $N \equiv N(n)$ represent the clumping parameter. We will now construct a random graph $G(n, N, X, r)$ via the following steps:

1. Vertex set be $[n] = \{1, 2, \ldots, n\}$.

2. Let $\{X_k\}_{k\geq 0}$ be the Markov chain constructed from graphon κ.

3. We divide the interval $[0,1]$ in n equal intervals, call them $A_{n,i} = [\frac{i-1}{n}, \frac{i}{n})$, where $1 \leq i \leq n-1$ and $A_{n,n} = [\frac{n-1}{n}, 1]$. For $1 \leq i, j \leq n$ with $i \neq j$, define

$$I_n(i,j) = \begin{cases} 1 & \text{if there exists } 0 \leq p < N \text{ such that either} \\ & X_p \in A_{n,i} \text{ and } X_{p+1} \in A_{n,j} \text{ or } X_{p+1} \in A_{n,j} \text{ and } X_p \in A_{n,i} \\ 0 & \text{otherwise} \end{cases}$$

4. We construct the random graph $G(n, N, X, \kappa)$, for $1 \leq i, j \leq n$ with $i \neq j$, we join i and j if $I_n(i,j) = 1$, and otherwise we do not join.

If we choose $N(n) = o(n^2)$, then the graph $G(n, N, X, r)$ will be a sparse random graph sequence. Let us call $G^n \equiv G(n, N, X, r)$.

We shall make the following assumptions on N.

Assumption 2: There are constants α and λ, where $0 < \alpha < 1$ and $\lambda > 0$, such that the sequence $N \equiv N(n)$ satisfies

$$\lim_{n\to\infty} \frac{N}{n^{1+\alpha}} = \lambda$$

Then we have the following result.

Theorem 2.1.1. *Under Assumption 1 and Assumption 2, and if* $0 < \alpha < 1$ *then we have*

$$\delta_\square \left(\frac{n^2}{N} G^n, \frac{r}{\|r\|_1} \right) \to 0 \quad (\text{ as } n \to \infty).$$

Proof. See [3], Theorem 3.1. $\square$

2.2 Sparse graph dynamic and Main result

In this section we will state Theorem 2.2.3 which is our main result. Recall weak convergence of graphons on the metric space $(\widetilde{\mathcal{W}}_1, \delta_\square)$ from Section 1.2. We will assume **(R1)**, **(H1)** and **(Y1)** as described in Section 1.2.

Weighted dynamic graph from Moran model. Consider n vertices which follow the $m-$multi type Moran model defined in Section 1.1.2. $\tau_i^{m,n}(s)$ denote the type of individual i divided by $m+1$, at time ns. At time ns if the type of individual i equals k, where $0 \le k \le m$, then $\tau_i^{m,n}(s) = \frac{k}{m+1}$. We first fix m and n. We have vertex set $[n] = \{1, 2, \ldots, n\}$, these vertices follow the Moran model.

Let, $h^{m,n}(s)$ be the graph on vertices $[n]$ at time $s \ge 0$, whose edge weights are

$$\beta_{ij}(h^{m,n}(s)) := r(H^{m,n}(s; \tau_i^{m,n}(s)), H^{m,n}(s; \tau_j^{m,n}(s))). \tag{2.2.1}$$

Let, $K^{m,n}(s)$ be the graph on vertices $[n]$ at time $s \ge 0$, whose edge weights are

$$\beta_{ij}(K^{m,n}(s)) := \min\{\rho_n r(H^{m,n}(s; \tau_i^{m,n}(s)), H^{m,n}(s; \tau_j^{m,n}(s))), 1\} \tag{2.2.2}$$

where $\{\rho_n\}_{n\ge1}$ is a fixed sequence such that $\rho_n \to 0$ and $n\rho_n \to \infty$.

We denote the graphon corresponding to the graph $K^{m,n}(s)$ by $\kappa_{m,n}(s) \equiv \kappa_{K^{m,n}}(s)$. At fixed time s the graphon $\kappa_{m,n}(s)$ depends on Moran model as in the previous chapter.

Lemma 2.2.1. *Fix m, n. Then $\kappa^{m,n} \in D([0,\infty), \widetilde{\mathcal{W}}_1)$.*

Proof. We have $Y^{m,n} \in D([0,\infty), \mathcal{S}^m)$ and $H^{m,n} \in D([0,\infty), \mathcal{L})$. Then using continuity of r and (1.3.32) we have the paths $\kappa^{m,n}(\cdot) \equiv \kappa_{K^{m,n}}(\cdot)$ are right continuous and have left limits. So we can conclude $\kappa^{m,n} \in D([0,\infty), \widetilde{\mathcal{W}}_1)$.

□

Dynamic sparse graph sequence: To model systems where interactions between individuals evolve dynamically over time we shall construct sparse graph sequence using a Markov chain. The connection probabilities and the graph structure evolve dynamically. We construct the dynamic sparse graph sequence $G^{m,n}(s)$ on n vertices for each time $s \ge 0$ based on the construction of a sparse random graph sequence from a graphon $\kappa_{K^{m,n}}(s)$ as done earlier. In the previous Chapter, the connection probability of individuals i and j in the graph $G^{m,n}$ were independent of all other pairs. In contrast, here, the connection probabilities in $G^{m,n}$ will be assigned via a Markov chain, introducing dependence among the connections.

Let $\mathbb{P}'$ denote conditional probability given $H^{m,n}, Y^{m,n}, H^m, Y^m$ $\forall n \ge 1$ then

$$\mathbb{P}' = \mathbb{P}(\cdot | H^{m,n}, Y^{m,n}, H^m, Y^m; n \ge 1). \tag{2.2.3}$$

We define a Markov chain $X^{m,n}(s) \equiv \{X_l^{m,n}(s)\}_{l\ge0}$ taking values in $[0,1]$. Let the transition probabilities be given by

$$\mathbb{P}'[X_{l+1}^{m,n}(s) \in dy | X_l^{m,n}(s) = x] = \frac{\kappa_{m,n}(s; x, y)}{\int_0^1 \kappa_{m,n}(s; z, x) dz} dy. \tag{2.2.4}$$

Since $\kappa_{m,n}(s)$ is symmetric, the Markov chain is time reversible with stationary distribution

$$\pi_s^{m,n}(dx) = \frac{\int_0^1 \kappa_{m,n}(s;x,u)du}{\int_0^1\int_0^1 \kappa_{m,n}(s;u,v)dudv}dx.$$

We assume that $X_0^{m,n}(s)$ has distribution $\pi_s^{m,n}$ under $\mathbb{P}'$. This means the chain is stationary. Then

$$\mathbb{P}'[X_l^{m,n}(s) \in dx, X_{l+1}^{m,n}(s) \in dy] = \pi_s^{m,n}(dx)\frac{\kappa_{m,n}(s;x,y)}{\int_0^1 \kappa_{m,n}(s;x,z)dz}dy = \frac{\kappa_{m,n}(s;x,y)}{\int_0^1\int_0^1 \kappa_{m,n}(s;u,v)dudv}dxdy. \quad (2.2.5)$$

Fix $s \geq 0$. Let $X_0^{m,n}(s), \dots, X_N^{m,n}(s)$ be a realisation of the Markov chain up to time N where N is a function of n.

Let $n \geq 1$. For $1 \leq i,j \leq n$ with $i \neq j$ define

$$I_{m,n}(i,j)(s) = \begin{cases} 1 & \text{if there exists } 0 \leq p < N \text{ such that either} \\ & X_p^{m,n}(s) \in A_{n,i} \text{ and } X_{p+1}^{m,n}(s) \in A_{n,j} \text{ or } X_{p+1}^{m,n}(s) \in A_{n,j} \text{ and } X_p^{m,n}(s) \in A_{n,i} \\ 0 & \text{otherwise.} \end{cases}$$

We now construct the graph $G^{m,n}(s) \equiv G(n,m,N,X^{m,n},r,s)$ on vertex set $[n] = \{1,2,\dots,n\}$ at time s. For $1 \leq i,j \leq n$ with $i \neq j$, we join i and j if $I_{m,n}(i,j)(s) = 1$, and otherwise we do not join. If we choose $N(n) = o(n^2)$ then $G(n,m,N,X,r,s)$ will be sparse graph sequence.

Lemma 2.2.2. *Fix m,n. Then $G^{m,n} \in D([0,\infty), \widetilde{\mathcal{W}}_1)$.*

Proof. Using uniform random variables we are going to construct for every $s \geq 0$ the Markov chain $X^{m,n}(s)$ in a way that (2.2.4) holds and $X_0^{m,n}(s)$ has distribution $\pi_s^{m,n}$. Let $\{U_n\}_{n\geq 0}$ be a sequence of i.i.d uniform random variables. Assume that $\{U_n\}_{n\geq 0}$ and $H^{m,n}, Y^{m,n}, H^m, Y^m$ are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $s \geq 0$ define $F_{s,0}^{m,n} : \Omega \times [0,1] \to [0,1]$ given by

$$F_{s,0}^{m,n}(\cdot, a) := \int_0^a \pi_s^{m,n}(\cdot)(dx).$$

Recall $\bar{F}$ from (1.2.1). We can construct $X_0^{m,n}(s)$ by defining

$$X_0^{m,n}(s) := \overline{F_{s,0}^{m,n}}(\cdot,\cdot) \circ (\cdot, U_0)$$

where $(\cdot, U_0) : \Omega \to \Omega \times [0,1]$ is given by

$$(\cdot, U_0)(\omega) = (\omega, U_0(\omega))$$

for $\omega \in \Omega$. $\overline{F_{s,0}^{m,n}}(\cdot,\cdot)$ is calculated on second coordinated. It is easy to verify that $X_0^{m,n}(s)$ has distribution $\pi_s^{m,n}$ under $\mathbb{P}'$. Define $F_{s,1}^{m,n} : [0,1] \times [0,1] \to [0,1]$ given by

$$F_{s,1}^{m,n}(\cdot, a) = \frac{\int_0^a \kappa_{m,n}(s;\cdot,y)dy}{\int_0^1 \kappa_{m,n}(s;\cdot,y)dy}.$$

We construct $X_1^{m,n}(s)$ by defining

$$X_1^{m,n}(s) := \overline{F_{s,1}^{m,n}}(X_0^{m,n}(s), \cdot) \circ (\cdot, U_1)$$

where $(\cdot, U_1) : \Omega \to \Omega \times [0,1]$ is given by

$$(\cdot, U_1)(\omega) = (\omega, U_1(\omega))$$

for $\omega \in \Omega$. $\overline{F^{m,n}_{s,1}}(X^{m,n}_0(s), \cdot)$ is calculated on second coordinated. It is easy to see that (2.2.5) holds for $l = 0$. We can similarly construct $X^{m,n}_l(s)$ for $l \geq 2$.

Let $\omega \in \Omega$ such that $U_0(\omega) = u$. We can see that $\overline{F^{m,n}_{s,0}}(\omega, \cdot)$ is a function of r, $Y^{m,n}(s, \omega)$ and $H^{m,n}(s, \omega)$ such that $X^{m,n}_0(\omega) \in D([0,\infty), [0,1])$ (using $Y^{m,n} \in D([0,\infty), \mathcal{S}^m)$ and $H^{m,n} \in D([0,\infty), \mathcal{L})$). Similarly we can show $X^{m,n}_l \in D([0,\infty), [0,1])$ $\forall l \geq 1$. Then from the definition of $I_{m,n}$ we can conclude $G^{m,n} \in D([0,\infty), \widetilde{\mathcal{W}}_1)$.

□

Now we proceed towards the main result. We make the following assumptions on r and N. Assumption **(K1)**: There is a constant $\delta > 0$ and an integrable function $\phi : [0,1] \to \mathbb{R}_+$ such that

$$0 < \delta \leq \frac{r(x,y)}{\int_0^1 r(x,z)dz} \leq \phi(y) \quad 0 \leq x, y \leq 1 \tag{2.2.6}$$

Assumption **(N1)**: There are constants α and λ, where $0 < \alpha < 1$ and $\lambda > 0$. The sequence $N \equiv N(n)$ and $\{\rho_n\}_{n\geq 1}$ satisfies

$$\lim_{n\to\infty} \frac{N}{n^{1+\alpha}} = \lambda \text{ and } \lim_{n\to\infty} \frac{n}{\rho_n^2 N} = 0 \tag{2.2.7}$$

where $\{\rho_n\}_{n\geq 1}$ is a fixed sequence such that $\rho_n \to 0$ and $n\rho_n \to \infty$. Recall $F^m, \bar{F}$ from (1.1.6) and (1.2.1).

Recall, " $\Rightarrow$ " denotes weak convergence for stochastic processes taking values in $D([0,\infty), \widetilde{\mathcal{W}}_1)$, the set of cádlág paths in $\widetilde{\mathcal{W}}_1$.

Theorem 2.2.3. *Let $G^{m,n}$ be constructed as above. Assume* (**R1**), (**H1**) *and* (**Y1**)*. Then under assumptions* (**K1**), (**N1**) *we have*

$$\frac{n^2}{N} G^{m,n} \Rightarrow \frac{\tilde{h^m}}{\|\tilde{h^m}\|_1} \quad (\text{ as } n \to \infty).$$

where for $s \geq 0$ the equivalence class $\tilde{h}^m(s)$ has a representative $h^m(s)$ of the form

$$h^m(s; x, y) = \begin{cases} r(H^m(s; \bar{F}^m(s;x)), H^m(s; \bar{F}^m(s;x))) & \text{if } (x,y) \in [0,1) \times [0,1) \\ 0 & \text{otherwise} \end{cases}$$

We will finish this section with an elementary example to understand our main result, Theorem 1.2.1. We will explain the result in the case of $m = 1$. The limiting graphon h^1 will be an 2×2-block graphon, whose block boundaries are influenced by Wright-Fisher diffusion. And the block heights will depend on H^1.

Example 2.2.4. *Recall Example 1.2.2, where the edges were independent. In this example, which is similar in spirit the edges are dependent on each other.*

Let

$$H^{1,n}(s, u) = H^1(s; u) = u$$

and $r : [0,1] \times [0,1] \to [0,\infty)$ be a function which is symmetric and integrable under Lebesgue measure. r is continuous at (x,y) if $(x,y) \in [0,1) \times [0,1)$ and $r(x,y) = 0$ if $x = 1$ or $y = 1$. Then

$$\beta_{ij}(K^{1,n}(s)) = \min\{\rho_n r(\tau_i^{1,n}(s), \tau_j^{1,n}(s)), 1\}.$$

It follows that at time s, $G^{1,n}(s)$ is the random graph with vertex set $[n] = \{1,2,\dots,n\}$, in which, vertices i and j are connected by an edge if

$$I_{1,n}(i,j)(s) = \begin{cases} 1 & \text{if there exists } 0 \le p < N \text{ such that either} \\ & X_p^{m,n}(s) \in A_{n,i} \text{ and } X_{p+1}^{m,n}(s) \in A_{n,j} \text{ or } X_{p+1}^{m,n}(s) \in A_{n,j} \text{ and } X_p^{m,n}(s) \in A_{n,i} \\ 0 & \text{otherwise} \end{cases}$$

Then by Theorem 2.2.3, as $n \to \infty$, the graph process $\frac{n^2}{N} G^{1,n}$ converges to $\frac{\tilde{h}^1}{\|\tilde{h}^1\|_1}$. Here, the connection probability of individuals i and j does not depend on the rest of the population.

Then r is being evaluated at the points $\{0, \frac{1}{2}\} \times \{0, \frac{1}{2}\}$. Then without loss of generality, we can take r as a 2×2 matrix, namely,

$$r = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

which gives the representation

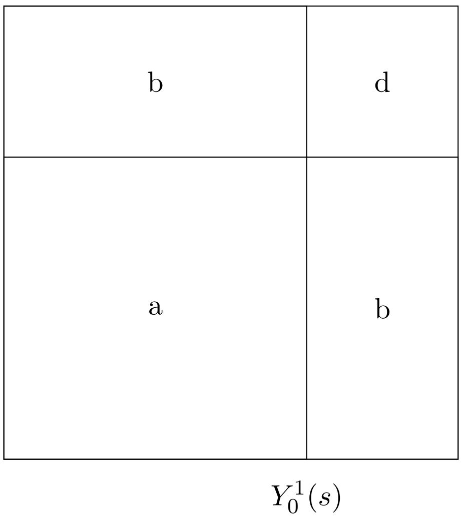

FIGURE 2.1: Graphical representaion of $h^1(s)$, with $Y_0^1(s)$ the fraction of Type 0 in the population at time s.

$$h^1(s;x,y) = \begin{cases} a & \text{if } (x,y) \in [0, Y_0^1(s)) \times [0, Y_0^1(s)) \\ d & \text{if } (x,y) \in [Y_0^1(s), 1) \times [Y_0^1(s), 1) \\ b & \text{if } (x,y) \in [Y_0^1(s), 1) \times [0, Y_0^1(s)) \\ b & \text{if } (x,y) \in [Y_0^1(s), 1) \times [0, Y_0^1(s)) \end{cases}$$

Similarly for general m, r will be evaluated at the points $\left\{0, \frac{1}{m+1}, \dots, \frac{m}{m+1}\right\} \times \left\{0, \frac{1}{m+1}, \dots, \frac{m}{m+1}\right\}$, then we can view r as a matrix $(r_{ij})_{0 \le i,j \le m}$.

2.3 Proof of Theorem 2.2.3

We will use Remark 1.3.1 to prove the Theorem 2.2.3. Indeed from Remark 1.3.1 to prove the Theorem 2.2.3 it is enough to show:

(S1) Whenever $s \geq 0$ is a point at which $\tilde{h}^m$ is continuous, we have $\mathbb{P}$-a.s

$$\frac{n^2}{N} G^{m,n}(s) \to \tilde{h^m}(s) \text{ as } n \to \infty \tag{2.3.1}$$

and

(S2)

$$\left\{ \frac{n^2}{N} G^{m,n} \right\}_{n \geq 1} \text{ is tight in } \widetilde{W}_1 \tag{2.3.2}$$

hold. We first set up a few preliminaries before we prove (S1) and (S2).

Recall $H^{m,n}, H^m$ as in **(H1)** and $Y^{m,n}, Y^m$ as in **(Y1)** and in Section 1.2. Then (1.1.5) implies $Y^{m,n} \Rightarrow Y^m$ as $n \to \infty$. Using Skorohod representation, we may assume there exists a probability space on which we can define distributionally equivalent $Y^{m,n}, Y^m, H^{m,n}, H^m$ in such a way that

$$(Y^{m,n}, H^{m,n}) \to (Y^m, H^m) \text{ as } n \to \infty \quad \mathbb{P} - a.s. \tag{2.3.3}$$

As that r is integrable (see **(R1)**) then using **(K1)** we can get

$$c_1(n) \leq \frac{\kappa_{m,n}(s; x, y)}{\int_0^1 \kappa_{m,n}(s; z, x) dz} \leq c_2(n) \tag{2.3.4}$$

for $s \geq 0$ where $c_1(n) = \rho_n \|r\|_1 \delta^2$ and $c_2(n) = \frac{1}{\rho_n \|r\|_1 \delta^2}$. Recall $\mathbb{P}'$ from (2.2.3), $I_{m,n}(s)$ and $K^{m,n}(s)$ from previous section. Before proving the Theorem 2.2.3 we introduce two intermediate weighted graphs and two definitions.

The first graph is denoted by notation $\mathbb{E}' G^{m,n}(s)$ at time $s \geq 0$; it is a weighted graph on vertices $[n]$ with edge weights

$$\beta_{ij}(\mathbb{E}' G^{m,n}(s)) = \frac{1}{2} \mathbb{E}' I_{m,n}(i,j)(s). \tag{2.3.5}$$

where $\mathbb{E}'$ in the right hand side of the above equation is the expectation with respect to $\mathbb{P}'$. The second graph is denoted by $H'_{m,n}(s)$ at time $s \geq 0$, is the weighted graph on the vertices $[n]$ with edge weights

$$\beta_{ij}(H'_{m,n}(s)) = \frac{n^2}{N} \left(1 - \exp \left(-\frac{2N}{n^2} \mu_{m,n}(s)(i,j) \right) \right), \tag{2.3.6}$$

where $\mu_{m,n}(s)(i,j) = n^2 \int_{A_{n,i}} \int_{A_{n,j}} \frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1} dx dy$.

We conclude the preliminaries with two definitions that we shall use in the proof.

Definition 2.3.1 ($\frac{1}{N}$-Hamming-Lipchitz)**.** *A function $f : X \to \mathbb{R}$, where X is a Hamming space equipped wit Hamming distance metric d_X, is said to be $(1/N)$-Hamming-Lipschitz if it satisfies the following property:*

$$|f(x), f(x')| \leq \frac{1}{N} d_X(x, x'), \quad \forall x, x' \in X,$$

where $N > 0$ is a scaling parameter.

Definition 2.3.2 (size biased distribution)**.** *Given a non-negative random variable W with distribution $dF(w)$ and mean λ, W^{size} is said to be the W-size biased distribution if it has distribution $\frac{wdF(w)}{\lambda}$.*

Proof of (S1). Recall for $s \geq 0$ the equivalence class $\tilde{h}^m(s)$ has a representative $h^m(s)$ of the form

$$h^m(s;x,y) = \begin{cases} r(H^m(s;\bar{F}^m(s;x)), H^m(s;\bar{F}^m(s;x))) & \text{if } (x,y) \in [0,1) \times [0,1) \\ 0 & \textit{otherwise} \end{cases}$$

Using triangle inequality we have

$$\begin{aligned} &\delta_\square\left(\frac{n^2}{N}G^{m,n}(s), \frac{\tilde{h}^m(s)}{\|\tilde{h}^m(s)\|_1}\right) \\ \leq &\delta_\square\left(\frac{n^2}{N}G^{m,n}(s), \frac{n^2}{N}\mathbb{E}'G^{m,n}(s)\right) + \delta_\square\left(\frac{n^2}{N}\mathbb{E}'G^{m,n}(s), H'_{m,n}(s)\right) + \delta_\square\left(H'_{m,n}(s), \frac{h^m(s)}{\|h^m(s)\|_1}\right). \end{aligned} \tag{2.3.7}$$

Using part (i), part (ii), part (iii) of the claim below and triangle inequality the proof of **(S1)** follows.

Claim 1: Fix $s \geq 0$ such that s is a point at which $\tilde{h}^m$ is continuous. Then we have

(i) $\mathbb{P}$-almost surely

$$\lim_{n\to\infty} \delta_\square\left(\frac{n^2}{N}G^{m,n}(s), \frac{n^2}{N}\mathbb{E}'G^{m,n}(s)\right) = 0.$$

(ii) $\mathbb{P}$-almost surely

$$\lim_{n\to\infty} \delta_\square\left(\frac{n^2}{N}\mathbb{E}'G^{m,n}(s), H'_{m,n}(s)\right) = 0.$$

(iii) $\mathbb{P}$-almost surely

$$\lim_{n\to\infty} \delta_\square\left(H'_{m,n}(s), \frac{\tilde{h}^m(s)}{\|\tilde{h}^m(s)\|_1}\right) = 0.$$

We defer the proof of the claim in Section 2.3.1. The key tools in proving (i) are a concentration inequality and results on mixing times for Markov chains. The proof of (ii) is done via an application Stein-Chen method. □

Proof of (S2). Let $T > 0$ be fixed. Observe that when the vertices resample their type according to a follows a Poisson point process with rate n, then the number of resampling event in the interval $[0,T]$, follows a Poisson distribution of mean nT. Let, $A_{n,m}$ be the event that the process saw no more that n^2 such resampling events in the interval $[0,T]$. If $Z \sim \text{Poisson}(nT)$ then we have,

$$\mathbb{P}[A^c_{n,m}] = \mathbb{P}[Z > n^2] \leq \frac{\mathbb{E}[Z^2]}{n^4} \leq \frac{2T^2}{n^2}. \tag{2.3.8}$$

On the event $A_{n,m}$ we apply a union bound over all resampling events and using bound from (2.3.13) we get for $\eta > 0$,

$$\mathbb{P}'\left[\sup_{s\in[0,T]} d_{\square}\left(\frac{n^2}{N}G^{m,n}(s), \frac{n^2}{N}\mathbb{E}'G^{m,n}(s)\right) > \epsilon \mid A_{n,m}\right]$$
$$\leq n^2 \exp\left(-\frac{2N\rho_n^2\|r\|_1^2\delta^4\epsilon^2}{9\ln\frac{1}{4}} + (2n+1)\ln(2)\right). \tag{2.3.9}$$

Claim 2: $\left\{\frac{n^2}{N}\mathbb{E}'G^{m,n}\right\}_{n\geq 1}$ is tight in $\widetilde{W}_1$.

We will prove **Claim 2** in Section 2.3.1. Then using (2.3.8) and (2.3.9), we can conclude $\left\{\frac{n^2}{N}G^{m,n}\right\}_{n\geq 1}$ is tight (See condition (b) of Ethier and Kurtz [17, Corollary 7.4, p.129] and Ethier and Kurtz[17, Inequality (6.3), p.122]). This conclude the proof of the Theorem 2.2.3.

□

2.3.1 Proof of Claim 1 and Claim 2

Recall $Y^{m,n}, Y^m$ as in (1.1.5) and $H^{m,n}, H^m$ as in **(H1)** in Section 1.2. Using Skorohod representation as in 2.3.3 we can construct $Y^{m,n}, Y^m, H^{m,n}, H^m$ in such a way that

$$(Y^{m,n}, H^{m,n}) \to (Y^m, H^m) \text{ as } n\to\infty \quad \mathbb{P}-a.s. \tag{2.3.10}$$

*Proof of **Claim 1**.* Fix $s\geq 0$ such that s is a point at which $\tilde{h}^m$ is continuous. We will first prove (i). Observe that

$$d_{\square}\left(\frac{n^2}{N}G^{m,n}(s), \frac{n^2}{N}\mathbb{E}'G^{m,n}(s)\right)$$
$$= \sup_{S=\cup_{m=1}^k A_{n,j_m}, T=\cup_{m=1}^l A_{n,j_m}} \left|\sum_{i:\frac{i}{n}\in S}\sum_{j:j\in\frac{j}{n}\in T}\frac{n^2}{N}(I_{m,n}(s)(i,j) - \mathbb{E}'I_{m,n}(s)(i,j))\mathrm{Vol}(A_{n,i})\mathrm{Vol}(A_{n,j})\right|$$
$$= \sup_{S=\cup_{m=1}^k A_{n,j_m}, T=\cup_{m=1}^l A_{n,j_m}} \left|f_{S,T}(X^{m,n}(s)) - \mathbb{E}'f_{S,T}(X^{m,n}(s))\right|,$$

where $f_{S,T}(X^{m,n}(s)) = f_{S,T}(X_0^{m,n}(s),\ldots,X_N^{m,n}(s)) = \frac{1}{2N}\sum_{i:\frac{i}{n}\in S}\sum_{j:\frac{j}{n}\in T} I_{m,n}(s)(i,j)$. Note that $X^{m,n}(s)$ is Harris recurrent by **(K1)**. We will now calculate an upper bound of mixing time (see Paulin [28, Definition 1.3]) t^s_{mix}. Note that t^s_{mix} depends on n. Using (2.3.4) we get

$$\mathbb{P}'[X_1^{m,n}(s)\in dy \mid X_0^{m,n}(s)=x] = \frac{\kappa_{m,n}(s;x,y)}{\int_0^1 \kappa_{m,n}(s;z,x)dz}dy \geq \rho_n\|r\|_1\delta^2 dy.$$

Using (2.3.4) we have

$$\mathbb{P}'[X_1^{m,n}(s)\in dy \mid X_0^{m,n}(s)=x] \geq \rho_n^2\|r\|_1^2\delta^4\pi_s^{m,n}(dy).$$

Let $a_n = \rho_n^2\|r\|_1^2\delta^4$ and $\theta_n = (1-a_n)$. Then for large n we have $0<\theta_n<1$. Using above we can write

$$\mathbb{P}'[X_1^{m,n}(s)\in dy \mid X_0^{m,n}(s)=x] = (1-\theta_n)\pi_s^{m,n}(dy) + \theta_n q_s^1(x,y)dy$$

where $q_s^1(x,y) \geq 0$ and $\int_0^1 q_s^1(x,y)dy = 1$.

We can inductively define for $k, l \geq 1$

$$q_s^{k+l}(x,y) = \int_0^1 q_s^k(x,z) q_s^l(z,y) dz.$$

Then using induction we can easily show that,

$$\mathbb{P}'[X_k^{m,n}(s) \in dy \mid X_0^{m,n}(s) = x] = (1-\theta_n^k)\pi_s^{m,n}(dy) + \theta_n^k q_s^k(x,y)dy.$$

From above we can write

$$\mathbb{P}'[X_k^{m,n}(s) \in dy \mid X_0^{m,n}(s) = x] - \pi_s^{m,n}(dy) = \theta_n^k(q_s^k(x,y)dy - \pi_s^{m,n}(dy)).$$

Observe that $q_s^k(x,\cdot)$ and $\pi_s^{m,n}(\cdot)$ are probability distributions. Then from above we have

$$d_{TV}(\mathbb{P}'[X_k^{m,n}(s) \in \cdot \mid X_0^{m,n}(s) = x], \pi_s^{m,n}(\cdot)) \leq \theta_n^k$$

where d_{TV} denotes the total variation distance. Recall t_{mix}^s from [28, Definition 1.3]. Then from above we get

$$t_{mix}^s < \min\left\{k \geq 1 : \theta_n^k \leq \frac{1}{4}\right\}.$$

This implies $t_{mix}^s < \frac{\ln(\frac{1}{4})}{\ln(\theta_n})$. Let $\epsilon > 0$ be given. Now changing one point in $f(X^{m,n}(s))$ will change f by at most 2 edges ; that is $f_{S,T}$ is $\frac{1}{N}$-Hamming-Lipchitz. Therefore using Paulin [28, Corollary 2.11] and $t_{mix}^s < \frac{\ln(\frac{1}{4})}{\ln(\theta_n)}$ we get

$$\mathbb{P}'\left[|f_{S,T}(X^{m,n}(s)) - \mathbb{E}' f_{S,T}(X^{m,n}(s))| > \epsilon\right] \leq 2\exp\left(-\frac{2N\epsilon^2}{9t_{mix}^s}\right) < 2\exp\left(-\frac{2N\ln(\theta_n)\epsilon^2}{9\ln(\frac{1}{4})}\right).$$

Using the union bound we have,

$$\begin{aligned}
&\mathbb{P}'\left[\sup_{S=\cup_{m=1}^k A_{n,j_m}, T=\cup_{m=1}^l A_{n,j_m}} |f_{S,T}(X^{m,n}(s)) - \mathbb{E}' f_{S,T}(X^{m,n}(s))| > \epsilon\right] \\
&\leq 2^{2n+1}\exp\left(-\frac{2N\ln(\theta_n)\epsilon^2}{9\ln(\frac{1}{4})}\right) \\
&= \exp\left(-\frac{2N\ln(\theta_n)\epsilon^2}{9\ln(\frac{1}{4})} + (2n+1)\ln(2)\right). \qquad (2.3.11)
\end{aligned}$$

For large n we have, $\ln(\theta_n) \leq -\rho_n^2\|r\|_1^2\delta^4$. This implies

$$\frac{2N\ln(\theta_n)\epsilon^2}{9\ln(\frac{1}{4})} - (2n+1)\ln(2) \geq -\frac{2N\rho_n^2\|r\|_1^2\delta^4\epsilon^2}{9\ln(\frac{1}{4})} - (2n+1)\ln(2). \qquad (2.3.12)$$

Using the above inequality we have

$$\begin{aligned}
&\mathbb{P}'\left[\sup_{S=\cup_{m=1}^{k}A_{n,j_m},T=\cup_{m=1}^{l}A_{n,j_m}}|f_{S,T}(X^{m,n}(s))-\mathbb{E}'f_{S,T}(X^{m,n}(s))|>\epsilon\right]\\
&\leq 2^{2n+1}\exp\left(-\frac{2N\ln(\theta_n)\epsilon^2}{9\ln(\frac{1}{4})}\right)\\
&=\exp\left(\frac{2N\rho_n^2\|r\|_1^2\delta^4\epsilon^2}{9\ln(\frac{1}{4})}+(2n+1)\ln(2)\right). \qquad (2.3.13)
\end{aligned}$$

Then using the above, **(N1)**, (2.3.13) and Borel-Cantelli lemma we have $\mathbb{P}$-a.s.

$$\lim_{n\to\infty}\delta_{\square}\left(\frac{n^2}{N}G^{m,n}(s),\frac{n^2}{N}\mathbb{E}'G^{m,n}(s)\right)=0.$$

So we have proved (i).

Now we prove (ii). Recall $\mathbb{E}'G^{m,n}(s)$ from (2.3.5) and $\mu_{m,n}$ from (2.3.6). We define,

$$E_{m,n}(s)(i,j)=\sum_{k=1}^{N}J_k(s),$$

where $J_k(s)=I_{[(X_{k-1}^{m,n}(s),X_k^{m,n}(s)])\in\{A_{n,ij},A_{n,ji}\}]}$ with $A_{n,ij}=A_{n,i}\times A_{n,j}$. Note that

$$\mathbb{E}'E_{m,n}(s)(i,j)=2N\int_{A_{n,i}}\int_{A_{n,j}}\frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1}dxdy=\frac{2N}{n^2}\mu_{m,n}(s)(i,j).$$

Clearly, $I_{m,n}(s)(i,j)=I_{[E_{m,n}(s)(i,j)>0]}$. We have

$$\begin{aligned}
&d_1\left(\frac{n^2}{N}\mathbb{E}'G^{m,n}(s),H'_{m,n}(s)\right)\\
=&\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{n^2}{N}\mathrm{Vol}(A_{n,i})\mathrm{Vol}(A_{n,j})\left|\mathbb{E}'I_{m,n}(s)(i,j)-(1-\exp(-\mathbb{E}'E_{m,n}(s)(i,j)))\right|.
\end{aligned}$$

Then the above implies

$$\begin{aligned}
&d_{\square}\left(\frac{n^2}{N}\mathbb{E}'G^{m,n}(s),H'_{m,n}(s)\right)\\
\leq&\frac{1}{2N}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\mathbb{E}'I_{m,n}(s)(i,j)-(1-\exp(-\mathbb{E}'E_{m,n}(s)(i,j)))\right|\\
=&\frac{1}{2N}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\mathbb{P}'[E_{m,n}(s)(i,j)=0]-\mathbb{P}'[Z_{m,n}(s)(i,j)=0]\right| \qquad (2.3.14)
\end{aligned}$$

where $Z_{m,n}(s)(i,j)\overset{d}{=}\text{Poisson}\left(\frac{2N}{n^2}\mu_{m,n}(s)(i,j)\right)$ under $\mathbb{P}'$. Let $E_{m,n}^{size}(s)(i,j)$ be a random variable having the size-bias distribution of $E_{m,n}(s)(i,j)$. Then the Stein-Chen method(see Barbour et al.[5, Theorem 1.B], yields under $\mathbb{P}'$,

$$d_{TV}\left(\mathcal{L}(E_{m,n}(s)(i,j)),\text{Poisson}\left(\frac{2N}{n^2}\mu_{m,n}(s)(i,j)\right)\right)\leq\mathbb{E}'\left|E_{m,n}(s)(i,j)-(E_{m,n}^{size}(s)(i,j)-1)\right|, \qquad (2.3.15)$$

where $\mathcal{L}(E_{m,n}(s)(i,j))$ refers to the law of random variable $E_{m,n}(s)(i,j)$ under $\mathbb{P}'$ and d_{TV} denotes the total variation distance.

Recall $c_1(n)$ and $c_2(n)$ from (2.3.4). Then using [3, Lemma 4.2] we have,

$$\begin{aligned} d_\square\left(\frac{n^2}{N}\mathbb{E}'G^{m,n}(s), H'_{m,n}(s)\right) &\le \frac{(1+2c_1(n))n}{N} + \frac{nc_2(n)}{Nc_1(n)} \\ &\le \frac{(1+2\|r\|_1\delta^2\rho_n)n}{N} + \frac{n}{\delta^4\|r\|_1{}^2\rho_n^2 N}. \end{aligned} \tag{2.3.16}$$

Then using Assumption **(N1)** we can conclude $\mathbb{P}$-a.s.

$$\lim_{n\to\infty}\delta_\square\left(\frac{n^2}{N}\mathbb{E}'G^{m,n}(s), H'_{m,n}(s)\right) = 0.$$

It remains to prove (iii). Recall $H'_{m,n}$ from (2.3.6). Using triangle inequality we have

$$\begin{aligned} &d_\square\left(H'_{m,n}(s), \frac{h^m(s)}{\|h^m(s)\|_1}\right) \\ &\le \left\|g_{m,n}(s) - \frac{h^{m,n}(s)}{\|h^{m,n}(s)\|_1}\right\|_1 + \left\|\frac{h^{m,n}(s)}{\|h^{m,n}(s)\|_1} - \frac{h^m(s)}{\|h^m(s)\|_1}\right\|_1 \end{aligned} \tag{2.3.17}$$

where $g_{m,n}(s)$ is the graphon associated with the graph $H'_{m,n}(s)$ given by

$$g_{m,n}(s;x,y) = \frac{n^2}{N}\left(1-\exp\left(-\frac{2N}{n^2}\frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1}\right)\right).$$

From (1.3.16) and (1.3.17) we have $\mathbb{P}-$a.s.

$$\left\|\frac{\kappa_{m,n}(s)}{\|\kappa_{m,n}(s)\|_1} - \frac{h^m(s)}{\|h^m(s)\|_1}\right\|_1 \to 0 \text{ as } n\to\infty.$$

It is remaining to show that $\mathbb{P}$-a.s.

$$\left\|g_{m,n}(s) - \frac{\kappa_{m,n}(s)}{\|\kappa_{m,n}(s)\|_1}\right\|_1 \to 0 \text{ as } n\to\infty.$$

It is equivalent to show that, for arbitrary $\epsilon>0$, we have a $n_0\in\mathbb{N}$ such that

$$\left\|g_{m,n}(s) - \frac{\kappa_{m,n}(s)}{\|\kappa_{m,n}(s)\|_1}\right\|_1 < 2\epsilon$$

for all $n\ge n_0$. Through Taylor approximation, we get, $0\le x-(1-e^{-x)}\le\min\{x,x^2\}$. Hence, we have for any $x,y\in\mathbb{R}$ that

$$\begin{aligned} \left|g_{m,n}(s;x,y) - \frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1}\right| &= \frac{n^2}{N}\left|\left(1-\exp\left(-\frac{2N}{n^2}\frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1}\right)\right) - \frac{2N}{n^2}\frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1}\right| \\ &\le \frac{n^2}{2N}\min\left\{\frac{2N}{n^2}\frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1}, \left(\frac{2N}{n^2}\frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1}\right)^2\right\} \\ &= \min\left\{1, \frac{2N}{n^2}\frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1}\right\}\frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1}. \end{aligned} \tag{2.3.18}$$

Integrating both side of (2.3.18) we get,

$$\left\| g_{m,n}(s) - \frac{\kappa_{m,n}(s)}{\|\kappa_{m,n}(s)\|_1} \right\|_1 \leq \int_0^1 \int_0^1 \min\left\{1, \frac{2N}{n^2} \frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1}\right\} \frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1} dxdy. \tag{2.3.19}$$

Let f be a graphon and $\tau > 0$. Then we define the graphon $f \wedge \tau$ as $(f \wedge \tau)(x,y) = f(x,y) \wedge \tau$. We also define graphon $(f)_n$ as

$$(f)_n(u,v) = n^2 \int_{A_{n,i}} \int_{A_{n,j}} \frac{f(u,v)}{\|f(u,v)\|_1} \quad for \quad x,y \in A_{n,i} \times A_{n,j}.$$

Then from (2.3.19) we have

$$\begin{aligned}\left\| g_{m,n}(s) - \frac{\kappa_{m,n}(s)}{\|\kappa_{m,n}(s)\|_1} \right\|_1 \leq & \int_0^1 \int_0^1 \min\left\{1, \frac{2N}{n^2} \frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1}\right\} \left(\frac{h^m(s)}{\|h^m(s)\|_1} \wedge \tau\right)_n (x,y) dxdy \\ & + \left\| \frac{\kappa_{m,n}(s)}{\|\kappa_{m,n}(s)\|_1} - \left(\frac{h^m(s)}{\|h^m(s)\|_1} \wedge \tau\right)_n \right\|_1 . \end{aligned} \tag{2.3.20}$$

Observe that, $\left(\frac{\kappa_{m,n}(s)}{\|\kappa_{m,n}(s)\|_1}\right)_n (u,v) = \frac{\kappa_{m,n}(s;u,v)}{\|\kappa_{m,n}(s;u,v)\|_1}$. Then it can be seen through contraction property that,

$$\left\| \frac{\kappa_{m,n}(s)}{\|\kappa_{m,n}(s)\|_1} - \left(\frac{h^m(s)}{\|h^m(s)\|_1} \wedge \tau\right)_n \right\|_1 \leq \left\| \frac{\kappa_{m,n}(s)}{\|\kappa_{m,n}(s)\|_1} - \frac{h^m(s)}{\|h^m(s)\|_1} \wedge \tau \right\|_1 .$$

Let $\epsilon > 0$, then there exists $\tau > 0$ such that

$$\left\| \frac{h^m(s)}{\|h^m(s)\|_1} - \frac{h^m(s)}{\|h^m(s)\|_1} \wedge \tau \right\|_1 < \epsilon.$$

Then

$$\min\left\{1, \frac{2N}{n^2} \frac{\kappa_{m,n}(s;x,y)}{\kappa_{m,n}(s)}\right\} \left(\frac{h^m(s)}{\|h^m(s)\|_1} \wedge \tau\right)_n (x,y) dxdy$$

converges to 0 pointwise and is bounded by τ, using dominated convergence we can conclude that there exists $n_0 > 0$ such that

$$\int_0^1 \int_0^1 \min\left\{1, \frac{2N}{n^2} \frac{\kappa_{m,n}(s;x,y)}{\kappa_{m,n}(s)}\right\} \left(\frac{h^m(s)}{\|h^m(s)\|_1} \wedge \tau\right)_n (x,y) dxdy < \epsilon$$

for all $n \geq n_0$. Therefore we have

$$\left\| g_{m,n}(s) - \frac{\kappa_{m,n}(s)}{\|\kappa_{m,n}(s)\|_1} \right\|_1 < 2\epsilon \text{ for all } n \geq n_0.$$

Then we can conclude $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \delta_\square \left(H'_{m,n}(s), \frac{\tilde{h}^m(s)}{\|\tilde{h}^m(s)\|_1} \right) = 0.$$

This concludes the proof of **Claim 1**.

□

Now we prove **Claim 2**.

*Proof of **Claim 2**.* Recall $(\widetilde{\mathcal{W}}_1, \delta_\square)$ which is a compact space (see Borges et al.[7, Theorem 2.13]), so $(\widetilde{\mathcal{W}}_1, \delta_\square)$ is also complete and separable. We denote by $D = D([0,\infty, \widetilde{\mathcal{W}}_1)$ the set of cádlág paths in $\widetilde{\mathcal{W}}_1$. Then under suitable metric the space D is complete and separable, because $(\widetilde{\mathcal{W}}_1, \delta_\square)$ is also complete and separable. Recall $H'_{m,n}$ and $\mu_{m,n}$ from (2.3.6). Let $s, t \geq 0$. Then

$$\begin{aligned}
&d_\square(H'_{m,n}(s), H'_{m,n}(t)) \\
&= \sup_{u,v\in\{0,1\}^n} \left| \sum_{i,j} u_i v_j \frac{1}{N}\left(1 - \exp\left(-\frac{2N}{n^2}\mu_{m,n}(s)(i,j)\right)\right) - \sum_{i,j} u_i v_j \frac{1}{N}\left(1 - \exp\left(-\frac{2N}{n^2}\mu_{m,n}(t)(i,j)\right)\right)\right| \\
&= \sup_{u,v\in\{0,1\}^n} \left| \frac{1}{N}\sum_{i,j} u_i v_j \exp\left(-\frac{2N}{n^2}\mu_{m,n}(s)(i,j)\right) - \frac{1}{N}\sum_{i,j} u_i v_j \exp\left(-\frac{2N}{n^2}\mu_{m,n}(t)(i,j)\right)\right|. \qquad (2.3.21)
\end{aligned}$$

Recall $Z_{m,n}(s)(i,j) \stackrel{d}{=} \text{Poisson}\left(\frac{2N}{n^2}\mu_{m,n}(s)(i,j)\right)$ under $\mathbb{P}'$. The the above implies

$$\begin{aligned}
&d_\square(H'_{m,n}(s), H'_{m,n}(t)) \\
&= \sup_{u,v\in\{0,1\}^n} \left| \frac{1}{N}\sum_{i,j} u_i v_j \mathbb{P}'[Z_{m,n}(s)(i,j) = 0] - \frac{1}{N}\sum_{i,j} u_i v_j \mathbb{P}'[Z_{m,n}(t)(i,j) = 0]\right| \\
&= \sup_{u,v\in\{0,1\}^n} \left| \frac{1}{N}\mathbb{P}'\left[\sum_{i,j} u_i v_j Z_{m,n}(s)(i,j) = 0\right] - \frac{1}{N}\mathbb{P}'\left[\sum_{i,j} u_i v_j Z_{m,n}(t)(i,j) = 0\right]\right|. \qquad (2.3.22)
\end{aligned}$$

Observe that $\sum_{i,j} u_i v_j Z_{m,n}(s)(i,j) \stackrel{d}{=} \text{Poisson}\left(\sum_{i,j} u_i v_j \frac{2N}{n^2}\mu_{m,n}(s)(i,j)\right)$ under $\mathbb{P}'$. Then from the above and [30, Theorem 1] we have

$$\begin{aligned}
&d_\square(H'_{m,n}(s), H'_{m,n}(t)) \\
&\leq \left| \frac{2}{n^2}\sum_{i,j} u_i v_i \mu_{m,n}(s)(i,j) - \frac{2}{n^2}\sum_{i,j} u_i v_i \mu_{m,n}(t)(i,j)\right|.
\end{aligned}$$

Then (2.3.21) and the above implies

$$\begin{aligned}
&d_\square(H'_{m,n}(s), H'_{m,n}(t)) \\
&\leq \sup_{u,v\in\{0,1\}^n} \left| \sum_{i,j} 2u_i v_i \int_{A_{n,i}}\int_{A_{n,j}} \frac{\kappa_{m,n}(s;x,y)}{\|\kappa_{m,n}(s)\|_1} dxdy - \sum_{i,j} 2u_i v_i \int_{A_{n,i}}\int_{A_{n,j}} \frac{\kappa_{m,n}(t;x,y)}{\|\kappa_{m,n}(t)\|_1} dxdy\right| \\
&\leq d_\square(K^{m,n}(s), K^{m,n}(t)) \qquad (2.3.23)
\end{aligned}$$

Using (1.3.32) we showed in Chapter 1 that $\{K^{m,n}\}_{n\geq 1}$ is tight in $\widetilde{W}_1$. Then from tightness of $\{K^{m,n}\}_{n\geq 1}$ and (2.3.23) we can conclude that $\{H'_{m,n}\}_{n\geq 1}$ is tight in $\widetilde{W}_1$. (See condition (b) of Ethier and Kurtz [17, Corollary 7.4] and Ethier and Kurtz[17, Inequality (6.3)].)

We have shown in (2.3.16) that,

$$d_\square\left(\frac{n^2}{N}\mathbb{E}'G^{m,n}(s), H'_{m,n}(s)\right) \leq \frac{(1+2\delta)n}{2N} + \frac{2n}{\|r\|_1^2\delta^4\rho_n^2 N}.$$

Then using tightness of $\{H'_{m,n}\}_{n\geq 1}$ and the above we can conclude $\left\{\frac{n^2}{N}\mathbb{E}'G^{m,n}\right\}_{n\geq 1}$ is tight in $\widetilde{W}_1$.

(See condition (b) of Ethier and Kurtz[17, Corollary 7.4] and Ethier and Kurtz[17, Inequality (6.3)]. So **Claim 2** is proved. □

Chapter 3

A trajectorial representation of certain non-linear equation

In [37] Sznitman considered a system of N Brownian spheres of radius $N^{-\frac{1}{d-2}}$ in $\mathbb{R}^d$, where $d \geq 3$. The center of these spheres evolve according to independent Brownian motions, initially distributed as $u_0(dx) = V(x)dx$. The spheres are destroyed as soon as a collision occurs between any two of them. We will refer the centers as particles and the spheres as balls of radius $N^{-\frac{1}{d-2}}$ around the particles. The term "alive" indicates that the particle has not been destroyed through above interactions. As $N \to \infty$, the author showed that the probability of each particle being alive in the system is given by the solution, $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}_+$, of the nonlinear equation

$$\partial_t u = \frac{1}{2}\Delta u - c_d u^2, u(0, \cdot) = u_0 \tag{3.0.1}$$

where $u_0 : \mathbb{R}^d \to \mathbb{R}_+$ and c_d is a constant depending on dimension.

Further, a collision tree was constructed by analyzing the interactions between N spheres while "neglecting" the destruction of spheres caused by collisions. By focusing on the collision tree, Sznitman derived the limiting behavior of the annihilating Brownian sphere system. We shall explain the chain reaction via an example. Let $S^1, S^2, \ldots, S^N$ be N Brownian spheres. We say the Brownian sphere S^2 has a chain reaction leading to the Brownian sphere S^1 before time t if the Brownian sphere S^2 has a collision with another sphere which later has a collision with a sphere and goes on and finally a sphere which collides with Brownian sphere S^1 before time t. Here we are mentioning the collisions but we ignore the destruction of the particles. Using the concept of chain reaction the author had shown that under finite time t the system can be represented as a tree structure where the nodes of the tree represent a path of finite length. We call these trees collision trees.

However in [38] Sznitman took a diffrent approach, first constructing the space of trees where nodes are finite length Brownian particle paths and then built the interactions on the trees. Under the interaction the trees emerge as collision trees. One starts with a Brownian particle (the ancestor) B_s running until time t, with initial density u_0. Then one constructs trajectories having a collision with the ancestor as follows: One picks conditionally on B_s a Poisson distribution of points on $[0, t]$. Now conditionally on B_s and on following time points $0 < t_1 < t_2 < \ldots < t_n < t$ one constructs independent Brownian bridges, $W_1, \ldots, W_n$. Now $W_l, 1 \leq l \leq n$ has the law of a Brownian motion on time $[0, t_l]$ with initial density u_0, conditioned to be equal to B_{t_l} at time t_l. These constitute the first generation. Now one perform the same thing on each of the $W_l, 1 \leq l \leq n$, as we have just described on B. And one obtains the second generation and one goes on like this. The second step is now to construct the interaction (telling whether a trajectory in the tree is already destroyed by the time it meets its direct ancestor, that is for instance time t_l for W_l). It can be shown that the trees are almost surely finite under a suitable measure. Then one defines an interaction on the trees which informs us whether a particle will be dead and or alive. One considers the end branches in the tree, that is those which do not have descendants. These are said to be alive. Now recursively one can tell for each particle, whether it is alive or dead at the time it meets its direct ancestor, by stating that if one of its direct descendants is alive, then the particle is dead. For instance in the case of B, our basic ancestor, it is dead if one of the $W_1, ..., W_n$ is alive. The interaction transform the trees into collision trees. The result is now that the probability $u(\cdot,\cdot)$ of B staying alive satisfies

$$\partial_t u = \frac{1}{2}\Delta u - u^2, u(0,\cdot) = u_0$$

where $u_0 : \mathbb{R}^d \to \mathbb{R}_+$. $u(t,x)$ gives the probability of B is still alive at time t and located at position x. Note that $u(\cdot,\cdot)$ is not an asymptotic probability.

In this chapter, we aim to construct a trajectorial representation for the solution $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}_+$ of the nonlinear equation

$$\partial_t u = \frac{1}{2}\Delta u - u^{1+\alpha}, \quad u(0,\cdot) = u_0, \tag{3.0.2}$$

where $u_0 : \mathbb{R}^d \to \mathbb{R}_+$ and $0 < \alpha < 1$.

The introduction of the exponent $1+\alpha$, where $\alpha \in (0,1)$, modifies the interaction mechanism among particles. Here, the particles follow Brownian motion as their primary mode of movement, while α-stable subordinators introduce an additional influence on their dynamics. This extension differs from the model in [38], where particle dynamics are governed solely by Brownian motion, making the current framework more intricate and reflective of the nonlinear term's effect.

Using techniques developed in part in [38] we will construct the trajectorial representation. We will be constructing the representation in two ways. One way is to use approximation of stable subordinators (see Section 3.2) and the other way is to use stable subordinators directly (see Section 3.4).

3.1 Preliminaries

In this Section we will discuss the necessary preliminaries required to state and prove our main result. We discuss laws of Brownian motion and it's conditional laws related to it. We define the basic tree.

We consider a Brownian motion $(C(\mathbb{R}_+, \mathbb{R}^d), (X_t)_{t\geq 0}, P^x, x \in \mathbb{R}^d)$ with continuous trajectories on $\mathbb{R}^d$, and transition density $p_t(x,y)$ with respect to Lebesgue measure m on $\mathbb{R}^d$, satisfying Chapman-Kolmogorov equation:

$$p_{t+s}(x,y) = \int_{\mathbb{R}^d} p_t(x,z)p_s(z,y)m(dz) \;\; t,s > 0; \quad x,y \in \mathbb{R}^d \text{ and} \tag{3.1.1}$$

$$\int_{\mathbb{R}^d} p_t(x,y)m(dx) \leq C_T \leq \infty \text{ for } t \leq T \tag{3.1.2}$$

We suppose we are given initial law $u_0(dx) = V(x)m(dx), V$ is bounded and we take

$$V(t,y) = \int_{\mathbb{R}^d} p_t(x,y)V(x)m(dx) > 0, \text{ for } t > 0. \tag{3.1.3}$$

We denote by P the law of Brownian motion with initial law u_0, and by P^t the image of this law on $C([0,t],\mathbb{R}^d)$, for $t > 0$. We assume we can define the probabilities $P^{t,x}$ on $C([0,t],\mathbb{R}^d)$, for $t > 0, x \in \mathbb{R}^d$ (defining the conditional law of P^t given $X_t = x$), satisfying (see [16], Section 3)

$$P^{t,x}(X_t = x) = 1, \tag{3.1.4}$$

$$P^t = \int P^{t,x} u_t(dx), \text{ for } u_t(dx) = V(t,x)m(dx), \tag{3.1.5}$$

$$P^{t,x}[A] = V(t,x)^{-1} E^t[p_{t-s}(X_s,x)1_A] \text{ for } 0 \le s < t, A \in \sigma(X_u, u \le s). \tag{3.1.6}$$

3.1.1 Basic tree

Definition 3.1.1. *A tree is a subset π of $U = \cup_{n\ge 0}\mathbb{N}^n$, satisfying:*

(i) $\varnothing \in \pi$.

(ii) u belongs to π whenever $uj, j \in \mathbb{N}$ belongs to π.

(iii) $u \in \pi$ then $\exists\ \nu_u \in \mathbb{N}$ such that $uj \in \pi$ for $1 \le j \le \nu_u$.

$\varnothing \in \pi$ is called the root of the tree π. The $u \in U$ that belongs to the tree π are called nodes of the tree π. The pair of nodes of the form (u, uj) where $u \in U$ and $j \in \mathbb{N}$ are called branches of the tree π. ν_u is called number of children of the node u in the tree π (see [27, Section 2]). This labeling is called Ulam-Harris labeling.

We denote by $C = \cup_{t>0} C([0,t],\mathbb{R}^d)$. Let $\tau : C \to (0,\infty)$ be a map such that

$$\tau(\psi) = t \tag{3.1.7}$$

if $\psi \in C([0,t],\mathbb{R}^d)$. We call $\tau(\psi)$ the length of the path ψ. We can identify C with $(0,\infty) \times C([0,1],\mathbb{R}^d)$, through the map $(t,\psi) \to \psi(\frac{u}{t}), 0 \le u \le t$.

Let $\psi \in C$ then we denote $\psi(s) = \psi_s \quad \forall s \in [0,\tau(\psi)]$.

Recall, P^t the image of the law P on $C([0,t],\mathbb{R}^d)$, for $t > 0$. Let $\mathcal{C}_t$ be the corresponding σ-algebra on $C([0,t],\mathbb{R}^d)$. Define

$$\mathcal{C} = \sigma\{\mathcal{C}_t : t \ge 0\}. \tag{3.1.8}$$

Then $\mathcal{C}$ is a σ-algebra on the space C.

3.2 Approximation approach towards trajectorial representation of $\partial_t u = \frac{1}{2}\Delta u - u^{1+\alpha}, 0 < \alpha < 1$

In this section we will first construct marked trees as done by Neveu in [27]. We obtain marked trees by attaching each node of a tree with a mark identified by finite length paths and Poisson point processes approximated from α-stable subordinators. Then for every $\epsilon > 0$ we will construct a probability measure R^{ϵ}_{ψ} on the space of marked trees, see Theorem 3.2.3. Under R^{ϵ}_{ψ}, the marked trees will have finitely many generations a.s., see Lemma 3.2.4. The finite length paths represent trajectories of the particles following those path. As in [38], we construct the marked trees which represent the collision tree and put an interaction on the marked trees, see Section 3.2.2. The interaction on marked trees depict collisions between particles. The interactions are $\{0, 1\}$ valued function Z associated with each node u. We start with nodes which have no children. We assign the Z value 1 to these nodes and then define the Z values of ancestors of these nodes inductively as discussed in Section 3.2.2. If Z value of a node is 1 then it is considered alive, else destroyed.

The root path is the finite path which is associated as a mark with the root.Then the probability of the event that the root path is alive is equal to the probability of the event that the Z value of the root path is 1. We call $u_\epsilon(x, t)$ the probability of a Brownian particle surviving under the interaction at time $t \in [0, T]$ and at point $x \in \mathbb{R}^d$.

Then the main result of this section, Theorem 3.2.6 states, as ϵ tends 0, we have u_ϵ goes to u where $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}_+$ satisfies

$$\partial_t u = \frac{1}{2}\Delta u - u^{1+\alpha}, u(0, \cdot) = u_0. \tag{3.2.1}$$

Recall that $C = \cup_{t>0} C([0, t], \mathbb{R}^d)$ and denote $A :=$ Space of all point measures on $\mathbb{R}_+ \times \mathbb{R}_+$. Let $\mathcal{A}$ be the smallest σ-algebra on A containing all sets of the form $\{m \in A \mid m(F) \in B\}$ for $F \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_+), B \in \mathcal{B}([0, \infty]$ where $\mathcal{B}$ denotes Borel-σ-algebra. (see [29]).

Define, $N^u_1 = \sum_{l=1}^{\infty} \delta_{(H^u_l, Y^u_l)}$, where H^u_l and Y^u_l denote the $l-$th hitting point and weigth associated to node u respectively. Note that (H^u_l, Y^u_l) takes values in $\mathbb{R}_+ \times \mathbb{R}_+$. With abuse of notation we denote $\nu_\varnothing$ as ν, $N^\varnothing_1$ as N_1, $H^\varnothing_l$ as H_l and $Y^\varnothing_l$ as Y_l.

Now we define marked trees.

Definition 3.2.1 (Marked tree)**.** *Let π be a tree as in Definition 3.1.1. A marked tree ω is of the form $\omega = (\pi, ((N^u_1 = \sum_{l\geq 1} \delta_{(H^u_l, Y^u_l)}, \phi^u), u \in \pi)))$, where N^u_1 denote the mark on node u from A. And ϕ^u denote the mark on node u from C. ϕ^u represents the path attached to node u and N^u_1 represents the point measure attached to node u. We call Ω be the set of all marked trees.*

We denote by G_n the n-th generation of the tree, that is to say, $\pi \cap \mathbb{N}^n$. Denote

$$\mathcal{F} = \sigma\{N^u_1, \phi^u, \Omega^u : u \in U\} \text{ and } \mathcal{F}_n = \sigma\{N^u_1, \phi^u, \Omega^u : |u| \leq n\}. \tag{3.2.2}$$

Here, $\Omega_u = \{\omega \in \Omega : u \in \omega\}$. Now we define the translation of a marked tree at node u.

Definition 3.2.2 (Translation of a marked tree)**.** *$T_u(\omega)$ be the translation of of marked tree ω. $T_u(\omega) = (\pi = \{v : v \in U \text{ and } uv \in \Omega\}, ((N^a_1, \phi^a), a \in \pi)))$.*

3.2.1 Results on space Marked trees

Now we state our first result related to the approximation approach. We construct a probability measure on the space of marked trees. Under this measure the marked trees will be finite. The path $\phi^{uj}, 1 \leq j \leq \mu$, will represent the trajectory of the particle which hit the path ϕ^u at time $\tau(\phi^{uj})$(the probabilities on the tree will be such that $\tau(\phi^u) > \tau(\phi^{uj})$ a.s.).

Proposition 3.2.3. *Fix $\psi \in C$. Let ψ be a path of length t. Then for each $\epsilon > 0$, $\exists$ a probability measure R^{ϵ}_{ψ} on Ω such that:*

(i) $R^{\epsilon}_{\psi}[\phi^{\varnothing} = \psi] = 1$

(ii) $N_1 = \sum_{l \geq 1} \delta_{(H_l, Y_l)}$ *is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\lambda_\epsilon \otimes Q_\epsilon$, where $\lambda_\epsilon = \alpha\Gamma(\alpha)\epsilon^{-\alpha}$, $Q_\epsilon(A) = \int_A \pi_\epsilon(x)dx$, A is Lebesgue measurable. And*

$$\pi_\epsilon(x) = \frac{\alpha}{\Gamma(1-\alpha)}(x-\epsilon)^{-\alpha}x^{-1}1_{\{x \geq \epsilon\}} \tag{3.2.3}$$

where $\alpha \in (0,1)$ and Γ is Gamma function.

(iii) $E^{\epsilon}_{\psi}[\prod_{u \in G_n} f_u \circ T_u | \mathcal{F}_n] = \prod_{u \in G_n} E^{\epsilon}_{\phi^u}[f_u]$ *where E^{ϵ}_{ψ} is the expectation under R^{ϵ}_{ψ}.*

(iv) Under, $R^{\epsilon}_{\psi}[\cdot|N_1]$ we have, $N = \sum_{l=0}^{\nu} \delta_{\phi^l}$ is Poisson random measure with intensity

$$\mu_N(\cdot) = \sum_{i=1}^{N_1[(0,t)\times\mathbb{R}_+]} V(H_i, \psi_{H_i}) P^{H_i, \psi_{H_i}}(\cdot) Y_i \tag{3.2.4}$$

We prove the above Theorem in Section 3.3.1. Now we state few properties of R^{ϵ}_{ψ}. These properties are needed for our main result.

Lemma 3.2.4. *For every $\epsilon > 0$, under R^{ϵ}_{ψ} the marked trees have finitely many generations a.s..*

Lemma 3.2.5. *For every $\epsilon > 0$, under $R^{\epsilon}_{\psi}[\cdot|N_1]$, $M = \sum_{l=1}^{\nu} \delta_{T_l}$ is Poisson random measure with intensity*

$$\mu_M(\cdot) = \sum_{i=1}^{N_1[(0,t)\times\mathbb{R}_+]} V(H_i, \psi_{H_i}) R_{\epsilon}^{H_i, \psi_{H_i}}(\cdot) Y_i \tag{3.2.5}$$

where $R^{t,x}_{\epsilon} = \int R^{\epsilon}_{\psi} P^{t,x}(d\psi)$

We prove these Lemmas in Section 3.3.3. Now we can define an interaction on the marked trees since marked trees are finite a.s. under R^{ϵ}_{ψ}.

3.2.2 Interaction on marked tree and Main result

We stated in Lemma 3.2.4 that there exists finitely many generations in a marked tree under R^{ϵ}_{ψ}. So we can construct an interaction on the marked trees. Let $Z^u, u \in \pi$ be marks on our basic tree, which will tell us whether ϕ^u is alive or not at time $\tau(\phi^u)$, in the sense that it has collided with trajectories $\phi^{u,l+1}$ for $1 \leq l \leq \nu_u$ that were all alive at the time of collision $\tau(\phi^{u,l})$.

We define $D_u(\omega)$, for $u \in \omega$, to be the number of generations descending from u, that is $\max\{|v|, uv \in \omega\}$, which is R^{ϵ}_{ψ}-a.s. finite. We set for $u \in \pi : Z^u = 1$ if $D_u = 0$, and by induction

$$Z^u = (1 - \sum_{l=1}^{\nu_u} Z^{u,l})_+, \tag{3.2.6}$$

in other words, Z^u is zero as soon as a collision with a path with mark Z equals to 1 happens.

Now we give an pictorial example to understand the interaction.

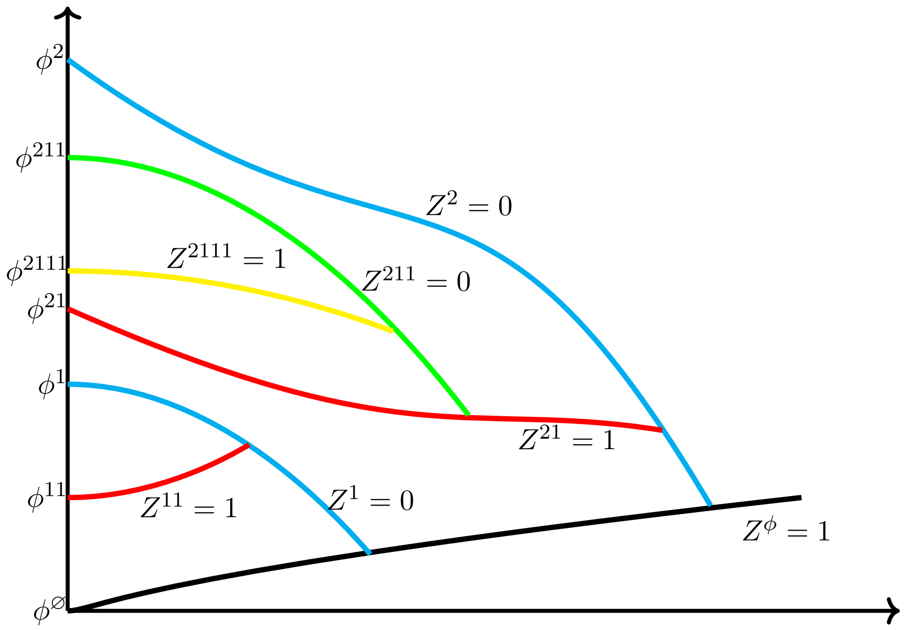

FIGURE 3.1: An example of the interaction on $\mathbb{R}^2$

In the Figure 3.1 the root path is coloured by black and denoted by $\phi^{\varnothing}$. The second generation paths are coloured by light blue. So there are two paths in the second generation, denoted by ϕ^1 and ϕ^2. The third generation paths are coloured by red. So there are two paths in the second generation, denoted by ϕ^{11} and ϕ^{21}. The third generation paths are coloured by green. So there is one path in the third generation, denoted by ϕ^{211}. The fourth generation paths are coloured by yellow. So there is one path in the fourth generation, denoted by ϕ^{2111}. We have used the equation (3.2.6) to calculate the Z values of each path in each generation mentioned above. We can see in the above example the root path has Z value 1, which means it survives the interaction. If the path has length t, then it is alive till time t.

Recall P^t and $P^{t,x}$ from (3.1.4) and (3.1.5) respectively. We denote by R^t_ϵ and $R^{t,x}_\epsilon$, probability measures on Ω defined by

$$R^t_\epsilon(\cdot) = \int R^\epsilon_\psi(\cdot)P^t(d\psi) \text{ and } R^{t,x}_\epsilon(\cdot) = \int R^\epsilon_\psi(\cdot)P^{t,x}(d\psi). \tag{3.2.7}$$

Define for $\epsilon > 0$ and $\eta > 0$,

$$\Phi_\epsilon(\eta) = \int_0^\infty (1 - e^{-\eta x})\pi_\epsilon(x)dx, \Phi(\eta) = \eta^\alpha, \tag{3.2.8}$$

where

$$\pi_\epsilon(x) = \frac{\alpha}{\Gamma(1-\alpha)}(x-\epsilon)^{-\alpha}x^{-1}1_{x\geq\epsilon}.$$

Define

$$\Psi_\epsilon(\eta) = \eta\Phi_\epsilon(\eta), \Psi(\eta) = \eta^{1+\alpha}. \tag{3.2.9}$$

Set

$$u_\epsilon(t,x) = V(t,x)R^{t,x}_\epsilon[Z^{\varnothing} = 1] \tag{3.2.10}$$

for $t > 0$ and $u_\epsilon(0,x) = V(x)$. $u_\epsilon(x,t)$ the probability of a Brownian particle surviving under the interaction at time $t \in [0,T]$ and at point $x \in \mathbb{R}^d$. For convenience, we will write ψ for $\phi^{\varnothing}$ and Z for $Z^{\varnothing}$, in what follows. Now we can state our main result of this section.

Theorem 3.2.6. *The limit*

$$\lim_{\epsilon \to 0} u_\epsilon(t,x) = u(t,x)$$

exists for every $t \geq 0$ and $x \in \mathbb{R}^d$, and the convergence is uniform on the set $[0,T] \times \mathbb{R}^d$ where $u_\epsilon(t,x)$ satisfies

$$u_\epsilon(t,x) = V(t,x) - \int_0^t ds \int dy\, p_{t-s}(y,x)\Psi_\epsilon(u_\epsilon(s,y)) \tag{3.2.11}$$

Furthermore, $u(t,x)$, $0 \leq t \leq T, x \in E$, is the unique solution in $L^\infty([0,T] \times E, dsdm(x))$ of the equation :

$$w(t,x) = V(t,x) - \int_0^t \int_E p_{t-s}(y,x) w^{1+\alpha}(s,y) dsdm(y), \quad dt \otimes dm - a.s. \tag{3.2.12}$$

where $0 < \alpha < 1$.

We will prove the theorem in the next section. The equation (3.2.12) is an integral form of the pde (3.2.1).

3.3 Proof of Theorem 3.2.6

In this section we first prove the existence of the measure R^ϵ_ψ, Theorem 3.2.3. Then we prove that, under R^ϵ_ψ we have marked trees are finite (Lemma 3.2.4) and variables $\{T_i\}$ follows Poisson Random Measure (Lemma 3.2.5). Then we prove our main result, Theorem 3.2.6.

3.3.1 Proof of Proposition 3.2.3

Before proving Theorem 3.2.3 let us define necessary measure space and discuss the proof strategy. Recall $A :=$ Space of all point measures on $\mathbb{R}_+ \times \mathbb{R}_+$, $C = \cup_{t>0} C([0,t], \mathbb{R}^d)$ and $\mathcal{A}$, $\mathcal{C}$ are σ-algebras on A, C respectively. Let U be as in Definition 3.1.1. We consider the product space $\Omega^\star = (A \times C \times \mathbb{N})^U$. For $u \in U$ define

$$(N_1^u)^\star : \Omega^\star \to A, (\phi^u)^\star : \Omega^\star \to C \text{ and } \nu_u^\star : \Omega^\star \to \mathbb{N} \tag{3.3.1}$$

be projection maps. Define

$$\mathcal{F}^\star = \sigma\{(N_1^u)^\star, (\phi^u)^\star, \nu_u^\star : u \in U\} \text{ and } \mathcal{F}_n^\star = \sigma\{(N_1^u)^\star, (\phi^u)^\star, \nu_u^\star : |u| \leq n\}. \tag{3.3.2}$$

Construction of $\mathbb{P}^\star_\psi$ on $(\Omega^\star, \mathcal{F}^\star)$ for $\psi \in C$: First for each $u \in U$, we will define measure $\mathbb{P}^u$ on $(A \times C \times \mathbb{N})$.

Fix $\epsilon > 0$. Let,

$$J_1^u = \sum_{l=1}^\infty \delta_{(H_l^u, Y_l^u)}$$

and $(J_1^u)_{u \in U}$ are i.i.d. Poisson point process on some measure space $(\Omega', \mathcal{F}', \mathbb{P}')$ taking values in A with intensity $\lambda_\epsilon \times Q_\epsilon$ for all $u \in U$. Recall $V(\cdot,\cdot)$ and $P^{t,x}$ from (3.1.3) and (3.1.4) repectively. Define for

$\sigma \in C$ and $u \in U$

$$\mu_u^\sigma(\cdot) = \sum_{l=1}^{N_1^u[(0,\tau(\sigma))\times\mathbb{R}_+]} V(H_l^u, \sigma_{H_l^u}) P^{H_l^u, \sigma_{H_l^u}}(\cdot) Y_l^u, \tag{3.3.3}$$

which is a measure on C. We will abuse of notation and write $\mu_\varnothing^\psi$ as μ_ψ. Now we define probability measure $\mathbb{P}^u$ for all $u \in U$ on $(A \times C \times \mathbb{N})$. Let $u = u_1 u_2 ... u_p$ where $u_l \in \mathbb{N}$ for $l = 1$ to p. Let, $u^l = u_1 u_2 ... u_l$ for all $l = 1$ to p. Let E, F belong to $\sigma-$algebra on A, C respectively and $I \subset \mathbb{N}$. Define

$$\begin{aligned}\mathbb{P}^u(E, F, I) = \frac{1}{Q_p} \int_{\Omega'} d\mathbb{P}' \int_{\Omega'} d\mathbb{P}' \ldots \int_{\Omega'} d\mathbb{P}' \int_{(J_1^u)^{-1}(E)} d\mathbb{P}' \\ \int_{\tilde{F}_p} \mu_\varnothing^\psi(d\sigma_1) \mu_{u^1}^{\sigma_1}(d\sigma_2) \ldots \mu_{u^{p-2}}^{\sigma_{p-2}}(d\sigma_{p-1}) \mu_{u^{p-1}}^{\sigma_{p-1}}(d\sigma_p) \sum_{k \in I} \frac{e^{-\mu_u^{\sigma_p}(C)} (\mu_u^{\sigma_p}(C))^k}{k!}.\end{aligned} \tag{3.3.4}$$

where

$$\tilde{F}_p = \{(f_1, f_2, \ldots, f_p) \in C^{p-1} \times F : \tau(f_1) > \tau(f_2) > \ldots > \tau(f_p)\},$$

$$\tilde{C}_p = \{(f_1, f_2, \ldots, f_p) \in C^p : \tau(f_1) > \tau(f_2) > \ldots > \tau(f_p)\}$$

and

$$Q_p = \int_{\tilde{C}_p} \mu_\varnothing^\psi(d\sigma_1) \mu_{u^1}^{\sigma_1}(d\sigma_2) \ldots \mu_{u^{p-2}}^{\sigma_{p-2}}(d\sigma_{p-1}) \mu_{u^{p-1}}^{\sigma_{p-1}}(d\sigma_p).$$

For $u = \varnothing$ define

$$\mathbb{P}^\varnothing(E, F, I) = 1_{\{\psi \in F\}} \int_{(J^\varnothing)^{-1}(E)} d\mathbb{P}' \sum_{k \in I} \frac{e^{-\mu_\psi(C)} (\mu_\psi(C))^k}{k!}. \tag{3.3.5}$$

We define

$$\mathbb{P}_\psi^\star = (\mathbb{P}^u)^{\otimes u \in U}. \tag{3.3.6}$$

We have defined $\mathbb{P}^u$ in a way that $\mathbb{P}^u(E, F, I)$ gives us the probability that $(N_1^u)^\star \in E$, $(\phi^u)^\star \in F$ and $(\nu_u)^\star \in I$. Then the probability measure $\mathbb{P}_\psi^\star$ on $\Omega^\star$ helps us in finding the behaviour of the random varables $((N_1^u)^\star, (\phi^u)^\star, (\nu_u)^\star)$ for all $u \in U$.

We will abuse notation and write μ_ψ for $\mu_\varnothing^\psi$. Recall from Ω is the set of all marked trees (see Definition 3.4.2). We define map $t_m : \Omega^\star \to \Omega$ given by

$$t_m(\omega^\star) = (\pi, ((N_1^u, \phi^u), u \in \pi))). \tag{3.3.7}$$

with $\pi = \{u = j_1 ... j_p : p \in \mathbb{N}, j_{k+1} \leq \nu_{j_1 \ldots j_k}^\star(\omega^\star) \text{ if } 1 \leq k < p\}$ and $N_1^u(t_m(\omega^\star)) = (N_1^u)^\star(\omega^\star)$, $\phi^u(t_m(\omega^\star)) = (\phi^u)^\star(\omega^\star)$ for $u \in \pi$. The map $t_m : (\Omega^\star, \mathcal{F}^\star) \to (\Omega, \mathcal{F})$ is measurable and not injective. This map takes any element $\omega^\star \in \Omega^\star$ and brings out a tree structure from $\omega^\star$.

We define

$$R_\psi^\epsilon = \mathbb{P}_\psi^\star \circ t_m^{-1}. \tag{3.3.8}$$

Define $T_u^\star : \Omega^\star \to \Omega^\star$ given by

$$T_u^\star(\omega^\star) = \{\sigma^\star \in \Omega^\star : \sigma^\star(v) = \omega^\star(uv) \quad \forall v \in U\}. \tag{3.3.9}$$

$T_u^\star$ is a translation map on $\Omega^\star$. It shifts $\omega^\star \in \Omega^\star$ to $u \in U$. Recall the definition of T_u from Definition

3.2.2 and $\Omega_u = \{\omega \in \Omega : u \in \omega\}$. We first state certain lemmas related to T_u and $T_u^\star$, these lemmas will be used show that R_ψ^ϵ satisfies (iii) of Proposition 3.2.3.

Lemma 3.3.1. *$T_u^\star$ satisfies the following*

(a) On $t_m^{-1}(\Omega_u)$ we have,

$$T_u \circ t_m = t_m \circ T_u^\star.$$

(b) On $(\Omega^\star, \mathcal{F}^\star, \mathbb{P}_\psi^\star)$, $\forall n \in \mathbb{N}$ fixed, the variables $T_u^\star$ with values in $\Omega^\star$, indices u of length n are independent of each other.

(c) T_u^* is independent of $t_m^{-1}(\Omega_v)$ for $|v| \leq n = |u|$.

(d) $T_u^\star$ are independent of $\sigma(\nu_v^\star, (N_1^u)^\star, (\phi^v)^\star : |v| < n = |u|)$. That is $T_u^\star$ is independent of $t_m^{-1}(\mathcal{F}_n)$.

These lemmas are required to show part (iii) of Theorem 3.2.3. We shall defer the proof of the lemma and prove Theorem 3.2.3 assuming the lemma. The lemmas will be proved after proving the theorem.

Proof of Proposition 3.2.3. First we prove part (i) of the theorem. From (3.3.7) we have,

$$t_m^{-1}[\phi^\varnothing = \psi] = (A \times \{\psi\} \times \mathbb{N}) \times \prod_{u \in U - \{\varnothing\}} (A \times C \times \mathbb{N}).$$

Recall R_ψ^ϵ as in (3.3.8) then we have

$$R_\psi^\epsilon[\phi^\varnothing = \psi] = \mathbb{P}_\psi^\star \circ t_m^{-1}[\phi^\varnothing = \psi].$$

From (3.3.6), the above and $\mathbb{P}^u(A, C, \mathbb{N}) = 1$ for $u \in U \setminus \{\varnothing\}$ we get,

$$R_\psi^\epsilon[\phi^\varnothing = \psi] = \mathbb{P}^\varnothing(A, \{\psi\}, \mathbb{N}) \times \prod_{u \in U \setminus \{\varnothing\}} \mathbb{P}^u(A, C, \mathbb{N}) = \mathbb{P}^\varnothing(A, \{\psi\}, \mathbb{N}).$$

From (3.3.5) we have,

$$\mathbb{P}^\varnothing(E, F, k) = \int_{(J^\varnothing)^{-1}(E)} d\mathbb{P}' \frac{e^{-\mu_\psi(C)}(\mu_\psi(C))^k}{k!} \tag{3.3.10}$$

where E, F belong to $\sigma-$algebra on A, C respectively and $k \in \mathbb{N}$. Then from above equality we get $\mathbb{P}^\varnothing(A, \{\psi\}, \mathbb{N}) = 1$, this implies

$$R_\psi^\epsilon[\phi^\varnothing = \psi] = 1.$$

We conclude part (i) of Proposition 3.2.3 holds. Now we show part (ii) of the proposition. Let E belong to $\sigma-$algebra on A. Then from (3.3.8) we have

$$R_\psi^\epsilon[N_1 \in E] = \mathbb{P}_\psi^\star \circ t_m^{-1}[N_1 \in E].$$

From (3.3.6) and above we get

$$R_\psi^\epsilon[N_1 \in E] = \mathbb{P}^\varnothing(E, C, \mathbb{N}) \times \prod_{u \in U - \{\varnothing\}} \mathbb{P}^u(A, C, \mathbb{N}) = \mathbb{P}^\varnothing(E, C, \mathbb{N}).$$

Then using (3.3.10) and above we get

$$R_\psi^\epsilon[N_1 \in E] = \mathbb{P}' \circ (J_1^\varnothing)^{-1}(E)$$

$\forall E$ in the σ-algebra of A. So $N_1 \overset{d}{=} J_1^{\varnothing}$. $J_1^{\varnothing} = \sum_{l=1}^{\infty} \delta_{(E_l^{\varnothing}, F_l^{\varnothing})}$ is Poisson process with intensity $\lambda_\epsilon \times Q_\epsilon$, so we conclude $N_1 = \sum_{l \geq 1} \delta_{(H_l, Y_l)}$ is a Poisson process with intensity $\lambda_\epsilon \otimes Q_\epsilon$, where $\lambda_\epsilon = \alpha\Gamma(\alpha)\epsilon^{-\alpha}$, $Q_\epsilon(A) = \int_A \pi_\epsilon(x)dx$, A is Lebesgue measurable. We can conclude part (ii) of the Proposition 3.2.3 holds.

Now we show part (iii) of the theorem. Let $B \subset \mathbb{N}^{*n}$ and recall G_n is the n-th generation of the tree. Observe that $\{\omega^\star \in \Omega^\star : B \subset G_n \circ t_m\} = \cap_{u \in B} t_m^{-1}(\Omega_u) \in t_m^{-1}(\mathcal{F}_n)$ where $\mathcal{F}_n$ is as in (3.2.2). So $G_n \circ t_m$ is $t_m^{-1}(\mathcal{F}_n)$ measurable. From part (a) of the Lemma 3.3.1 we have

$$\prod_{u \in B} f_u \circ t_m \circ T_u^\star = \prod_{u \in B} f_u \circ T_u \circ t_m \tag{3.3.11}$$

on $\cap_{u \in B} t_m^{-1}(\Omega_u)$. Now $G_n \circ t_m$ is $t_m^{-1}(\mathcal{F}_n)$ measurable, so applying (3.3.11) on

$$E_{\mathbb{P}_\psi^\star}\left[\prod_{u \in G_n \circ t_m} f_u \circ t_m \circ T_u^\star \mid t_m^{-1}(\mathcal{F}_n)\right]$$

we have,

$$E_{\mathbb{P}_\psi^\star}\left[\prod_{u \in G_n \circ t_m} f_u \circ t_m \circ T_u^\star \mid t_m^{-1}(\mathcal{F}_n)\right] = E_{\mathbb{P}_\psi^\star}\left[\prod_{u \in G_n \circ t_m} f_u \circ T_u \circ t_m \mid t_m^{-1}(\mathcal{F}_n)\right].$$

Then first applying part (d) of the Lemma 3.3.1 and then part (b) of the Lemma 3.3.1 on the left side of the above equality we get

$$\prod_{u \in G_n \circ t_m} E_{\mathbb{P}_\psi^\star}\left[f_u \circ t_m \circ T_u^\star\right] = E_{\mathbb{P}_\psi^\star}\left[\prod_{u \in G_n \circ t_m} f_u \circ T_u \circ t_m \mid t_m^{-1}(\mathcal{F}_n)\right]. \tag{3.3.12}$$

Now for $u \in G_n \circ t_m$ we have

$$\mathbb{P}_\psi^\star \circ (T_u^\star)^{-1} = \mathbb{P}_{(\phi^u)^\star}^\star. \tag{3.3.13}$$

Then applying the above on the left side of (3.3.12) we get

$$E_{\mathbb{P}_\psi^\star}\left[\prod_{u \in G_n \circ t_m} f_u \circ T_u \circ t_m \mid t_m^{-1}(\mathcal{F}_n)\right] = \prod_{u \in G_n \circ t_m} E_{\mathbb{P}_{(\phi^u)^\star}^\star}\left[f_u \circ t_m\right].$$

Then using (3.3.8) in both sides of the above equality and using the relation $(\phi_u)^\star = \phi_u \circ t_m$ on the right side of the equality we get,

$$E_{R_\psi^\epsilon}\left[\prod_{u \in G_n} f_u \circ T_u \mid \mathcal{F}_n\right] = \prod_{u \in G_n} E_{R_{\phi^u}^\epsilon}\left[f_u\right].$$

So part (iii) of Theorem 3.2.3 holds. Now it remains to prove part (iv) of Theorem 3.2.3.

Let N be as in Theorem 3.2.3. We show that $\nu \sim Poisson(\mu_\psi(C))$ and $R_\psi[\phi_l \in F \mid L] = \frac{\mu_\psi(F)}{\mu_\psi(C)}$ where $l \in \mathbb{N}$ and F belong to $\sigma-$algebra on C. This will imply part (iv) of the theorem.

From (3.3.8) and using the relations $N_1 \circ t_m = (N_1^{\varnothing})^\star$ and $\phi_l \circ t_m = (\phi_l)^\star$ we have

$$\begin{aligned} R_\psi[\phi_l \in F \mid N_1] &= \mathbb{P}_\psi^\star \circ t_m^{-1}[\phi_l \in F \mid N_1] \\ &= \mathbb{P}_\psi^\star[\phi_l \circ t_m \in F \mid N_1 \circ t_m] \\ &= \mathbb{P}_\psi^\star[(\phi_l)^\star \in F \mid (N_1^{\varnothing})^\star] \end{aligned}$$

Now first applying (3.3.6) to the last term in the above equation we have,

$$R_\psi[\phi_l \in F \mid N_1] = (\mathbb{P}^u)^{\otimes u \in U}[(\phi_l)^\star \in F \mid (N_1^\varnothing)^\star] \tag{3.3.14}$$

From (3.3.4) we have

$$\mathbb{P}^l(E, F, k) = \int_{\Omega'} d\mathbb{P}' \int_F \frac{\mu_\psi(d\sigma_1)}{\mu_\psi(C)} \int_{(J^l)^{-1}(E)} d\mathbb{P}' \frac{e^{-\mu_l^1(C)}(\mu_l^1(C))^k}{k!}. \tag{3.3.15}$$

Then using above in (3.3.14) we have

$$R_\psi[\phi_l \in F \mid N_1] = \frac{\mu_\psi(F)}{\mu_\psi(C)}.$$

From (3.3.8) we have

$$\begin{aligned} R_\psi[\nu = k \mid N_1] =& \mathbb{P}_\psi^\star \circ t_m^{-1}[\nu = k \mid N_1] \\ =& \mathbb{P}_\psi^\star[\nu \circ t_m = k \mid N_1 \circ t_m] \\ =& \mathbb{P}_\psi^\star[(\nu)^\star = k \mid (N_1^\varnothing)^\star]. \end{aligned}$$

Now first applying (3.3.6) to the last term in the above equation we have,

$$R_\psi[\nu = k \mid N_1] = (\mathbb{P}^u)^{\otimes u \in U}[(\phi_l)^\star \in F \mid (N_1^\varnothing)^\star] \tag{3.3.16}$$

Then using (3.3.15) in (3.3.16) we have

$$R_\psi[\nu = k \mid N_1] = \frac{(\mu_\psi(C))^k e^{-\mu_\psi(C)}}{k!}.$$

We can conclude that $\nu \sim Poisson(\mu_\psi(C))$. Hence part (iv) of the Theorem 3.2.3 is proved. □

3.3.2 Proof of Theorem 3.2.6

In this Section we will prove our main result, Theorem 3.2.6. First we calculate the probability that the path ψ does not collide with any other path in it's lifetime. In other words we calculate the probability of the event $R_\psi^\epsilon[Z = 1]$. Then we use the relation (3.2.10) to show (3.2.11).

Proof of Theorem 3.2.6. Let u_ϵ be as in (3.2.10). Recall from (3.1.3) that V is bounded implies that $V(\cdot, \cdot)$ is a bounded function on $[0, T] \times \mathbb{R}^d$. Further as $R^{t,x}$ is a probability measure for all $t \in [0, T], x \in \mathbb{R}^d$. Consequently by definition u_ϵ is bounded.

Recall from Theorem 3.2.3, for $\psi \in C = \cup_{t>0} C([0, t], \mathbb{R}^d)$, R_ψ^ϵ is a probability measure on the space of marked trees and N_1 is a poisson process. Note that,

$$R_\psi^\epsilon[Z = 1 \mid N_1] = R_\psi^\epsilon\left[\sum_{l=1}^{\nu} 1_{(Z^l=1)} = 0 \mid N_1\right], \tag{3.3.17}$$

where ν is the random number of children of the root path ψ on random trees and for each l from 1 to ν, Z_l is the interaction term from the interaction as in (3.2.6). Let M be the Poisson random measure defined as in Lemma 3.2.5 with intensity μ_M, then we have

$$R^{\epsilon}_{\psi}\left[\sum_{l=1}^{\nu} 1_{(Z^l=1)} = 0 \mid N_1\right] = R_{\psi}[M(Z=1)=0 \mid N_1]$$
$$= \exp\left\{-\sum_{l=0}^{N_1[(0,t)\times\mathbb{R}_+]} V(H_i,\psi_{H_i}) R_{\epsilon}^{H_i,\psi_{H_i}}(Z=1)Y_i\right\}.$$

Recall Φ_ϵ, Φ from (3.2.8) and Ψ_ϵ, Ψ from (3.2.9). Taking expectation in above and using (3.3.17) we get,

$$\begin{aligned} R^{\epsilon}_{\psi}[Z=1] &= \sum_{n=0}^{\infty} \frac{e^{-\lambda_\epsilon t}(\lambda_\epsilon t)^n}{n!} (E^{\epsilon}_{\psi}[\exp\{-V(H_1,\psi_{H_1})R_{\epsilon}^{H_1,\psi_{H_1}}(Z=1)Y_1\}])^n \\ &= \exp\left\{-\lambda\left\{\int_0^t\int_0^\infty (1-\exp\{-V(s,\psi_s)R_{\epsilon}^{s,\psi_s}[Z=1]y\})\pi_\epsilon(y)dyds\right\}\right\} \\ &= \exp\left\{-\int_0^t \Phi_\epsilon(V(s,\psi_s)R_{\epsilon}^{s,\psi_s}[Z=1])ds\right\} \end{aligned} \tag{3.3.18}$$

Now by definition

$$u_\epsilon(t,x) = V(t,x)(1-(1-R_\epsilon^{t,x}[Z=1])) = V(t,x) - V(t,x)\int(1-R^{\epsilon}_{\psi}[Z=1])P^{t,x}(d\psi).$$

From (3.3.18) and the above we have

$$u_\epsilon(t,x) = V(t,x) - V(t,x)\int\left(1-\exp\left\{-\int_0^t \Phi_\epsilon(V(s,\psi_s)R_{\epsilon}^{s,\psi_s}[Z=1])ds\right\}\right)P^{t,x}(d\psi).$$

Observe that for all $t>0$,

$$\begin{aligned} &\left(1-\exp\left\{-\int_0^t \Phi_\epsilon(V(s,\psi_s)R^{s,\psi_s}[Z=1])ds\right\}\right) \\ &= \int_0^t -\frac{d}{ds}\left(\exp\left\{-\int_0^s \Phi_\epsilon(V(r,\psi_r)R^{r,\psi_r}[Z=1])dr\right\}\right)ds. \end{aligned}$$

Consequently using the above and Fubini's theorem we have,

$$\begin{aligned} &u_\epsilon(t,x) \\ =&V(t,x) - V(t,x)\int\int_0^t -\frac{d}{ds}\left(\exp\left\{-\int_0^s \Phi_\epsilon(V(r,\psi_r)R_{\epsilon}^{r,\psi_r}[Z=1])dr\right\}\right)P^{t,x}(d\psi) \\ =&V(t,x) - V(t,x)\int_0^t\int \Phi_\epsilon(V(s,\psi_s)R_{\epsilon}^{s,\psi_s}[Z=1])\exp\left\{-\int_0^s \Phi_\epsilon(V(r,\psi_r)R_{\epsilon}^{r,\psi_r}[Z=1])dr\right\}P^{t,x}(d\psi)ds. \end{aligned}$$

Applying (3.2.10) to the first term in the integral above we have

$$u_\epsilon(t,x) = V(t,x) - V(t,x)\int_0^t\int \Phi_\epsilon(u_\epsilon(s,\psi_s))\exp\left\{-\int_0^s \Phi_\epsilon(V(r,\psi_r)R_{\epsilon}^{r,\psi_r}[Z=1])dr\right\}P^{t,x}(d\psi)ds.$$

Applying (3.1.6) first and then (3.1.5) to the above we have

$$
\begin{aligned}
&u_\epsilon(t,x)\\
&= V(t,x) - \int_0^t \int \int p_{t-s}(\psi_s,x)\Phi_\epsilon(u_\epsilon(s,\psi_s)) \exp\left\{-\int_0^s \Phi_\epsilon(V(r,\psi_r)R_\epsilon^{r,\psi_r}[Z=1])dr\right\} P^t(d\psi)ds\\
&= V(t,x)-\\
&\int_0^t \int \int p_{t-s}(\psi_s,x)\Phi_\epsilon(u_\epsilon(s,\psi_s)) \exp\left\{-\int_0^s (V(r,\psi_r)R_\epsilon^{r,\psi_r}[Z=1])dr\right\} V(s,y)P^{s,y}(d\psi)m(dy)ds.
\end{aligned}
$$

Using the definition of $P^{s,y}$ from (3.1.4) and $R_\epsilon^{t,x}$ from (3.2.7) and applying (3.3.18) again we obtain

$$
u_\epsilon(t,x) = V(t,x) - \int_0^t \int p_{t-s}(y,x)\Psi_\epsilon(u(s,y))m(dy)ds.
$$

Now we will show $\lim_{\epsilon\to 0} u_\epsilon(t,x) = u(t,x)\ \forall (t,x) \in [0,T]\times\mathbb{R}^d$. Let w^ϵ and w be two bounded solutions on $[0,T]\times\mathbb{R}^d$ of (3.2.11) and (3.2.12) respectively. Let $t\in[0,T]$, set

$$
g_\epsilon(t) = \sup_{s\times y\in[0,t]\times\mathbb{R}^d} |w^\epsilon(s,y) - w(s,y)|. \tag{3.3.19}
$$

Then, $g_\epsilon : [0,T]\to\mathbb{R}_+$ is an increasing function and for fixed $t\in[0,T]$ we have,

$$
\sup_{x\in\mathbb{R}^d} |w^\epsilon(t,x) - w(t,x)| \le g_\epsilon(t).
$$

As w^ϵ and w are two bounded solutions on $[0,T]\times\mathbb{R}^d$ of (3.2.11) and (3.2.12) respectively, we also have for $t\in[0,T]$,

$$
\sup_{x\in\mathbb{R}^d} |w^\epsilon(t,x) - w(t,x)| \le \sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y,x)\big|\Psi_\epsilon(w^\epsilon(s,y)) - \Psi(w(s,y))\big| dsm(dy). \tag{3.3.20}
$$

Let $\Psi : [0,\infty)\to[0,\infty)$ be given by $\Psi(\eta) = \eta^{1+\alpha}$. Then Ψ is Lipschitz on compact subsets of $\mathbb{R}_+$ and w_1, w_2 being two bounded solutions on $[0,T]\times\mathbb{R}^d$, we can write

$$
\big|\Psi_\epsilon(w^\epsilon(s,y)) - \Psi(w(s,y))\big| \le c_1\big|w^\epsilon(s,y) - w(s,y)\big| + b(\epsilon) \tag{3.3.21}
$$

for $(s,y)\in[0,T]\times\mathbb{R}^d$, some constant $c_1>0$ and $b(\epsilon)\to 0$ as $\epsilon\to 0$. Now applying the above inequality to the right hand side of (3.3.20) we get

$$
\sup_{x\in\mathbb{R}^d} |w_1(t,x) - w_2(t,x)| \le c_1 \sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y,x)\big|w^\epsilon(s,y) - w(s,y)\big| dsm(dy) + b(\epsilon)T
$$

Using the bound on $p_t(\cdot,\cdot)$ from (3.1.2) for $t\in[0,T]$ and (3.3.18) we have

$$
\sup_{x\in\mathbb{R}^d} |w_1(t,x) - w_2(t,x)| \le c_2 \int_0^t g_\epsilon(s)ds + b(\epsilon)T
$$

where $c_2 > 0$ is some constant. Now taking supremum over $t \in [0, T]$ in the left side of above inequality and then using (3.3.23) we get

$$g_\epsilon(t) \leq c_2 \int_0^t g_\epsilon(s)ds + b(\epsilon)T. \tag{3.3.22}$$

If $g_\epsilon : [0, T] \to \mathbb{R}_+$ is a continuous function then from (3.3.22) and Grönwall's Lemma we can conclude $\lim_{\epsilon \to 0} g_\epsilon(t) = 0$. Consequently

$$\lim_{\epsilon \to 0} u_\epsilon(t, x) = u(t, x)$$

follow $\forall (t, x) \in [0, T] \times \mathbb{R}^d$. So it remains to show that $g_\epsilon : [0, T] \to \mathbb{R}_+$ is a continuous function.

Let $t_1, t_2 \in [0, T]$ and w_1^ϵ and w_2^ϵ be two bounded solutions on $[0, T] \times \mathbb{R}^d$ of (3.2.12). Then applying the boundedness property of w_1^ϵ and w_2^ϵ and (3.1.2) we get

$$|g_\epsilon(t_1) - g_\epsilon(t_2)| \leq c_3 |t_1 - t_2|$$

where c_3 is some constant. Above equation implies e is Lipschitz continuous, hence g_ϵ is continuous. So u satisfies (3.2.12). Now we show uniqueness of the non-negative solution of (3.2.12). Let w_1 and w_2 be two bounded solutions on $[0, T] \times \mathbb{R}^d$ of (3.2.12). Let $t \in [0, T]$, set

$$e(t) = \sup_{s \times y \in [0,t] \times \mathbb{R}^d} |w_1(s, y) - w_2(s, y)|. \tag{3.3.23}$$

Then, $e : [0, T] \to \mathbb{R}_+$ is an increasing function and for fixed $t \in [0, T]$ we have,

$$\sup_{x \in \mathbb{R}^d} |w_1(t, x) - w_2(t, x)| \leq e(t).$$

As w_1 and w_2 are two bounded solutions on $[0, T] \times \mathbb{R}^d$ of (3.2.12), we also have for $t \in [0, T]$,

$$\sup_{x \in \mathbb{R}^d} |w_1(t, x) - w_2(t, x)| \leq \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y, x) \left| w_1^{1+\alpha}(s, y) - w_2^{1+\alpha}(s, y) \right| dsm(dy). \tag{3.3.24}$$

Let $\Psi : [0, \infty) \to [0, \infty)$ be given by $\Psi(\eta) = \eta^{1+\alpha}$. Then Ψ is Lipschitz on compact subsets of $\mathbb{R}_+$ and w_1, w_2 being two bounded solutions on $[0, T] \times \mathbb{R}^d$, we can write

$$\left| w_1^{1+\alpha}(s, y) - w_2^{1+\alpha}(s, y) \right| \leq c_1 \left| w_1(s, y) - w_2(s, y) \right| \tag{3.3.25}$$

for $(s, y) \in [0, T] \times \mathbb{R}^d$ and some constant $c_1 > 0$. Now applying the above inequality to the right hand side of (3.3.24) we get

$$\sup_{x \in \mathbb{R}^d} |w_1(t, x) - w_2(t, x)| \leq c_1 \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y, x) \left| w_1(s, y) - w_2(s, y) \right| dsm(dy).$$

Using the bound on $p_t(\cdot, \cdot)$ from (3.1.2) for $t \in [0, T]$ and (3.3.23) we have

$$\sup_{x \in \mathbb{R}^d} |w_1(t, x) - w_2(t, x)| \leq c_1 \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y, x) e(s) dsm(dy) \leq c_2 \int_0^t e(s)ds.$$

where $c_2 > 0$ is some constant. Now taking supremum over $t \in [0,T]$ in the left side of above inequality and then using (3.3.23) we get

$$e(t) \leq c_2 \int_0^t e(s)ds. \tag{3.3.26}$$

If $e : [0,T] \to \mathbb{R}_+$ is a continuous function then from (3.3.26) and Grönwall's inequality we can conclude $e(t) = 0$. Consequently uniqueness of the non-negative solution of (3.2.12) will follow. So it remains to show that $e : [0,T] \to \mathbb{R}_+$ is a continuous function.

Let $t_1, t_2 \in [0,T]$ and w_1 and w_2 be two bounded solutions on $[0,T] \times \mathbb{R}^d$ of (3.2.12). Then applying the boundedness property of w_1 and w_2 and (3.1.2) we get

$$|e(t_1) - e(t_2)| \leq c_3 |t_1 - t_2|$$

where c_3 is some constant. Above equation implies e is Lipschitz continuous, hence e is continuous. We can conclude uniqueness of the non-negative solution of (3.2.12) holds.

□

3.3.3 Proof of Lemma 3.2.4, Lemma 3.2.5 and Lemma 3.3.1

In this section we prove that, under R_ψ we have marked trees are finite (Lemma 3.2.4) and variables $\{T_i\}$ follows Poisson Random Measure (Lemma 3.2.5). Then we prove Lemma 3.3.1 which is related to different properties of $T_u^\star$.

Proof of Lemma 3.2.4. For $0 \leq p \leq n$ and $c = \sup\limits_{(s,y)\in[0,\tau(\psi)]\times\mathbb{R}^d} V(\cdot,\cdot)$. We define

$$v_p = E_\psi^\epsilon \left[\sum_{u \in G_{n-p}} \frac{1}{p!} (c\tau(\phi^u))^p \mid N_1 \right].$$

We will show $v_0 \leq v_n$, this will imply marked trees are finite under R_ψ^ϵ. Let $0 \leq p < n$, then from (3.3.2) and using property of conditional expectation we have,

$$v_p = E_\psi^\epsilon \left[E_\psi^\epsilon \left[\sum_{u \in G_{n-p-1}, uj \in G_{n-p}} \frac{1}{p!} (c\tau(\phi^{uj}))^p \mid \mathcal{F}_{n-p-1} \right] \mid N_1 \right].$$

From (iii) of Proposition 3.2.3 and above we get

$$v_p = E_\psi^\epsilon \left[\sum_{u \in G_{n-p-1}} E_{\phi^u}^\epsilon \left[\sum_{1 \leq j \leq \nu} \frac{1}{p!} (c\tau(\phi^{uj}))^p \right] \mid N_1 \right].$$

From construction of measure R_ψ^ϵ we have $\tau(\phi^u) > \tau(\phi^{uj})$ a.s.. Then from the above we have

$$v_p \leq E_\psi^\epsilon \left[\sum_{u \in G_{n-p-1}} \int_0^{\tau(\phi^u)} \frac{1}{p!} (cs)^p c ds \mid N_1 \right] = v_{p+1}.$$

We can conclude, $v_0 \leq v_n$, so $E^{\epsilon}_{\psi}[card(G_n)|N_1] \leq \frac{1}{n!}(ct)^n$, and $E^{\epsilon}_{\psi}[\sum_n card(G_n)|N_1] \leq e^{ct}$, where $card(G_n)$ is the cardinality of the set G_n. Then, $E^{\epsilon}_{\psi}[\sum_n card(G_n)] \leq e^{ct}$ follows. This implies under R_ψ the marked trees are finite a.s.. □

Proof of Lemma 3.2.5. Let F be a positive measurable function on Ω. M be as in Lemma 3.2.5. If we can show

$$E^{\epsilon}_{\psi}[\exp\{-\langle M,F\rangle\}|N_1] = \exp\left\{\sum_{i=1}^{N_1[(0,t)\times\mathbb{R}_+]} V(H_i,\psi_{H_i})Y_i\int(\exp\{-F\}-1)R^{H_i,\psi_{H_i}}_{\epsilon}(d\phi^{\varnothing})\right\} \tag{3.3.27}$$

then from [29, Proposition 3.6] we can conclude M is a Poisson random measure with intensity

$$\mu_M(\cdot) = \sum_{i=1}^{N_1[(0,t)\times\mathbb{R}_+]} V(H_i,\psi_{H_i})R^{H_i,\psi_{H_i}}_{\epsilon}(\cdot)Y_i \tag{3.3.28}$$

where $R^{t,x} = \int R_\psi P^{t,x}(d\psi)$. Then, from (3.3.2) and using property of conditional expectation we have

$$E^{\epsilon}_{\psi}\left[\exp\{-\langle M,F\rangle\} \mid N_1\right] = E^{\epsilon}_{\psi}\left[E^{\epsilon}_{\psi}\left[\prod_{l=1}^{\nu}\exp\{-F(T_l)\} \mid \mathcal{F}_1\right] \mid N_1\right].$$

Now (iii) of Proposition 3.2.3 and above implies

$$\begin{aligned}E^{\epsilon}_{\psi}[\exp\{-\langle M,F\rangle\} \mid N_1] &= E^{\epsilon}_{\psi}\left[\prod_{l=1}^{\nu}E_{\phi^l}\left[\exp\{-F\}\right] \mid N_1\right]\\ &= E^{\epsilon}_{\psi}\left[\exp\{-\sum_{l=1}^{\nu}-\ln\{E_{\phi^l}[\exp\{-F\}]\}\} \mid N_1\right].\end{aligned}$$

Recall N as in Proposition 3.2.3. Then from above we get,

$$\begin{aligned}E_\psi[\exp\{-\langle M,F\rangle\} \mid L] &= E_\psi[\exp\{-\langle N,-\ln\{E_{\phi^{\varnothing}}[\exp\{-F\}]\}\rangle\} \mid L]\\ &= \exp\left\{\sum_{i=1}^{N_1(0,t)\times\mathbb{R}_+} V(H_i,\psi_{H_i})Y_i\int(\exp\{-F\}-1)R^{H_i,\psi_{H_i}}_{\epsilon}(d\phi^{\varnothing})\right\}\end{aligned}$$

Consequently we can say (3.3.27) holds. □

Proof of Lemma 3.3.1. We first prove part (a) of the lemma. Let $\omega^\star \in \Omega^\star$. For $u \in U$, let $\nu^\star_u$ and $T^\star_u$ be as in (3.3.1) and (3.3.9) respectively. Then we have

$$\nu^\star_v(T^\star_u(\omega^\star)) = \nu^\star_v(\sigma^\star) = \nu^\star_{uv}(\omega^\star). \tag{3.3.29}$$

Let $\omega^\star \in t_m^{-1}(\Omega_u)$ then we have,

$$t_m \circ T^\star_u(\omega^\star) = \{(v = j_1 ... j_p, N^v_1, \phi^v) \in T^\star_u(\omega^\star) : p \in \mathbb{N}, j_{k+1} \leq \nu^\star_{j_1...j_k}(T^\star_u(\omega^\star))\}.$$

From (3.3.29) and above we get,

$$t_m \circ T^\star_u(\omega^\star) = \{(v = j_1 ... j_p, N^v_1, \phi^v) : p \in \mathbb{N}, j_{k+1} \leq \nu^\star_{uj_1...j_k}(\omega^\star) \text{ if } 0 \leq k < p\}.$$

Then from above and for t_m be as in (3.3.7) we have,

$$t_m \circ T_u^\star(\omega^\star) = \{(v = j_1 ... j_p, N_1^v, \phi^v) : (uv, N_1^{uv}, \phi^{uv}) \in t_m(\omega^\star)\}.$$

Consequently from Definition 3.2.2 we have

$$t_m \circ T_u^\star(\omega^\star) = T_u \circ t_m(\omega^\star).$$

So we can conclude that on $t_m^{-1}(\Omega_u)$ we have

$$T_u \circ t_m = t_m \circ T_u^\star.$$

Next we prove part (b) of the lemma. $\mathcal{F}^\star$ be as in (3.3.2). Let $A_1, B_1 \in \mathcal{F}^\star$ and $u \neq v \in U$ with same length. Then define

$$C_u = \{a \in U : a = ul_1 \dots l_k \text{ where } l_i \in \mathbb{N} \text{ for } 1 \leq i \leq k\}, \quad S = \{\omega^\star : T_u^\star(\omega^\star) \in A_1, T_v^\star(\omega^\star) \in B_1)\}, \tag{3.3.30}$$

and

$$S_1 = \{\omega^\star|_{C_u} : \omega^\star \in S\}, \quad S_2 = \{\omega^\star|_{C_v} : \omega^\star \in S\}. \tag{3.3.31}$$

Observe that $C_u \cap C_v = \varnothing$ since $u \neq v$ with same length. Then

$$\mathbb{P}_\psi^\star(T_u^\star \in A_1, T_v^\star \in B_1) = \left(\prod_{a \in U \setminus (C_u \cup C_v)} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a \in C_u}(S_1) \times (\mathbb{P}^a)^{\otimes a \in C_v}(S_2).$$

From (3.3.6) and above we get,

$$\mathbb{P}_\psi^\star(T_u^\star \in A_1, T_v^\star \in B_1) = (\mathbb{P}^a)^{\otimes a \in C_u}(S_1) \times (\mathbb{P}^a)^{\otimes a \in C_v}(S_2).$$

Then first applying (3.3.6) in the above and then using (3.3.9) we get,

$$\begin{aligned}
&\mathbb{P}_\psi^\star(T_u^\star \in A_1, T_v^\star \in B_1) \\
&= \left(\left(\prod_{a \in U \setminus C_u} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a \in C_u}(S_1)\right) \times \left(\left(\prod_{a \in U \setminus C_v} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a \in C_v}(S_2)\right) \\
&= \mathbb{P}_\psi^\star(T_u^\star \in A_1)\mathbb{P}_\psi^\star(T_v^\star \in B_1).
\end{aligned}$$

Consequently we have,

$$\mathbb{P}_\psi^\star(T_u^\star \in A_1, T_v^\star \in B_1) = \mathbb{P}_\psi^\star(T_u^\star \in A_1)\mathbb{P}_\psi^\star(T_v^\star \in B_1).$$

So we can conclude that the variables $T_u^\star$ are independent of each other for indices u of length n .

Now we prove part (c) of the lemma. Let $u, v \in U$ and $|v| \leq n = |u|$. Let $v = j_1 j_2 ... j_p$ where $p \leq n$ and $j_l \in \mathbb{N}$ for $l = 1$ to p. Let

$$J = \{j_1 j_2 ... j_l : l = 1 \text{ to } p-1\} \text{ and } S_3 = \{\omega^\star|_J : \omega^\star \in t_m^{-1}(\Omega_v)\} \tag{3.3.32}$$

t_m be as in (3.3.7) and then from above we get,

$$t_m^{-1}(\Omega_v) = (A \times C \times \mathbb{N})^{\otimes U\setminus J} \times S_3. \tag{3.3.33}$$

Let $A_1 \in \mathcal{F}^\star$. Then (3.3.30), (3.3.31) and above implies

$$\mathbb{P}^\star_\psi(T^\star_u \in A_1, t_m^{-1}(\Omega_v)) = \mathbb{P}^\star_\psi((A \times C \times \mathbb{N})^{\otimes U\setminus C_u} \otimes S_1, (A \times C \times \mathbb{N})^{\otimes U\setminus J} \otimes S_3).$$

Observe that $C_u \cap J = \varnothing$. Then (3.3.6) and above implies

$$\mathbb{P}^\star_\psi(T^\star_u \in A_1, t_m^{-1}(\Omega_v)) = (\mathbb{P}^u)^{\otimes a\in C_u}(S_1) \times (\mathbb{P}^u)^{\otimes a\in J}(S_3).$$

From (3.3.6) and above we have

$$\begin{aligned}&\mathbb{P}^\star_\psi(T^\star_u \in A_1, t_m^{-1}(\Omega_v))\\ =&\left(\left(\prod_{a\in U\setminus C_u} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^u)^{\otimes a\in C_u}(S_1)\right) \times \left(\left(\prod_{a\in U\setminus J} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^u)^{\otimes a\in J}(S_3)\right)\end{aligned}$$

Now (3.3.33) and above gives

$$\mathbb{P}^\star_\psi(T^\star_u \in A_1, t_m^{-1}(\Omega_v)) = \mathbb{P}^\star_\psi(T^\star_u \in A_1)\mathbb{P}^\star_\psi(t_m^{-1}(\Omega_v)).$$

Consequently $T^\star_u$ is independent of $t_m^{-1}(\Omega_v)$ for $|v| \leq n = |u|$.

It is remaining to show part (d) of the lemma. $\mathcal{F}^\star$ be as in (3.3.2). Let $u, v \neq U, |v| < n = |u|$. Take $A_1, B_1 \in \mathcal{F}^\star$. Observe that $\{v\} \cap C_u = \varnothing$ since $|u| = n$. Then

$$\mathbb{P}^\star_\psi(T^\star_u \in A_1, \nu^\star_v \in B_1) = \left(\prod_{a\in U\setminus (C_u \cup \{v\}} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a\in C_u}(S_1) \times \mathbb{P}^v(B_1).$$

From (3.3.6) and above we get,

$$\mathbb{P}^\star_\psi(T^\star_u \in A_1, \nu^\star_v \in B_1) = (\mathbb{P}^a)^{\otimes a\in C_u}(S_1) \times \mathbb{P}^v(B_1).$$

Then first applying (3.3.6) in the above and then using (3.3.9) we get,

$$\begin{aligned}&\mathbb{P}^\star_\psi(T^\star_u \in A_1, \nu^\star_v \in B_1)\\ =&\left(\left(\prod_{a\in U\setminus C_u} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a\in C_u}(S_1)\right) \times \left(\left(\prod_{a\in U\setminus \{v\}} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a\in \{u\}}(B_1)\right)\\ =&\,\mathbb{P}^\star_\psi(T^\star_u \in A_1)\mathbb{P}^\star_\psi(\nu^\star_v \in B_1).\end{aligned}$$

Consequently we have,

$$\mathbb{P}^\star_\psi(T^\star_u \in A_1, \nu^\star_v \in B_1) = \mathbb{P}^\star_\psi(T^\star_u \in A_1)\mathbb{P}^\star_\psi(\nu^\star_v \in B_1).$$

So we can conclude that the variables $T^\star_u$ are independent of $\nu^\star_v$ where $|v| < n$. Similarly we can show

$T_u^\star$ are independent of $(N_1^v)^\star, (\phi^v)^\star$ where $|v| < n$. This will imply that $T_u^\star$ is independent of $t_m^{-1}(\mathcal{F}_n)$. Similarly we can show, $T_u^\star$ are independent of $\sigma((N_1^v)^\star, (\phi^v)^\star : |v| < n = |u|)$. This implies T_u^* is independent of $t_m^{-1}(\mathcal{F}_n)$. □

3.4 Trajectorial representation using stable subordinators of $\partial_t u = \frac{1}{2}\Delta u - u^{1+\alpha}, 0 < \alpha < 1$

In this section we will first construct marked trees. We get marked trees by associating each node of a tree with some mark. In this section the marks are represented by α-stable subordinators and finite paths. The finite paths represent trajectories of the particles following those path. Then we will construct a probability measure on the space of marked trees, see Theorem 3.4.7. Under this probability measure, the marked trees will have finitely many generations a.s., see Lemma 3.4.5.

Recall in [38], the author first constructed the collision tree and then built the interactions on the tree. Here the marked trees represent the collision tree and marked trees are finite, so we can construct an interaction on the marked trees, see Section 3.4.2. The interaction on marked trees depict collisions between particles.

As in [38] we construct the marked trees which represent the collision tree and put an interaction on the marked trees, see Section 3.4.2. The interaction on marked trees depict collisions between particles. The interactions are $\{0, 1\}$ valued function Z associated with each node u. We start with nodes which have no children. We assign the Z value 1 to these nodes and then define the Z values of ancestors of these nodes inductively as discussed in Section 3.4.2. If Z value of a node is 1 then it is considered alive, else destroyed.

The root path is the finite path which is associated as a mark with the root.Then the probability of the event that the root path is alive is equal to the probability of the event that the Z value of the root path is 1. We call $u(x, t)$ the probability of a Brownian particle surviving under the interaction at time $t \in [0, T]$ and at point $x \in \mathbb{R}^d$.

Then the main result of this section, Theorem 3.4.7 states, u satisfies $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}_+$ satisfies

$$\partial_t u = \frac{1}{2}\Delta u - u^{1+\alpha}, u(0, \cdot) = u_0. \tag{3.4.1}$$

Definition 3.4.1 ($\alpha-$stable subordinator)**.** *Let $D([0, \infty), \mathbb{R})$ be the set of cádlág paths (everywhere right continuous and everywhere left limits exist). We call a process $X \in D([0, \infty), \mathbb{R})$ an $\alpha-$stable subordinator where $0 < \alpha < 1$ if it is a Lévy process with non-decreasing, non-negative sample paths; $X_0 = 0$ and satisfies the self-similarity property*

$$\frac{X_t}{t^{\frac{1}{\alpha}}} \stackrel{d}{=} X_t \quad \forall t > 0.$$

Here the parameter α is called the exponent of the process $(X_t)_{t\geq 0}$.

Recall that $C = \cup_{t>0} C([0, t], \mathbb{R}^d)$ and denote $A := D([0, \infty), \mathbb{R}_+)$. Under suitable metric d (see Ethier and Kurtz [17, Pg. 117]) we will have (A, d) a metric space. Let $\mathcal{A}$ be the σ-algebra generated by d.

Definition 3.4.2 (Marked tree)**.** *We call ω a marked tree in $D := A \times C$ if it is of the form $\omega = (\pi, ((L^u, \phi^u), u \in \pi)))$, where L^u denote the mark on node u from A. And ϕ^u denote the mark on node u from C. ϕ^u represents the path attached to node u and L^u represents possible set of weighted hitting* *points on path ϕ^u. With abuse of notation we denote $\nu_\varnothing$ as ν and $L^\varnothing$ as L. We call Ω be the set of all marked trees.*

We denote by G_n the n-th generation of the tree, that is to say, $\pi \cap \mathbb{N}^n$. Define,

$$\mathcal{F} = \sigma\{L^u, \phi^u, \Omega^u : u \in U\} \text{ and } \mathcal{F}_n = \sigma\{L^u, \phi^u, \Omega^u : |u| \leq n\}. \tag{3.4.2}$$

where $|u|$ denote length of u, for $u \in U$. Here, $\Omega_u = \{\omega \in \Omega : u \in \omega\}$. Now we define the translation of a marked tree at node u.

Definition 3.4.3 (Translation map). *$T_u(\omega)$ be the translation of marked tree ω at node u. $T_u(\omega) = (\pi = \{v : v \in U \text{ and } uv \in \Omega\}, ((L^a, \phi^a), a \in \pi)))$.*

3.4.1 Results on space of marked trees

Now we state our first result related to the approximation approach. We construct a probability measure on the space of marked trees. Under this measure the marked trees will be finite. The path $\phi^{uj}, 1 \leq j \leq \mu$, will represent the trajectory of the particle which hit the path ϕ^u at time $\tau(\phi^{uj})$ (the probabilities on the tree will be such that $\tau(\phi^u) > \tau(\phi^{uj})$ a.s.).

Proposition 3.4.4. *Let $\psi \in C$ be a path of length t. Then $\exists$ a probability measure R_ψ on Ω such that:*

(i) $R_\psi[\phi^\varnothing = \psi] = 1$

(ii) *L is a $\alpha-$stable subordinator under R_ψ where $0 < \alpha < 1$.*

(iii) $E_\psi[\prod_{u \in G_n} f_u \circ T_u | \mathcal{F}_n] = \prod_{u \in G_n} E_{\phi^u}[f_u]$ *(Here E_ψ is the expectation under R_ψ).*

(iv) *Under, $R_\psi[\cdot|L]$ we have, $N = \sum_{l=0}^{\nu} \delta_{\phi^l}$ is Poisson random measure with intensity*

$$\mu_N(\cdot) = \int_0^{\tau(\psi)} V(s, \psi_s) P^{s,\psi_s}(\cdot) L(ds). \tag{3.4.3}$$

Under the probability measure R_ψ we will have marked trees finite almost surely and variables $\{T_i\}$ follows Poisson Random Measure.

Lemma 3.4.5. *Let ψ and R_ψ be as in Proposition 3.4.4. Then R_ψ a.s. the marked trees are finite.*

Lemma 3.4.6. *Let ψ and R_ψ be as in Proposition 3.4.4. Then under $R_\psi[\cdot|L]$, $M = \sum_{l=1}^{\nu} \delta_{T_l}$ is Poisson random measure with intensity*

$$\mu_M(\cdot) = \int_0^t V(s, \psi_s) R^{s,\psi_s}(\cdot) L(ds) \tag{3.4.4}$$

where $R^{t,x} = \int R_\psi P^{t,x}(d\psi)$.

3.4.2 Interaction on marked tree and Main result

We showed that marked trees are finite under R_ψ, recall Lemma 3.4.5. So there exists finitely many generations in a marked tree under R_ψ. Then we can construct an interaction on the marked trees as

Sznitman did in [38]. Let $Z^u, u \in \pi$ be marks on our basic tree, which will tell us whether ϕ^u is alive or not at time $\tau(\phi^u)$, in the sense that it has collided with trajectories $\phi^{u,l+1}$ for $1 \le l \le \nu_u$ that were all alive at the time of collision $\tau(\phi^{u,l})$. We define $D_u(\omega)$, for $u \in \omega$, to be the number of generations descending from u, that is $\max\{|v|, uv \in \omega\}$, which is R_ψ-a.s. finite. We set for $u \in \pi : Z^u = 1$ if $D_u = 0$, and by induction

$$Z^u = (1 - \sum_{l=1}^{\nu_u} Z^{u,l})_+, \tag{3.4.5}$$

in other words, Z^u is zero as soon as a collision with a path with mark Z equals to 1 happens.

Now we give an pictorial example to understand the interaction. In the Figure 3.2 the root path is

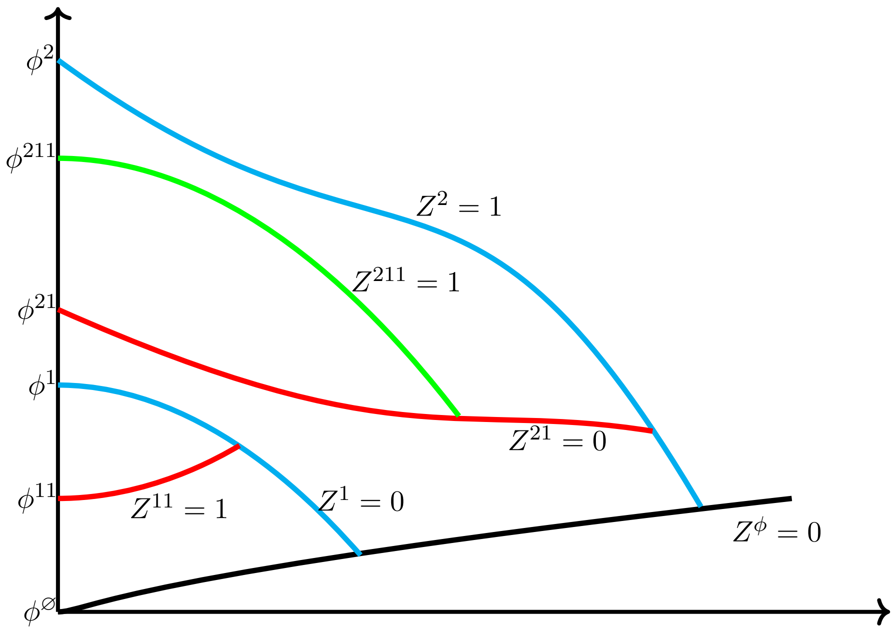

FIGURE 3.2: An example of the interaction on $\mathbb{R}^2$

coloured by black and denoted by $\phi^\varnothing$. The second generation paths are coloured by light blue. So there are two paths in the second generation, denoted by ϕ^1 and ϕ^2. The third generation paths are coloured by red. So there are two paths in the second generation, denoted by ϕ^{11} and ϕ^{21}. The third generation paths are coloured by green. So there is one path in the third generation, denoted by ϕ^{211}. We have used the equation (3.4.5) to calculate the Z values of each path in each generation mentioned above. We can see in the above example the root path has Z value 0, which means it survives the interaction. If the path has length t, then it is not alive till time t.

Recall P^t and $P^{t,x}$ from (3.1.4) and (3.1.5) respectively. We denote by R^t and $R^{t,x}$, the laws on Ω defined by

$$R^t = \int R_\psi P^t(d\psi) \text{ and } R^{t,x} = \int R_\psi P^{t,x}(d\psi). \tag{3.4.6}$$

Recall $V(\cdot)$ and $V(\cdot,\cdot)$ as in (3.1.3), set

$$u(t,x) = V(t,x) R^{t,x}[Z^\varnothing = 1] \tag{3.4.7}$$

for $t > 0$ and $u(0,x) = V(x)$. For convenience, we will write ψ for $\phi^\varnothing$ and Z for $Z^\varnothing$, in what follows. We now state the main result of this section.

Theorem 3.4.7. *Let $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be as in (3.4.7). Then u is bounded unique solution of the equation*

$$w(t,x) = V(t,x) - \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y,x) w^{1+\alpha}(s,y) ds dm(y), \tag{3.4.8}$$

on $[0,T] \times \mathbb{R}^d$ where $) < \alpha < 1$.

3.5 Proof of Theorem 3.4.7

In this section we first prove Proposition 3.4.4 which shows the existence of the measure R_ψ on marked trees under which the root path is ψ a.s., L is a α-stable subordinator and generational Markov property holds. Then we prove that, under R_ψ marked trees are finite(Lemma 3.4.5). Let $\{T_i\}$ be as in Definition 3.4.3, then under R_ψ, $\{T_i\}$ follows Poisson Random Measure (Lemma 3.4.6). Then using definition of u from (3.4.7) and using the above results we prove our main result Theorem 3.4.7.

3.5.1 Proof of Proposition 3.4.4

We shall begin by defining certain necessary auxiliary measure spaces and discuss the proof strategy. Recall $A = D([0,\infty), \mathbb{R})$, $C = \cup_{t>0} C([0,t], \mathbb{R}^d)$ and $\mathcal{A}$, $\mathcal{C}$ be σ-algebras on A and C respectively. Let U be as in Definition 3.1.1. We consider the product space $\Omega^\star = (A \times C \times \mathbb{N})^U$. Let $u \in U$ then define

$$(L^u)^\star : \Omega^\star \to A, \quad (\phi^u)^\star : \Omega^\star \to C \text{ and } \nu_u^\star : \Omega^\star \to \mathbb{N} \tag{3.5.1}$$

to be the projections of $\Omega^\star$ on A, C and $\mathbb{N}$ respectively. Define

$$\mathcal{F}^\star = \sigma\{(L^u)^\star, (\phi^u)^\star, \nu_u^\star : u \in U\} \text{ and } \mathcal{F}_n^\star = \sigma\{(L^u)^\star, (\phi^u)^\star, \nu_u^\star : |u| \leq n\}. \tag{3.5.2}$$

Construction of $\mathbb{P}_\psi^\star$ on $(\Omega^\star, \mathcal{F}^\star)$ for $\psi \in C$: First for each $u \in U$, we will define measure $\mathbb{P}^u$ on $(A \times C \times \mathbb{N})$. Fix a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and let $\{J^u \mid u \in U\}$ be a collection of i.i.d $\alpha-$stable subordinator on it and taking values in A. Recall $V(\cdot,\cdot)$ and $P^{t,x}$ from (3.1.3) and (3.1.4) respectively. For every $\sigma \in C$ and $u \in U$ define the measure $\mu_u^\sigma(\cdot)$ on $(C, \mathcal{C})$ by

$$\mu_u^\sigma(\cdot) = \int_0^{\tau(\sigma)} V(s, \sigma_s) P^{s,\sigma_s}(\cdot) J^u(ds).$$

Let $u = u_1 u_2 \ldots u_p$ where $u_l \in \mathbb{N}$ for $l = 1, \ldots, p$ and for some $p \geq 1$. Then define $u^l = u_1 u_2 \ldots u_l$ for all $l = 1$ to p. Let E, F belong to $\mathcal{A}$ and $\mathcal{C}$ respectively and $I \subset \mathbb{N}$. Define

$$\begin{aligned}\mathbb{P}^u(E,F,I) = \frac{1}{Q_p} \int_{\Omega'} d\mathbb{P}' \int_{\Omega'} d\mathbb{P}' \ldots \int_{\Omega'} d\mathbb{P}' \int_{(J^u)^{-1}(E)} d\mathbb{P}' \\ \int_{\tilde{F}_p} \mu_\varnothing^\psi(d\sigma_1) \mu_{u^1}^{\sigma_1}(d\sigma_2) \ldots \mu_{u^{p-2}}^{\sigma_{p-2}}(d\sigma_{p-1}) \mu_{u^{p-1}}^{\sigma_{p-1}}(d\sigma_p) \sum_{k \in I} \frac{e^{-\mu_u^{\sigma_p}(C)} (\mu_u^{\sigma_p}(C))^k}{k!}.\end{aligned} \tag{3.5.3}$$

where

$$\tilde{F}_p = \{(f_1, f_2, \ldots, f_p) \in C^{p-1} \times F : \tau(f_1) > \tau(f_2) > \ldots > \tau(f_p)\},$$

$$\tilde{C}_p = \{(f_1, f_2, \ldots, f_p) \in C^p : \tau(f_1) > \tau(f_2) > \ldots > \tau(f_p)\}$$

and

$$Q_p = \int_{\tilde{C}_p} \mu_{\varnothing}^{\psi}(d\sigma_1)\mu_{u^1}^{\sigma_1}(d\sigma_2)\ldots\mu_{u^{p-2}}^{\sigma_{p-2}}(d\sigma_{p-1})\mu_{u^{p-1}}^{\sigma_{p-1}}(d\sigma_p).$$

For $u = \varnothing$ define

$$\mathbb{P}^{\varnothing}(E, F, I) = 1_{\{\psi \in F\}} \int_{(J^{\varnothing})^{-1}(E)} d\mathbb{P}' \sum_{k \in I} \frac{e^{-\mu_\psi(C)}(\mu_\psi(C))^k}{k!}. \tag{3.5.4}$$

We define

$$\mathbb{P}_\psi^\star = (\mathbb{P}^u)^{\otimes\{u \in U\}}. \tag{3.5.5}$$

We have defined $\mathbb{P}^u$ in a way that $\mathbb{P}^u(E, F, I)$ gives us the probability that $(L^u)^\star \in E$, $(\phi^u)^\star \in F$ and $(\nu_u)^\star \in I$. Then the probability measure $\mathbb{P}_\psi^\star$ on $\Omega^\star$ helps us in finding the behaviour of the random varables $((L^u)^\star, (\phi^u)^\star, (\nu_u)^\star)$ for all $u \in U$.

We will abuse notation and write μ_ψ for $\mu_\varnothing^\psi$. Recall from Ω is the set of all marked trees (see Definition 3.4.2). We define $t_m : \Omega^\star \to \Omega$ (where m denotes marked) given by

$$t_m(\omega^\star) = (\pi, ((L^u, \phi^u), u \in \pi))), \tag{3.5.6}$$

with $\pi = \{u = j_1 \ldots j_p : p \in \mathbb{N}, j_{k+1} \leq \nu^\star_{j_1\ldots j_k}(\omega^\star) \text{ if } 1 \leq k < p\}$ and $L^u(t_m(\omega^\star)) = (L^u)^\star(\omega^\star)$, $\phi^u(t_m(\omega^\star)) = (\phi^u)^\star(\omega^\star)$ for $u \in \pi$.

The map $t_m : (\Omega^\star, \mathcal{F}^\star) \to (\Omega, \mathcal{F})$ is measurable but not injective. We define,

$$R_\psi = \mathbb{P}_\psi^\star \circ t_m^{-1}. \tag{3.5.7}$$

We will show R_ψ satisfies $(i), (ii), (iii)$ and (iv) of Proposition 3.4.4.

Define the translation map by $u \in U$ by $T_u^\star : \Omega^\star \to \Omega^\star$ as,

$$T_u^\star(\omega^\star) = \{\sigma^\star \in \Omega^\star : \sigma^\star(v) = \omega^\star(uv), \quad \forall v \in U\}. \tag{3.5.8}$$

Recall the definition of T_u from Definition 3.4.3 and $\Omega_u = \{\omega \in \Omega : u \in \omega\}$. We first state the lemma related to $T_u^\star$, this lemma will be used show that R_ψ satisfies (iii) of Proposition 3.4.4.

Lemma 3.5.1. *$T_u^\star$ satisfies the following*

(a) On $t_m^{-1}(\Omega_u)$ we have,

$$T_u \circ t_m = t_m \circ T_u^\star.$$

(b) On $(\Omega^\star, \mathcal{F}^\star, \mathbb{P}_\psi^\star)$, $n \in \mathbb{N}$ fixed, the variables $T_u^\star$ are independent of each other for indices u of length n.

(c) $T_u^\star$ is independent of $t_m^{-1}(\Omega_v)$ for $|v| \leq n = |u|$.

(d) $T_u^\star$ are independent of $\sigma(\nu_v^\star, (L^u)^\star, (\phi^v)^\star : |v| < n = |u|)$. More precisely, $T_u^\star$ is independent of $t_m^{-1}(\mathcal{F}_n)$.

This lemma is required to show part (iii) of Proposition 3.4.4. We shall defer the proof of the lemma and prove Proposition 3.4.4 assuming the lemma. The lemma will be proved after proving the proposition.

Proof of Proposition 3.4.4. First we prove part (i) of the proposition. From (3.5.6) we have,

$$t_m^{-1}[\phi^{\varnothing}=\psi]=(A\times\{\psi\}\times\mathbb{N})\times\prod_{u\in U-\{\varnothing\}}(A\times C\times\mathbb{N}).$$

Recall R_ψ as in (3.5.7) then we have

$$R_\psi[\phi^{\varnothing}=\psi]=\mathbb{P}^{\star}_{\psi}\circ t_m^{-1}[\phi^{\varnothing}=\psi].$$

From (3.5.5), the above and $\mathbb{P}^u(A,C,\mathbb{N})=1$ for $u\in U-\{\varnothing\}$ we get,

$$R_\psi[\phi^{\varnothing}=\psi]=\mathbb{P}^{\varnothing}(A,\{\psi\},\mathbb{N})\times\prod_{u\in U-\{\varnothing\}}\mathbb{P}^u(A,C,\mathbb{N})=\mathbb{P}^{\varnothing}(A,\{\psi\},\mathbb{N}).$$

From (3.5.4) we have,

$$\mathbb{P}^{\varnothing}(E,F,I)=\int_{(J^{\varnothing})^{-1}(E)}d\mathbb{P}'\sum_{k\in I}\frac{e^{-\mu_\psi(C)}(\mu_\psi(C))^k}{k!}\tag{3.5.9}$$

where E,F belong to $\mathcal{A}$ and $\mathcal{C}$ respectively and $I\subset\mathbb{N}$. Then from above equality we get $\mathbb{P}^{\varnothing}(A,\{\psi\},\mathbb{N})=1$, this implies

$$R_\psi[\phi^{\varnothing}=\psi]=1.$$

We conclude part (i) of Proposition 3.4.4 holds. Now we show part (ii) of the proposition. Let $E\in\mathcal{A}$. Then from (3.5.6) we have

$$t_m^{-1}[L\in E]=(E\times C\times\mathbb{N})\times\prod_{u\in U\setminus\{\varnothing\}}(A\times C\times\mathbb{N})$$

Then from (3.5.7) we have

$$R_\psi[L\in E]=\mathbb{P}^{\star}_{\psi}\circ t_m^{-1}[L\in E].$$

From (3.5.5), the above and $\mathbb{P}^u(A,C,\mathbb{N})=1$ for $u\in U\setminus\{\varnothing\}$ we get

$$R_\psi[L\in E]=\mathbb{P}^{\varnothing}(E,C,\mathbb{N})\times\prod_{u\in U\setminus\{\varnothing\}}\mathbb{P}^u(A,C,\mathbb{N})=\mathbb{P}^{\varnothing}(E,C,\mathbb{N}).$$

Then using (3.5.9) and above we get

$$R_\psi[L\in E]=\mathbb{P}'\circ(J^{\varnothing})^{-1}(E)$$

$\forall E$ in $\mathcal{A}$. So $L\overset{d}{=}J^{\varnothing}$. $J^{\varnothing}$ is an α-stable subordinator so we conclude L is an α-stable subordinator.

Now we show part (iii) of the proposition. Let $B\subset\mathbb{N}^{*n}$. Observe that $\{\omega^\star\in\Omega^\star : B\subset G_n\circ t_m\}=\cap_{u\in B}t_m^{-1}(\Omega_u)\in t_m^{-1}(\mathcal{F}_n)$ where $\mathcal{F}_n$ is as in (3.4.2). Recall G_n is the n-th generation of the tree. So $G_n\circ t_m$ is $t_m^{-1}(\mathcal{F}_n)$ measurable. From part (a) of Lemma 3.5.1 we have

$$\prod_{u\in B}f_u\circ t_m\circ T^\star_u=\prod_{u\in B}f_u\circ T_u\circ t_m\tag{3.5.10}$$

on $\cap_{u \in B} t_m^{-1}(\Omega_u)$. Now $G_n \circ t_m$ is $t_m^{-1}(\mathcal{F}_n)$ measurable, so applying (3.5.10) on

$$E_{\mathbb{P}_\psi^\star}\left[\prod_{u \in G_n \circ t_m} f_u \circ t_m \circ T_u^\star \mid t_m^{-1}(\mathcal{F}_n)\right]$$

we have,

$$E_{\mathbb{P}_\psi^\star}\left[\prod_{u \in G_n \circ t_m} f_u \circ t_m \circ T_u^\star \mid t_m^{-1}(\mathcal{F}_n)\right] = E_{\mathbb{P}_\psi^\star}\left[\prod_{u \in G_n \circ t_m} f_u \circ T_u \circ t_m \mid t_m^{-1}(\mathcal{F}_n)\right].$$

Recall $G_n \circ t_m$ is $t_m^{-1}(\mathcal{F}_n)$ measurable. Then first applying part (d) of the Lemma 3.5.1 and then part (b) of Lemma 3.5.1 on the left side of the above equality we get

$$\prod_{u \in G_n \circ t_m} E_{\mathbb{P}_\psi^\star}\left[f_u \circ t_m \circ T_u^\star\right] = E_{\mathbb{P}_\psi^\star}\left[\prod_{u \in G_n \circ t_m} f_u \circ T_u \circ t_m \mid t_m^{-1}(\mathcal{F}_n)\right]. \tag{3.5.11}$$

Now for $u \in G_n \circ t_m$ we have

$$\mathbb{P}_\psi^\star \circ (T_u^\star)^{-1} = \mathbb{P}_{(\phi^u)^\star}^\star. \tag{3.5.12}$$

Then applying the above on the left side of (3.5.11) we get

$$E_{\mathbb{P}_\psi^\star}\left[\prod_{u \in G_n \circ t_m} f_u \circ T_u \circ t_m \mid t_m^{-1}(\mathcal{F}_n)\right] = \prod_{u \in G_n \circ t_m} E_{\mathbb{P}_{(\phi^u)^\star}^\star}\left[f_u \circ t_m\right].$$

Then using (3.5.7) in both sides of the above equality and using the relation $(\phi^u)^\star = \phi^u \circ t_m$ on the right side of the equality we get,

$$E_{R_\psi}\left[\prod_{u \in G_n} f_u \circ T_u \mid \mathcal{F}_n\right] = \prod_{u \in G_n} E_{R_{\phi^u}}\left[f_u\right].$$

So part (iii) of Proposition 3.4.4 holds. It remains to show part (iv) of the Proposition 3.4.4. Let N be as in Proposition 3.4.4. We show that $\nu \sim Poisson(\mu_\psi(C))$ and $R_\psi[\phi^l \in F \mid L] = \frac{\mu_\psi(F)}{\mu_\psi(C)}$ where $l \in \mathbb{N}$ and $F \in \mathcal{C}$. This will imply part (iv) of the proposition.

Using the relation (3.5.7) we have

$$R_\psi[\phi^l \in F \mid L] = \mathbb{P}_\psi^\star \circ t_m^{-1}[\phi^l \in F \mid L].$$

Then taking t_m^{-1} inside the equation and using the relations $L \circ t_m = (L^\varnothing)^\star$ and $\phi^l \circ t_m = (\phi^l)^\star$ we get,

$$R_\psi[\phi^l \in F \mid L] = \mathbb{P}_\psi^\star[(\phi^l)^\star \in F \mid (L^\varnothing)^\star].$$

Now first applying (3.5.5) to the last term in the above equation we have,

$$R_\psi[\phi^l \in F \mid L] = (\mathbb{P}^u)^{\otimes u \in U}[(\phi^l)^\star \in F \mid (L^\varnothing)^\star] \tag{3.5.13}$$

From (3.5.3) we have

$$\mathbb{P}^l(E, F, I) = \frac{1}{Q_1} \int_{\Omega'} d\mathbb{P}' \int_{(J^l)^{-1}(E)} d\mathbb{P}' \int_{\tilde{F}_1} \frac{\mu_\psi(d\sigma_1)}{\mu_\psi(C)} \sum_{k \in \mathbb{N}} \frac{e^{-\mu_l^1(C)}(\mu_l^1(C))^k}{k!}. \tag{3.5.14}$$

Then using above in (3.5.13) we have

$$R_\psi[\phi^l \in F \mid L] = \frac{\mu_\psi(F)}{\mu_\psi(C)}.$$

Using the relation (3.5.7) we have

$$R_\psi[\nu = k \mid L] = \mathbb{P}^\star_\psi \circ t_m^{-1}[\nu = k \mid L].$$

Then taking t_m^{-1} inside the equation and using the relations $L \circ t_m = (L^\varnothing)^\star$ and $\nu \circ t_m = (\nu_\varnothing)^\star$ we get,

$$R_\psi[\nu = k \mid L] = \mathbb{P}^\star_\psi[(\nu_\varnothing)^\star = k \mid (L^\varnothing)^\star].$$

Now first applying (3.5.5) to the last term in the above equation we have,

$$R_\psi[\nu = k \mid L] = (\mathbb{P}^u)^{\otimes u \in U}[(\nu_\varnothing)^\star = k \mid (L^\varnothing)^\star] \tag{3.5.15}$$

Then using (3.5.4) in (3.5.15) we have

$$R_\psi[\nu = k \mid L] = \frac{(\mu_\psi(C))^k e^{-\mu_\psi(C)}}{k!}.$$

We can conclude that $\nu \sim Poisson(\mu_\psi(C))$ under $R_\psi[\cdot \mid L]$. Hence part (iv) of the Proposition 3.4.4 is proved.

□

3.5.2 Proof of Theorem 3.4.7

In this Section we will prove our main result, Theorem 3.4.7. First we calculate the probability that the path ψ does not collide with any other path in it's lifetime. In other words we calculate the probability of the event $R_\psi[Z = 1]$. Then we use the relation (3.4.7) to show (3.4.8).

Proof of Theorem 3.4.7. Let u be as in (3.4.7). Recall from (3.1.3) that V is bounded implies that $V(\cdot,\cdot)$ is a bounded function on $[0,T] \times \mathbb{R}^d$. Further as $R^{t,x}$ is a probability measure for all $t \in [0,T], x \in \mathbb{R}^d$. Consequently by definition u is bounded.

Recall from Proposition 3.4.4, for $\psi \in C = \cup_{t>0} C([0,t], \mathbb{R}^d)$, R_ψ is a probability measure on the space of marked trees and from Definition 3.4.2 L is α-stable subordinator. Note that,

$$R_\psi[Z = 1 \mid L] = R_\psi\left[\sum_{l=1}^{\nu} 1_{(Z^l=1)} = 0 \mid L\right], \tag{3.5.16}$$

where ν is the random number of children of the root path ψ on random trees and for each l from 1 to ν, Z_l is the interaction term from the interaction as in (3.4.5). Let M be the Poisson random measure defined as in Lemma 3.4.6 with intensity μ_M, then we have

$$R_\psi\left[\sum_{l=1}^{\nu} 1_{(Z^l=1)} = 0 \mid L\right] = R_\psi[M(Z=1) = 0 \mid L] = \exp\left\{-\int_0^t V(s,\psi_s) R^{s,\psi_s}[Z=1] L(ds)\right\}.$$

Taking expectation in above and using (3.5.16) we get,

$$R_\psi[Z=1] = \exp\left\{-\int_0^t \left(V(s,\psi_s)R^{s,\psi_s}[Z=1]\right)^\alpha ds\right\}. \tag{3.5.17}$$

Now by definition

$$u(t,x) = V(t,x)(1-(1-R^{t,x}[Z=1])) = V(t,x) - V(t,x)\int(1-R_\psi[Z=1])P^{t,x}(d\psi).$$

From (3.5.17) and the above we have

$$u(t,x) = V(t,x) - V(t,x)\int\left(1-\exp\left\{-\int_0^t \left(V(s,\psi_s)R^{s,\psi_s}[Z=1]\right)^\alpha ds\right\}\right)P^{t,x}(d\psi).$$

Observe that for all $t>0$,

$$\begin{aligned}&\left(1-\exp\left\{-\int_0^t \left(V(s,\psi_s)R^{s,\psi_s}[Z=1]\right)^\alpha ds\right\}\right)\\ &= \int_0^t -\frac{d}{ds}\left(\exp\left\{-\int_0^s \left(V(r,\psi_r)R^{r,\psi_r}[Z=1]\right)^\alpha dr\right\}\right)ds.\end{aligned}$$

Consequently using the above and Fubini's theorem we have,

$$\begin{aligned}&u(t,x)\\ =&V(t,x) - V(t,x)\int\int_0^t -\frac{d}{ds}\left(\exp\left\{-\int_0^s \left(V(r,\psi_r)R^{r,\psi_r}[Z=1]\right)^\alpha dr\right\}\right)P^{t,x}(d\psi)\\ =&V(t,x)-\\ &V(t,x)\int_0^t\int\left(V(s,\psi_s)R^{s,\psi_s}[Z=1]\right)^\alpha \exp\left\{-\int_0^s \left(V(r,\psi_r)R^{r,\psi_r}[Z=1]\right)^\alpha dr\right\}P^{t,x}(d\psi)ds.\end{aligned}$$

Applying (3.4.7) to the first term in the integral above we have

$$u(t,x) = V(t,x) - V(t,x)\int_0^t\int u(s,\psi_s)^\alpha \exp\left\{-\int_0^s \left(V(r,\psi_r)R^{r,\psi_r}[Z=1]\right)^\alpha dr\right\}P^{t,x}(d\psi)ds.$$

Applying (3.1.6) first and then (3.1.5) to the above we have

$$\begin{aligned}&u(t,x)\\ &= V(t,x) - \int_0^t\int\int p_{t-s}(\psi_s,x)u(s,\psi_s)^\alpha \exp\left\{-\int_0^s \left(V(r,\psi_r)R^{r,\psi_r}[Z=1]\right)^\alpha dr\right\}P^t(d\psi)ds\\ &= V(t,x)-\\ &\int_0^t\int\int p_{t-s}(\psi_s,x)u(s,\psi_s)^\alpha \exp\left\{-\int_0^s \left(V(r,\psi_r)R^{r,\psi_r}[Z=1]\right)^\alpha dr\right\}V(s,y)P^{s,y}(d\psi)m(dy)ds.\end{aligned}$$

Using the definition of $P^{s,y}$ from (3.1.4) and $R^{t,x}$ from (3.4.6) and applying (3.5.17) again we obtain

$$u(t,x) = V(t,x) - \int_0^t\int p_{t-s}(y,x)u(s,y)^{1+\alpha}m(dy)ds.$$

So u satisfies (3.4.8). Now we show uniqueness of the non-negative solution of (3.4.8). The rest of the argument including the below computation is very similar to that done between (3.3.23) to (3.3.22) but we present it again for completeness sake.

Let w_1 and w_2 be two bounded solutions on $[0,T]\times\mathbb{R}^d$ of (3.4.8). Let $t\in[0,T]$, set

$$e(t)=\sup_{s\times y\in[0,t]\times\mathbb{R}^d}|w_1(s,y)-w_2(s,y)|. \tag{3.5.18}$$

Then, $e:[0,T]\to\mathbb{R}_+$ is an increasing function and for fixed $t\in[0,T]$ we have,

$$\sup_{x\in\mathbb{R}^d}|w_1(t,x)-w_2(t,x)|\leq e(t).$$

As w_1 and w_2 are two bounded solutions on $[0,T]\times\mathbb{R}^d$ of (3.4.8), we also have for $t\in[0,T]$,

$$\sup_{x\in\mathbb{R}^d}|w_1(t,x)-w_2(t,x)|\leq\sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}p_{t-s}(y,x)\big|w_1^{1+\alpha}(s,y)-w_2^{1+\alpha}(s,y)\big|dsm(dy). \tag{3.5.19}$$

Let $\Psi:[0,\infty)\to[0,\infty)$ be given by $\Psi(\eta)=\eta^{1+\alpha}$. Then Ψ is Lipschitz on compact subsets of $\mathbb{R}_+$ and w_1, w_2 being two bounded solutions on $[0,T]\times\mathbb{R}^d$, we can write

$$\big|w_1^{1+\alpha}(s,y)-w_2^{1+\alpha}(s,y)\big|\leq c_1\big|w_1(s,y)-w_2(s,y)\big| \tag{3.5.20}$$

for $(s,y)\in[0,T]\times\mathbb{R}^d$ and some constant $c_1>0$. Now applying the above inequality to the right hand side of (3.5.19) we get

$$\sup_{x\in\mathbb{R}^d}|w_1(t,x)-w_2(t,x)|\leq c_1\sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}p_{t-s}(y,x)\big|w_1(s,y)-w_2(s,y)\big|dsm(dy).$$

Using the bound on $p_t(\cdot,\cdot)$ from (3.1.2) for $t\in[0,T]$ and (3.5.18) we have

$$\sup_{x\in\mathbb{R}^d}|w_1(t,x)-w_2(t,x)|\leq c_1\sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}p_{t-s}(y,x)e(s)dsm(dy)\leq c_2\int_0^t e(s)ds.$$

where $c_2>0$ is some constant. Now taking supremum over $t\in[0,T]$ in the left side of above inequality and then using (3.5.18) we get

$$e(t)\leq c_2\int_0^t e(s)ds. \tag{3.5.21}$$

If $e:[0,T]\to\mathbb{R}_+$ is a continuous function then from (3.5.21) and Grönwall's inequality we can conclude $e(t)=0$. Consequently uniqueness of the non-negative solution of (3.4.8) will follow. So it remains to show that $e:[0,T]\to\mathbb{R}_+$ is a continuous function.

Let $t_1,t_2\in[0,T]$ and w_1 and w_2 be two bounded solutions on $[0,T]\times\mathbb{R}^d$ of (3.4.8). Then applying the boundedness property of w_1 and w_2 and (3.1.2) we get

$$|e(t_1)-e(t_2)|\leq c_3|t_1-t_2|$$

where c_3 is some constant. Above equation implies e is Lipschitz continuous, hence e is continuous. We can conclude uniqueness of the non-negative solution of (3.4.8) holds.

□

3.5.3 Proof of Lemma 3.4.5, Lemma 3.4.6 and Lemma 3.5.1

In this section we prove that, under R_ψ we have marked trees are finite (Lemma 3.4.5) and variables $\{T_i\}$ follows Poisson Random Measure (Lemma 3.4.6). Then we prove Lemma 3.5.1 which states different properties of $T_u^\star$.

Proof of Lemma 3.4.5. Let ψ and R_ψ be as in Proposition 3.4.4. Recall $V(\cdot,\cdot)$ from (3.1.3), G_n is the n-th generation, τ be as in (3.1.7) and ϕ^u be as in Definition 3.1.1. . For p in $\{0,\dots,n\}$ and let

$$c = \sup_{(s,y)\in[0,\tau(\psi)]\times\mathbb{R}^d} V(s,y),$$

and define

$$v_p = E_\psi\left[\sum_{u\in G_{n-p}} \frac{1}{p!}(c\tau(\phi^u))^p \mid L\right].$$

We will first show $v_0 \le v_n$ and then using this we will show that marked trees are finite under R_ψ. Recall $\mathcal{F}_n$ from (3.4.2). Let $0 \le p < n$, then using property of conditional expectation we have,

$$v_p = E_\psi\left[E_\psi\left[\sum_{u\in G_{n-p-1}} \sum_{\{j\in\mathbb{N}: uj\in G_{n-p}\}} \frac{1}{p!}(c\tau(\phi^{uj}))^p \mid \mathcal{F}_{n-p-1}\right] \mid L\right].$$

From (iii) of Proposition 3.4.4 and above we get

$$v_p = E_\psi\left[\sum_{u\in G_{n-p-1}} E_{\phi^u}\left[\sum_{1\le j\le\nu} \frac{1}{p!}(c\tau(\phi^{uj}))^p\right] \mid L\right].$$

Recall $R_\psi = \mathbb{P}^\star_\psi \circ t_m^{-1}$. Then (3.5.3), (3.5.4) and (3.5.5) implies that under the measure R_ψ we have $\tau(\phi^u) > \tau(\phi^{uj})$ a.s.. Then from definition of Riemann integral and above we have

$$v_p \le E_\psi\left[\sum_{u\in G_{n-p-1}} \int_0^{\tau(\phi^u)} \frac{1}{p!}(cs)^p c ds \mid L\right] = v_{p+1}.$$

As $v_0 = E_\psi[card(G_n)|L]$ and $v_n = \frac{1}{n!}(ct)^n$ then from above we get

$$E_\psi[\sum_n card(G_n)|L] \le e^{ct}$$

where $card(G_n)$ is the cardinality of the set G_n. Then $E_\psi[\sum_n card(G_n)] \le e^{ct}$ follows. This implies under R_ψ the marked trees are finite a.s.. □

Proof of Lemma 3.4.6. Let F be a positive measurable function on Ω. M be as in Lemma 3.4.6. If we can show

$$E_\psi[\exp\{-\langle M,F\rangle\} \mid L] = \exp\left\{\int_0^t V(s,\psi_s)L(ds)\int(\exp\{-F\}-1)\,R^{s,\psi_s}(d\phi^\varnothing)\right\} \tag{3.5.22}$$

then from [29, Proposition 3.6] we can conclude M is a Poisson random measure with intensity

$$\mu_M(\cdot) = \int_0^t V(s,\psi_s) R^{s,\psi_s}(\cdot) L(ds) \tag{3.5.23}$$

where $R^{t,x} = \int R_\psi P^{t,x}(d\psi)$. We will now prove (3.5.22). From (3.4.2) and using property of conditional expectation we have

$$E_\psi[\exp\{-\langle M, F\rangle\} \mid L] = E_\psi\left[E_\psi\left[\prod_{l=1}^{\nu} \exp\{-F(T_l)\} \mid \mathcal{F}_1\right] \mid L\right].$$

Now (iii) of Proposition 3.4.4 and above implies

$$\begin{aligned} E_\psi[\exp\{-\langle M, F\rangle\} \mid L] &= E_\psi\left[\prod_{l=1}^{\nu} E_{\phi^l}[\exp\{-F\}] \mid L\right] \\ &= E_\psi\left[\exp\left\{-\sum_{l=1}^{\nu} -\ln\{E_{\phi^l}[\exp\{-F\}]\}\right\} \mid L\right]. \end{aligned}$$

Let N be as in Proposition 3.4.4. Then from above we get,

$$\begin{aligned} E_\psi[\exp\{-\langle M, F\rangle\} \mid L] &= E_\psi[\exp\{-\langle N, -\ln\{E_{\phi^\varnothing}[\exp\{-F\}]\}\rangle\} \mid L] \\ &= \exp\{\int_0^t V(s,\psi_s) L(ds) \int (\exp\{-F\} - 1)\, R^{s,\psi_s}(d\phi^\varnothing)\} \end{aligned}$$

Consequently we can say (3.5.22) holds. □

Proof of Lemma 3.5.1. We first prove part (a) of the lemma. Let $\Omega^\star$, $\nu^\star_u$ and $T^\star$ be as in early part of Section 3.5.1 and $u \in U$ be as in Definition 3.1.1. Let $\omega^\star \in \Omega^\star$. For $u \in U$ we have

$$\nu^\star_v(T^\star_u(\omega^\star)) = \nu^\star_v(\sigma^\star) = \nu^\star_{uv}(\omega^\star). \tag{3.5.24}$$

Let $\omega^\star \in t_m^{-1}(\Omega_u)$ then we have,

$$t_m \circ T^\star_u(\omega^\star) = \{(v = j_1 ... j_p, L^v, \phi^v) \in T^\star_u(\omega^\star) : p \in \mathbb{N}, j_{k+1} \le \nu^\star_{j_1...j_k}(T^\star_u(\omega^\star))\}.$$

From (3.5.24) and above we get,

$$t_m \circ T^\star_u(\omega^\star) = \{(v = j_1 ... j_p, L^v, \phi^v) : p \in \mathbb{N}, j_{k+1} \le \nu^\star_{uj_1...j_k}(\omega^\star) \text{ if } 0 \le k < p\}.$$

Then from above and for t be as in (3.5.6) we have,

$$t_m \circ T^\star_u(\omega^\star) = \{(v = j_1 ... j_p, L^v, \phi^v) : (uv, L^{uv}, \phi^{uv}) \in t_m(\omega^\star)\}.$$

Consequently from Definition 3.4.3 we have

$$t_m \circ T^\star_u(\omega^\star) = T_u \circ t_m(\omega^\star).$$

So we can conclude that on $t_m^{-1}(\Omega_u)$ we have

$$T_u \circ t_m = t_m \circ T_u^\star.$$

Next we prove part (b) of the lemma. $\mathcal{F}^\star$ be as in (3.5.2). Let $A_1, B_1 \in \mathcal{F}^\star$ and $u \neq v \in U$ with same length. Then define

$$C_u = \{a \in U : a = ul_1 \dots l_k \text{ where } l_i \in \mathbb{N} \text{ for } 1 \leq i \leq k\}, \quad S = \{\omega^\star : T_u^\star(\omega^\star) \in A_1, T_v^\star(\omega^\star) \in B_1)\}, \tag{3.5.25}$$

and

$$S_1 = \{\omega^\star|_{C_u} : \omega^\star \in S\}, \quad S_2 = \{\omega^\star|_{C_v} : \omega^\star \in S\}. \tag{3.5.26}$$

Observe that $C_u \cap C_v = \varnothing$ since $u \neq v$ with same length. Then

$$\mathbb{P}_\psi^\star(T_u^\star \in A_1, T_v^\star \in B_1) = \left(\prod_{a \in U-(C_u \cup C_v)} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a \in C_u}(S_1) \times (\mathbb{P}^a)^{\otimes a \in C_v}(S_2).$$

From (3.5.5) and above we get,

$$\mathbb{P}_\psi^\star(T_u^\star \in A_1, T_v^\star \in B_1) = (\mathbb{P}^a)^{\otimes a \in C_u}(S_1) \times (\mathbb{P}^a)^{\otimes a \in C_v}(S_2).$$

Then first applying (3.5.5) in the above and then using (3.5.8) we get,

$$\begin{aligned}\mathbb{P}_\psi^\star(T_u^\star \in A_1, T_v^\star \in B_1) = &\left(\left(\prod_{a \in U-C_u} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a \in C_u}(S_1)\right) \\ &\times \left(\left(\prod_{a \in U-C_v} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a \in C_v}(S_2)\right) \\ =&\mathbb{P}_\psi^\star(T_u^\star \in A_1)\mathbb{P}_\psi^\star(T_v^\star \in B_1).\end{aligned}$$

Consequently we have,

$$\mathbb{P}_\psi^\star(T_u^\star \in A_1, T_v^\star \in B_1) = \mathbb{P}_\psi^\star(T_u^\star \in A_1)\mathbb{P}_\psi^\star(T_v^\star \in B_1).$$

So we can conclude that the variables $T_u^\star$ are independent of each other for indices u of length n.

Now we prove part (c) of the lemma. Let $u, v \in U$ and $|v| \leq n = |u|$. Let $v = j_1 j_2 ... j_p$ where $p \leq n$ and $j_l \in \mathbb{N}$ for $l = 1$ to p. Let

$$J = \{j_1 j_2 ... j_l : l = 1 \text{ to } p-1\} \text{ and } S_3 = \{\omega^\star|_J : \omega^\star \in t_m^{-1}(\Omega_v)\} \tag{3.5.27}$$

t_m be as in (3.5.6) and then from above we get,

$$t_m^{-1}(\Omega_v) = (A \times C \times \mathbb{N})^{\otimes U \setminus J} \times S_3. \tag{3.5.28}$$

Let $A_1 \in \mathcal{F}^\star$. Then (3.5.25), (3.5.26) and above implies

$$\mathbb{P}_\psi^\star(T_u^\star \in A_1, t_m^{-1}(\Omega_v)) = \mathbb{P}_\psi^\star((A \times C \times \mathbb{N})^{\otimes U \setminus C_u} \otimes S_1, (A \times C \times \mathbb{N})^{\otimes U \setminus J} \otimes S_3).$$

Observe that $C_u \cap J = \varnothing$. Then (3.5.5) and above implies

$$\mathbb{P}^\star_\psi(T^\star_u \in A_1, t_m^{-1}(\Omega_v)) = (\mathbb{P}^u)^{\otimes a \in C_u}(S_1) \times (\mathbb{P}^u)^{\otimes a \in J}(S_3).$$

From (3.5.5) and above we have

$$\begin{aligned}\mathbb{P}^\star_\psi(T^\star_u \in A_1, t_m^{-1}(\Omega_v)) = &\left(\left(\prod_{a \in U \setminus C_u} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^u)^{\otimes a \in C_u}(S_1)\right) \\ &\times \left(\left(\prod_{a \in U \setminus J} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^u)^{\otimes a \in J}(S_3)\right)\end{aligned}$$

Now (3.5.28) and above gives

$$\mathbb{P}^\star_\psi(T^\star_u \in A_1, t_m^{-1}(\Omega_v)) = \mathbb{P}^\star_\psi(T^\star_u \in A_1)\mathbb{P}^\star_\psi(t_m^{-1}(\Omega_v)).$$

Consequently $T^\star_u$ is independent of $t_c^{-1}(\Omega_v)$ for $|v| \leq n = |u|$.

It is remaining to show part (d) of the lemma. $\mathcal{F}^\star$ be as in (3.5.2). Let $u, v \neq U, |v| < n = |u|$. Take $A_1, B_1 \in \mathcal{F}^\star$. Observe that $\{v\} \cap C_u = \varnothing$ since $|u| = n$. Then

$$\mathbb{P}^\star_\psi(T^\star_u \in A_1, \nu^\star_v \in B_1) = \left(\prod_{a \in U \setminus (C_u \cup \{v\}} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a \in C_u}(S_1) \times \mathbb{P}^v(B_1).$$

From (3.5.5) and above we get,

$$\mathbb{P}^\star_\psi(T^\star_u \in A_1, \nu^\star_v \in B_1) = (\mathbb{P}^a)^{\otimes a \in C_u}(S_1) \times \mathbb{P}^v(B_1).$$

Then first applying (3.5.5) in the above and then using (3.5.8) we get,

$$\begin{aligned}\mathbb{P}^\star_\psi(T^\star_u \in A_1, \nu^\star_v \in B_1) = &\left(\left(\prod_{a \in U \setminus C_u} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a \in C_u}(S_1)\right) \\ &\times \left(\left(\prod_{a \in U \setminus \{v\}} \mathbb{P}^a(A \times C \times \mathbb{N})\right) \times (\mathbb{P}^a)^{\otimes a \in \{u\}}(B_1)\right) \\ &= \mathbb{P}^\star_\psi(T^\star_u \in A_1)\mathbb{P}^\star_\psi(\nu^\star_v \in B_1).\end{aligned}$$

Consequently we have,

$$\mathbb{P}^\star_\psi(T^\star_u \in A_1, \nu^\star_v \in B_1) = \mathbb{P}^\star_\psi(T^\star_u \in A_1)\mathbb{P}^\star_\psi(\nu^\star_v \in B_1).$$

So we can conclude that the variables $T^\star_u$ are independent of $\nu^\star_v$ where $|v| < n$. Similarly we can show $T^\star_u$ are independent of $(L^v)^\star, (\phi^v)^\star$ where $|v| < n$. This will imply that $T^\star_u$ is independent of $t_m^{-1}(\mathcal{F}_n)$. $\square$

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