

RESTRICTED COLLECTION

STOCHASTIC PROCESSES

AND

MODELS

IN

TEXTILE RESEARCH

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P R E F A C E

This thesis is being submitted to the Indian Statistical Institute in support of the author's application for the degree of Doctor of Philosophy. The work presented here has been carried out during the period 1959-1962 under the supervision of Prof. K. Nagabhushanam (of the Andhra University) at the Ahmedabad Textile Industry's Research Association.

This thesis is concerned with the application of the theory of stationary stochastic processes and of stochastic models to some important theoretical problems of Textile Technology. Some of the results presented in this thesis have already appeared in an article by the author [18] and the rest have been submitted for publication.

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CHAPTER I

INTRODUCTION

Study of the density fluctuations in 'slivers' and 'yarns' and of the effect of 'drafting' on these fluctuations is a very important problem of textile technology. In 1945 Martindale [15] first introduced the probability reasoning in this study, but his investigations as well as the subsequent ones have not made a systematic use of probability methods. Further, the mechanism of the formation of 'drafting waves' [6] has not been satisfactorily explained. Also the empirical formula of Cavany and Foster [3] can not be explained by the earlier theories of drafting. We shall study in this work these problems in detail using the theory of stationary stochastic processes and give a satisfactory explanation of the phenomenon of 'drafting waves' and of Cavany and Foster's observations.

After preliminary processing cotton comes out in the form of a long, round, rope-like and non-twisted strand of fibres, called a 'sliver'. The sliver being very thick and loosely packed with fibres, is quite unsuitable for weaving. Subsequent processes aim at attenuating the thickness and increasing the strength of a sliver. Increase in strength is brought about by twisting and the

attenuation in thickness by a process called 'drafting'. Drafting consists essentially in drawing a sliver at one end while keeping the other end relatively fixed. This is accomplished in practice by feeding a sliver into the grip of a pair of rollers, called the back rollers, which deliver it to another pair, called the front rollers [see Fig. 3 of chapter II]. The front rollers rotate with a speed (v_2) greater than that (v_1) of the back rollers. The ratio v_2/v_1 is called 'draft' and the distance between the nips of the rollers, the roller-setting. The final product of spinning is called a yarn. Detailed technological information concerning spinning is given in chapter II.

Among the properties of slivers that are important from the technological point of view are the number of fibres $n(x)$, and the linear density $n(x)$ in the cross-section at x (being measured from a chosen origin on the sliver axis) of a sliver.

A completely uniform yarn (uniform in linear density or in the number of fibres), though very desirable from technological considerations, is never obtained in practice. Thus the need arises to measure and control the irregularity of yarn. This can be fulfilled only when it is known how the density fluctuations in a sliver are affected by drafting. The measures of irregularity that are commonly used are the coefficient of variation and the serial correlation coefficient of $n(x)$.

Martindale's [15] investigations show that some variation in the quality of a yarn is inevitable and that least variation can be expected when the arrangement of fibres is 'completely random'. Such a sliver is known as 'ideal sliver'. Subsequently, several other workers have calculated the serial-correlation coefficient and the spectrum of $n(x)$ for ideal slivers, without a systematic use of probability methods and sometimes by involving unnecessary assumptions. In chapter III we shall review in some detail the earlier work on the analysis of yarn irregularity and the theory of drafting. In chapter IV we describe briefly the properties of stationary processes that are relevant for the present work.

In chapter V we shall consider some stochastic models for sliver irregularity. We ^{shall} first give a precise mathematical definition of an ideal sliver and establish all its known properties in a unified manner. We shall also prove the central limit property of $n(x)$ and $m(x)$ for an ideal sliver. In addition to the ideal sliver we shall consider only briefly the Poisson model for a non-uniform sliver, the Hanning-Poisson model and the 'quasi-ideal sliver'.

In chapter VI we come to the theory of drafting of ideal slivers. In Part I we consider the drafting of ideal slivers in which all the fibres are of the same length ^{and} we introduce the concept of 'ideal drafting'. We show that the drafting waves are caused by the correlations in the transition times, when

CHAPTER II

- (1) each fibre after leaving the back rollers moves with the speed of back rollers until its speed abruptly changes to that of the front rollers and continues to move thereafter with the latter speed,
- (2) each fibre influences the motion of ^{another} only through its 'overlap'
- and (3) the joint distribution of the 'transition times' of two fibres has a specified form [Assumption A_2 of chapter VI]. Further, when the correlation coefficient of the transition times of two fibres depends linearly on the lag between their times of entry into the drafting zone, the expression for 'added irregularity' agrees closely well with the empirical formula of Cavaney and Foster [3]. In part II we generalize these considerations to the case where the fibres are not of the same length.

This operation, known as 'combing' may also be used to improve yarn quality.

This chapter is devoted to revisiting a simplified picture of the drafting process with special reference to cotton spinning. It contains several facts about the dimensions of various machines and mechanical

CHAPTER II

TECHNOLOGY OF SPINNING

2.1. An outline of the chapter

All natural fibres, except silk, are of short length. The process of converting fibres into a continuous long thread - yarn - is termed spinning.

The spinning process consists of four basic operations.

- (i) Opening (and cleaning) of fibres and converting them into a sheet form. This sheet is known as the 'lap'.
- (ii) Conversion of the lap into a thick rope like, but untwisted structure known technically as 'sliver'.
- (iii) Attenuation or thinning of this sliver in successive stages until it becomes thin enough to be used as yarn. This process of attenuation is called 'Drafting'.
- (iv) Twisting the drafted fibre strands to impart the necessary strength.

A fifth operation, known as 'Combing' may also be used as an aid to better yarn quality.

This chapter aims at providing a simplified picture of the spinning process with special reference to cotton spinning. It gives a general idea about the dimensions of various machines and materials

and will help in making the definitions of technical terms such as 'Sliver' and 'draft' more clear-cut. Finally, it indicates the importance of 'regularity' of yarn to its behaviour during processing and also to its end-use.

2.2. Fibres:

Fibres are the raw material for the manufacture of yarn. Fibres can be classified according to their origin as natural or man-made; the natural fibres into sub-classes of vegetable fibres, animal fibres etc. But for the present purpose, fibres may be classified into two broad groups; those with lengths upto $2\frac{1}{2}$ " , and those, such as wool and jute, with longer lengths. Only the shorter fibres, which can be processed on the same sequence of machines as that used for cotton spinning, will be considered here.

Any given sample of cotton contains fibres varying in length, diameter and linear density. The man-made fibres, on the other hand, are much more uniform in all these properties. These are manufactured in the form of continuous filaments and are cut to specified lengths. These fibres, therefore, can be considered to be constant in length.

In Table 1 are listed some relevant properties of a few fibres which can be processed on cotton spinning machinery.

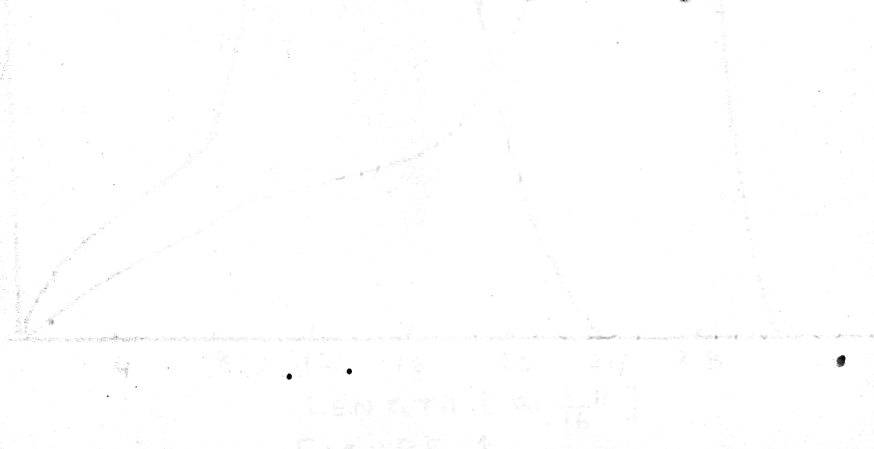


TABLE 1

Type of fibres	Average Length	C.V. of Length %	Average diameter μ (in microns)	Linear density μ gms/inch
Cotton (Indian)	$\frac{1}{2}$ " to 1"	20 to 35	14 to 22	3.0 to 5.5
Cotton (Egyptian)	1" to 2"	22 to 32	12 to 15	3.0 to 4.0
Glasswool Rayon	$1\frac{1}{4}$ " to $2\frac{1}{2}$ "	Negligible	10 to 43	4.2 to 8.4
Terylene	"	"	18 to 25	"
Nylon	"	"	14 to 43	"

Typical frequency distributions of lengths of Indian and Egyptian cottons are shown in Figure 1.

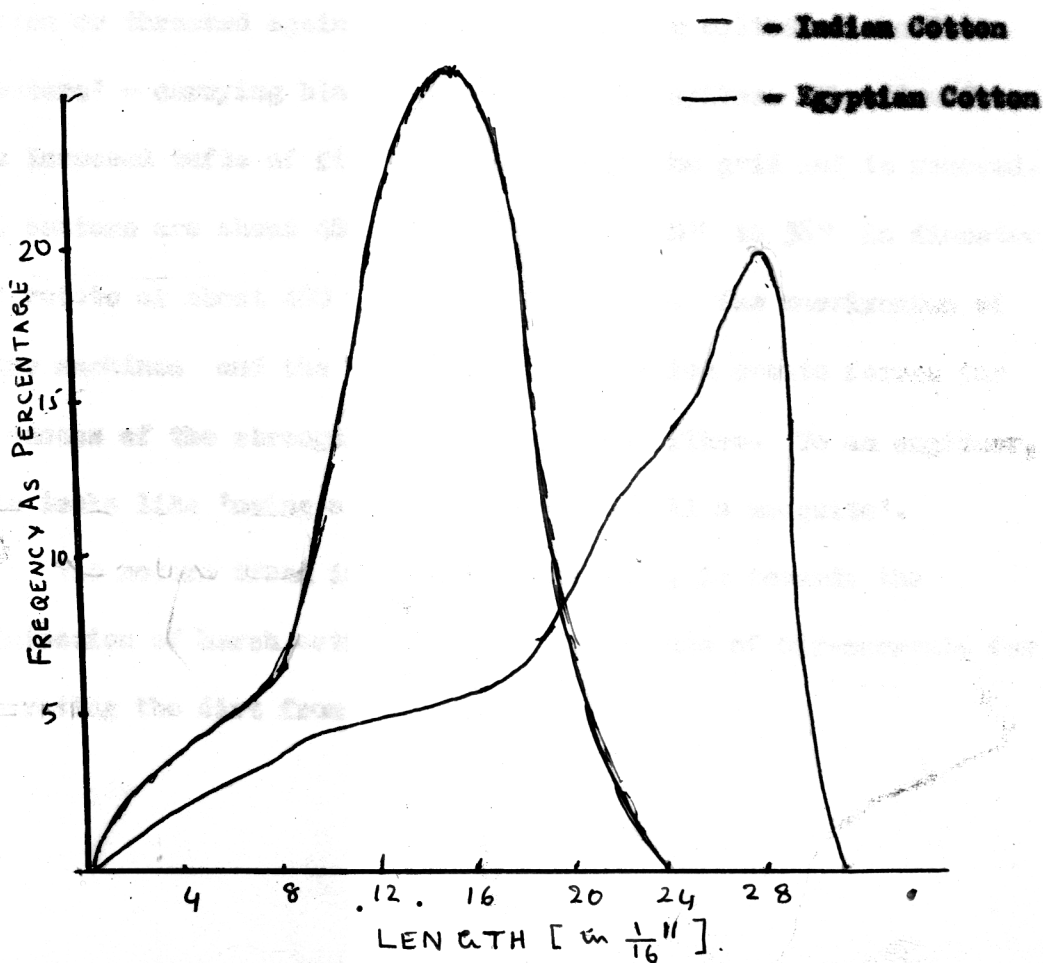


FIGURE 1

2.3. Preliminary Processings- The Blow Room

To the spinner, fibres are available in a highly compressed form. These have to be loosened, opened out and converted into a uniform thin sheet suitable for further processing. Cotton fibres, unlike the man-made fibres, contain dirt particles and need cleaning in addition to the opening. The sequence of about four to seven machines which is used for opening and cleaning the fibres and for converting them into a compressed sheet is collectively called the 'Blow-Room'. These machines are of two types. In one, the fibre-mass is torn and opened between lathies and rollers, both covered with spikes. The tearing action loosens and breaks the large fibre masses. In the second type of machines, the fibres are beaten or thrashed against a grid by rotating cylinders - called 'beaters' - carrying blades or covered with spikes. The dirt from the loosened tufts of fibres falls through the grid and is removed. The beaters are about 40" wide, range from 18" to 36" in diameter, and rotate at about 400 r.p.m. to 1000 r.p.m. The massiveness of these machines and the high speeds of rotation create forces far in excess of the strength of the individual fibre. To an engineer, this looks like 'using a sledge hammer to kill a mosquito'.

The modern trend in blow-room machines, is towards the elimination of harsh action and towards the use of air-currents for separating the dirt from the fibres.

When the preliminary cleaning and opening is completed, the fibres are converted into a 40" wide compressed sheet between 11 ounces and 16 ounces per running yard. Uniform weight of each successive yard is ensured by means of controlled feed and a mechanical type of self-regulating device. This sheet is, then, rolled into a cylinder, termed the 'lap'.

2.4. Carding

The machine that converts the lap into a sliver is known as the 'Card' and the process of conversion as 'Carding'. Figure 2 is a diagrammatic representation of a card and shows the passage of fibres through the machine.

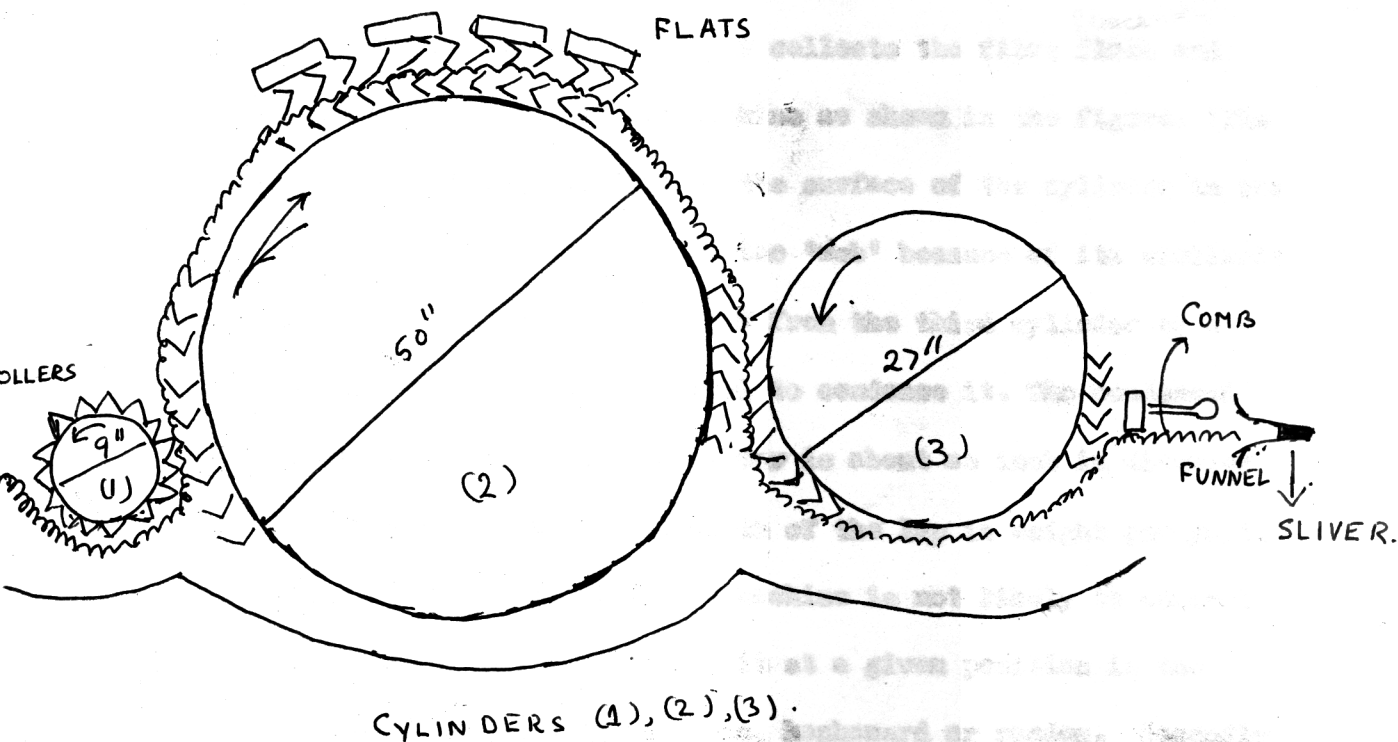
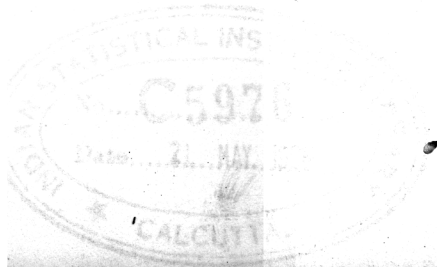


FIGURE 2

The lap is fed by means of a slowly rotating roller to a set of three fast moving cylinders, each 40" wide. The first cylinder covered with projections similar to the teeth of a carpenter's saw, tears the lap, thins it down and feeds it to the next cylinder, which is densely covered with wire projections, as shown. Across the top surface of this cylinder is a series of slowly moving bars, called 'flats' (40" x 2"), covered with similar wires. The gap between the cylinders and flats is about $5/1000''$ to $10/1000''$. The fibre ^{piece} mass is thus completely shredded and almost each individual fibre is separated from the other. The third cylinder is smaller in diameter and rotates slowly in opposite direction, but is otherwise similar to the second cylinder. It collects the fibre ^{piece} mass and carries it to the front of the machine as shown in the figure. The fibres have now been deposited on the surface of the cylinder in the form of a very thin sheet, called the 'web' because of its similarity to a cob-web. The web is ^{peeled} off from the third cylinder by a comb and is passed through a funnel to condense it. The condensed fibre mass is the sliver. The sliver is about an inch in diameter (non-compressed) and is one hundredth of the lap in weight per yard.

It will be noticed that the machine is not likely to control each fibre individually and place it at a given position in the sliver. The fibre placing is, thus, haphazard or random. Secondly



the transfer of the fibres from the second to the third cylinder bends the fibres. The fibres in the sliver are neither straight nor parallel to the sliver axis. A majority of the fibres are hooked () at one or both extremities. The next series of the machines straightens the fibres and aligns them along the sliver axis.

2.5. Drafting:

Drafting is the most important, and perhaps, the least understood operation in spinning. The operation of attenuating the sliver so as to reduce its linear density is called 'Drafting', and the degree of reduction in density or increase in length is termed 'Draft'. If a sliver becomes four times longer, or one-fourth in weight per unit length, then a draft of four is said to have been employed. Figure 3 illustrates the drafting operation.

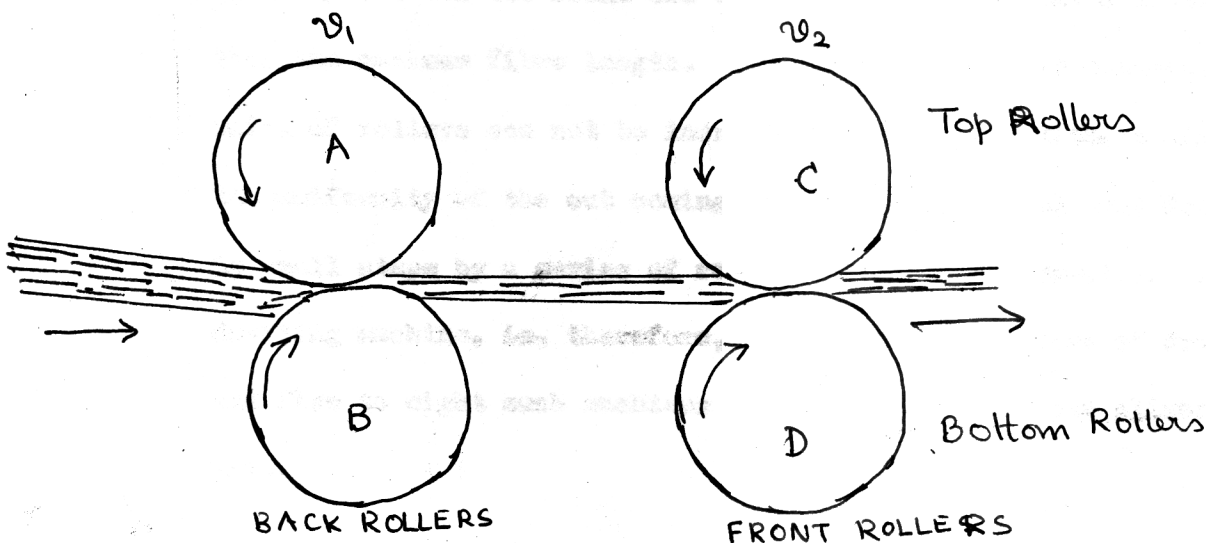
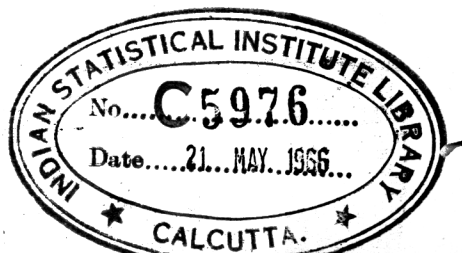


Figure 3



The drafting device consists of two pairs of rollers A, B and C, D; each roller is about 1" in diameter and 6" to 12" long. The bottom rollers B and D are fluted metal rollers and are driven by gears such that the surface speed (v_2) of D is more than that (v_1) of B. The top rollers A and C are covered with resilient material such as synthetic rubber and by means of springs or dead-weights, they are pressed against the respective bottom rollers, from which they derive their motion by frictional contact.

The sliver is fed into the grip of back rollers which deliver it to the front pair of rollers. [Note the direction of rotations in Figure 3]. The front pair of rollers revolve faster, and the sliver is, therefore, attenuated, the degree of attenuation being given by the ratio of the two velocities (v_2/v_1). This ratio is known as the draft between the two pairs of rollers. The distance between the front and back rollers is kept at a little more than the maximum fibre length. The draft between two consecutive pairs of rollers can not be increased indefinitely without affecting the uniformity of the out coming sliver. It is found that drafting in small steps by a series of rollers gives better results. Each drafting machine, is, therefore, has two to three zones of drafting and five to eight such machines are used to convert the sliver into yarn.

In the first two or three machines called 'Draw Frames', six (or eight) slivers are placed side by side and fed into the back rollers. The draft in the draw frames also is 6 (or 8). Thus the linear density of the outgoing sliver is the same as that of one of the incoming slivers. The process of feeding more than one sliver side by side, technically known as 'Doubling', results in more uniform weight per unit length along the length of the sliver.

The next three or four processes employ a doubling of two or more, but the draft increases on each of the successive machines. The sliver, during these stages, becomes quite thin and has to be twisted to stand the stresses of processing. This thin twisted sliver is known as a 'Rove' or 'Roving' and these machines as 'Fly Frames'. From the point of view of the present work, the three different forms of the fibre (sliver, roving and yarn) may be treated as identical.

The process of drafting, in addition to thinning the sliver to the desired extent, straightens the fibres and makes them parallel to the sliver axis. The roving is then fed to the final machine - the Ring Frame - on which the yarn is manufactured. Like the fly frames, the ring frames have, two functions, drafting and twisting. But the twist inserted in the yarn is ten to fifteen times that in the roving. The turns per inch of the yarn vary from 10 to 40 depending upon the thickness of yarn.

The measure of the thickness of yarn, roving or sliver is called the 'Count'. The higher the count, the thinner the yarn. Count is defined as the number of hanks (1 Hank = 840 yards) per pound. Thus a yarn count of 100 (written as 100^s) means that 84,000 yards of this yarn will weigh one pound. Table 2 gives two typical sets of machines sequences and process parameters. These should provide a rough idea as to how a required yarn count can be made from a given sliver by adjusting process particulars.

TABLE 2

Yarn count	Drafting process		Name of material delivered	Draft	No. of doublings	Count of material delivered	Turns per inch
(1)	(2)		(3)	(4)	(5)	(6)	(7)
30 ^s	Draw Frame	I	Sliver	6.0	6	0.16	mil
	Draw Frame	II	Sliver	6.0	6	0.16	mil
	Fly Frame	I	Roving	5.0	mil	0.80	0.9
	Fly Frame	II	Roving	6.2	2	2.50	2.1
	Ring Frame		Yarn	12.0	mil	30.00	34.6
100 ^s	Draw Frame	I	Sliver	6.0	6	0.18	mil
	Draw Frame	II	Sliver	6.0	6	0.18	mil
	Draw Frame	III	Sliver	6.0	6	0.18	mil
	Fly Frame	I	Roving	3.5	mil	0.63	0.6
	Fly Frame	II	Roving	5.0	2	1.57	1.2
	Fly Frame	III	Roving	6.0	2	4.70	2.4
	Fly Frame	IV	Roving	7.0	2	16.45	4.8
	Ring Frame		Yarn	12.2	2	100.00	36.0

In addition to the basic operations already described, a process known as 'Combing' is employed in making cotton yarn of counts higher than 40^s. This process eliminates the shorter fibres and helps in straightening the fibres and making them parallel to the sliver axis.

2.6. Count variation

All textile strands - laps, slivers, rovings and yarn - are produced in continuous length. The aim is always to obtain yarn as uniform as possible. Some non-uniformity or irregularity is inevitable and this is measured by the coefficient of variation (c.v.) of linear density of yarn. The C.V. is a function of the test length of the sliver or yarn; and, in general, is less for larger test lengths. Table 3 shows the extent of variation obtained for different test lengths or out lengths at various stages of processing.

TABLE 3
C.V. OF WEIGHT PER UNIT LENGTH

Material	Out-Length	Percent Coefficient of Variation
Lap	1 inch	10.0 to 16.0
	1 yard	1.0 to 3.0
Sliver	1 inch	3.0 to 6.0
	1 yard	1.5 to 4.5
Roving	1 inch	7.0 to 14.0
	15 yards	2.5 to 5.0
Yarn	1 inch	15.0 to 25.0
	120 yards	2.5 to 6.0

2.7. Need For A Regular Yarn

Irregular yarn is objectionable on the following grounds.

It appears uneven (shows thick and thin places) to the eye, and, when woven or knitted, results in fabrics of patchy appearance and uneven handle. Irregular yarn is also weaker and thus gives fabrics of lower strength and durability. In dyeing, the thick and thin portions of yarn absorb different amounts of dye, again resulting in patchy fabrics.

During the operation of winding rovings and yarns, on the respective passages on each machine, the material is subjected to tensile stresses. When the strength at any point on the material being wound is less than the minimum necessary to withstand at that point, it breaks. The main job of the machine operator, then, is to stop the machine, join the broken-ends and re-start the machine. Thus every break in the material results in a loss in production. Secondly, the lower the breakage rate, the lower the work load of the operator so that more machine units can be allotted to him. It is easy to see that greater irregularity will mean more weak places which, in turn, will cause more breaks and result in lower production per man-hour. In the ultimate analysis, then, the cost of yarn production is dependent on the regularity of the slivers, rovings and yarns.

Both from the point of view of lower manufacturing costs and improved quality of the end-product, namely fabrics, it is important to produce yarn of high regularity. The study of irregularity is, therefore, of considerable importance to the technologist.

The irregularity of a yarn is the variation in its

properties from one part to another.

The irregularity of a yarn is measured by the

coefficient of variation in the linear density

or the variation in the linear density

of a yarn of a given length of a

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CHAPTER III

ANALYSIS OF THE IRREGULARITY OF YARNS AND SLIVERS AND THE THEORY OF DRAFTING (A Brief Review)

3.1. An outline of the chapter:

It is common experience that some variation in the linear density of a sliver or yarn is inevitable.

The irregularity of a sliver or yarn is measured by the square of the coefficient of variation in its linear density. Also, behaviour of the serial correlation in the linear density brings out the pattern of fluctuations of density along the length of a sliver. These measures are very useful in studying the effect of drafting, which is a very important technological process, on the irregularity of the material drafted.

In this chapter we shall describe briefly the features and the results of the work that has been done in the last two decades in the analysis of yarn irregularity and the theory of drafting. Though the contributions of Kamah [Jour. Text. Inst. Transaction, 1951] and Grishin [Jour. Text. Inst. Transaction, 1954] to the theory of drafting are important they have not been considered here as they relate to aspects other than those taken up for study in this thesis.

3.2. Measures of Irregularity:

Let x be the distance measured along the length of a sliver from some chosen reference section. The following measures of irregularity of $m(x)$, the linear density of the sliver at the cross section at x have been extensively used by the earlier workers:

$$(a) \quad V_m = (C.V)^2 m(x)$$

$$= \frac{L^2}{\bar{m}^2} \left[\frac{1}{2L} \int_{-L}^L m^2(x) dx - \bar{m}^2 \right] \quad (3.2.1)$$

$$(b) \quad \beta(\xi) = \frac{L^2}{\bar{m}^2} \left[\frac{1}{2L} \int_{-L}^L m(x)m(x+\xi) dx - \bar{m}^2 \right] \quad (3.2.2)$$

$$(c) \quad B(L) \text{ [The standardised variance between pieces of length } L]$$

$$= \frac{L^2}{\bar{m}^2} \left[\frac{1}{2L} \int_{-L}^L m_1^2(x) dx - \bar{m}^2 \right] \quad (3.2.3)$$

and

$$(d) \quad V(L) \text{ [The standardised variance within pieces of length } L]$$

$$= V_m - B(L), \quad (3.2.4)$$

$$\text{where } \bar{m} = \frac{L^2}{\bar{m}^2} = \frac{1}{2L} \int_{-L}^L m(x) dx,$$

$(C.V)^2 m(x)$ stands for the square of the coefficient of variation of $m(x)$

$$\text{and } m_1(x) = \frac{1}{L} \int_x^{x+L} m(y) dy.$$

They have used also another measure, called the 'Power Spectrum' $g(w)$ in the analysis of yarn irregularity, and have defined it for $n(x)$ by

$$g(w) = \lim_{x \rightarrow \infty} \frac{1}{2x} \left| \frac{1}{\sqrt{2x}} \int_{-x}^x n(x) e^{-iwx} dx \right|^2 \quad (3.245)$$

Similar measures have been used for the analysis of the irregularity of $n(x)$, the number of fibres in the cross-section at x . We shall now proceed with the review of the work on the analysis of yarn irregularity.

3.3. Martindale's Contribution:

It has been pointed out in the last chapter that non-uniformity in the number of fibres in a cross-section is an undesirable quality from the spinning and weaving aspects. The most uniform sliver would be one which has the same number of fibres in each of its cross-sections. Further from a mathematical stand point, it is the simplest conceivable structure. The problem, then, is to find out if such a configuration can ever be realized. It is Martindale [15] who, in 1945, observed that such a uniform configuration in slivers can never be realized in practice. He observed, 'Neglecting variations of fibre thickness, a uniform yarn would have the same number of fibres

per cross-section at all points along it. The construction of such a sliver (yarn) is merely a question of arrangement of the fibre ends. To obtain uniformity of thickness, each time a fibre terminates another should be introduced to take its place. No mechanical contrivance, Girding, Combining, Drawing or Spinning machine, possesses this faculty and, therefore, no other arrangement of fibres will give a uniform yarn. This means that it will require some revolutionary change in Drawing and Spinning methods to do this'. Further he showed by qualitative argument that the ideal fibre arrangement that could be realized in practice is the completely random arrangements of fibre ends. That is, if we have a sliver of length L composed of N fibres, then in the ideal case, these N fibre ends would be uniformly and independently distributed along the length of the sliver. He deduced from these considerations, that for the ideal case $C.V.^2 n(x) = \frac{1}{N}$ where N is the average number of fibres in a cross-section and contended that for any actual sliver, the $C.V.^2 n(x)$ would be more than $\frac{1}{N}$. He even showed that, as the number of fibres N and the length L of the sliver tend to infinity in such a way that $\frac{N}{L} \rightarrow r$, then the number of fibres in a cross-section would tend to a Poisson variate with mean $r\bar{l}$ ($\bar{l}$ is the mean fibre length). However, he does not seem to have realized the important connection between his ideas on the arrangement of fibres in a sliver, and the Poisson process on the real line. Martindale's

arguments enable one to regard a sliver as a countable collection of fibres whose ends are distributed along the sliver axis according to some probability law; so viewed, Martindale's 'ideal sliver' is one for which the probability law is the Poisson Law.

3.4. Properties of Ideal Slivers. (Contributions of Spencer Smith, Todd, Cox, Townsend, Bruce Fujino and Kawabata).

Martindale's considerations show that the arrangement of the fibres in a sliver is random, and hence the number $n(x)$ of fibres in the cross-section at x (measured from an arbitrarily chosen reference section) is a random function of x . The question, then, arises as to whether the random function $n(x)$ is stationary; and if it is stationary, what the nature of the co-variance function will be and so on. Although, for an ideal sliver, the stationarity of $n(x)$ seems to be an obvious property, this fact has not been realized by the earlier workers. This is, then perhaps, due to the fact that they have not looked at these problems from the stand point of stochastic processes for one thing, and, for another, the concept of an ideal sliver, and hence that of the random arrangement of fibres in a sliver, has been invoked for the sole purpose of establishing that it was impossible to produce a 'uniform yarn' even under the most ideal conditions. When $n(x)$ is viewed as a random function, or equivalently as a stochastic process, the measures of irregularity given ⁱⁿ (3.2), all become the estimates

of the corresponding properties for $n(x)$. For example, if $n(x)$ is stationary, then (3.2.2) and (3.2.5) will be unbiased estimators of the autocovariance function and the spectral density of $n(x)$ respectively.

Spencer-Smith and Todd [17] first derived the expression for the serial correlation of $n(x)$ for ideal slivers, from elementary considerations. To do this, they assumed $n(x)$ satisfies the restriction

$$n(x + \varepsilon) = v_\varepsilon n(x) + n(\varepsilon) + \theta$$

where v_ε depends only on ε and θ is a disturbance function whose distribution approaches normal as the number of fibres per cross-section of the sliver increases. Using the above regression model, they obtained

$$\begin{aligned} r(\varepsilon) &= \text{Corr. } [n(x) n(x + \varepsilon)] \\ &= \frac{1}{l} \int_0^{l_{\max}} (l - \varepsilon) dP(l) \end{aligned} \quad (3.4.1)$$

where $P(l)$ is the distribution function of l , the fibre-length. This regression model is quite unnecessary. We will show in the next chapter that the condition that the given sliver is ideal is alone sufficient to deduce the above expression for the serial correlation of $n(x)$.

The use of correlograms for measuring yarn quality was first pointed out by Townsend and Cox. [5]. Further, they [21] have analysed the variations in the weights of pieces of length L of a sliver, and have obtained the following general relationship between $V(L)$ [cf. 3.2.4] and the serial correlation coefficient $r(\xi)$ of the sliver.

$$V(L) = V_{\infty} \left[1 - \frac{2}{L^2} \int_0^L (L - \xi) r(\xi) d\xi \right] \quad (3.4.2)$$

Using the foregoing relationship and the Spencer-Smith-Todd's expression for $r(\xi)$, Breny [1] computed $V(L)$ for an ideal sliver and showed that $V(L)$ is of the form

$$V(L) = V_{\infty} \left[1 - \frac{A}{L} + \frac{B}{L^2} \right] \quad \text{for } L \geq l_{\max} \quad (3.4.3)$$

where $l_{\max}$ is the length of the longest fibre in the sliver.

Recently, Fugino and Kawanata [12] [cf. 3.2.5] proved that the power spectrum of $n(x)$ for an ideal sliver is given by

$$g(v) = \frac{2}{\pi v^2} \int_0^{l_{\max}} (1 - \cos vl) dP(l) \quad (3.4.4)$$

All the above properties of ideal slivers, some of which have been proved under more restrictions than are necessary, can be proved without these additional assumptions and in a unified manner. This will

be seen in chapter V. 'Some yarn models'.

3.5. Some Yarn Models:

Before proceeding with the review of the theories of drafting, we shall first see how the study of yarn models, different from the ideal and uniform ones, is a topic of technological importance.

It has been observed that the $(C.V.)^2 m(x)$ [or $(C.V.)^2 n(x)$] for any actual sliver is much more than the Martindale's limit $\frac{1}{N} [1 + (C.V.)^2 a]$, 'a' being the linear density of a fibre and N , the average number of fibres per cross-section of the sliver, and that this limit is never attained in practice. This may, well, be due to the fact that the fibres in a sliver are not all straight and parallel to the sliver axis. Straightening and parallelising are known to reduce the deviation of $(C.V.)^2 m(x)$ from its ideal value, but even after this process the observed $(C.V.)^2 m(x)$ is higher than the ideal limit.

The foregoing considerations have led a number of workers to study different arrangements of fibre ends in a sliver. For example, Stig Lafgren [e.f. his book 'The Fibrogram' Gothenburg, 1950, Chap. III pp. 40-42] considers a model in which the number of fibre-ends (left or right) in the interval $(x, x+y)$ has a distribution with mean Ay and variance By and is uncorrelated with the number of fibre-ends in any interval disjoint with $(x, x+y)$. While this model can be made, by a proper choice of A and B , to yield a value of $(C.V.)^2 m(x)$

which is observed in practice, it has the disadvantage that it can not give us complete information regarding the distributions of the distances (measured from an arbitrarily chosen point on the sliver axis) of the fibre-ends lying in an interval, say (a, b) . Such information is very vital for a proper understanding of the effect of a process like drafting or the fibre-arrangement in a sliver. Besides the above, there are other models due to Vanden Abeele [22], Hannah and Rodden [13].

Vanden Abeele's model may be described as follows: Divide the total length of yarn into portions of length L each. Imagine that all the fibres in the yarn were first emitted by a spinneret^{*} having m holes. Consider a particular hole, say the i th and a portion, say j th, of length L and denote by P_{ij} , the sum of all the fibre-lengths emitted by the i th hole which are found in the j th portion. Then P_{ij} will be a random variable. Vanden Abeele considered various cases in which P_{ij} has forms similar to those of the analysis of variance. Vanden Abeele model has the same disadvantages as that of Stig Lofgren. Hannah and Rodden's model, unlike the preceding two, gives complete information about the fibre arrangement in their sliver, but their analysis of the model is not correct in all the essentials. This fact was pointed out by H. Remy [2], but he has not given the correct result. In chapter IV we shall consider this model in greater detail.

3.6. Notations of 'Added Irregularity' and 'Drafting Waves'

We have seen in chapter I that drafting, in addition to producing attenuation in the thickness of the sliver, straightens out the fibres and achieves a greater degree of parallelisation of fibres. While drafting serves these technologically important purposes, it is known to 'add to the irregularity of the input sliver' and also produce 'drafting waves'. Addition to irregularity means that $(C.V.)^2 \cdot n(x)$ for a drafted sliver is always more than that of the input. This statement must not be confused with the statement that a sliver becomes more regular after drafting; because the second statement is generally used to convey that the irregularity of a drafted sliver [as measured by the $(C.V.)^2 \cdot n(x)$ or $(C.V.)^2 \cdot n(x)$] is nearer to its ideal limit than the irregularity of the input sliver is to its ideal limit.

Cavaney and Foster [3] studied the behaviour of added irregularity with different drafts and roller-settings. Their data show that, in general, the added irregularity varies linearly with the draft D for fixed roller-settings and linearly with h , the roller-setting, for fixed D .

The explanation of 'drafting waves' is due to Foster, [6], [7] and [8]. His explanation runs somewhat on the following lines:

During the roller drafting of cotton, the rollers cannot be set much closer than the length of the longest fibres and consequently most of the fibres are for a time out of the control of the ^{rollers} fibres. If, after leaving the back rollers, the fibres always continue to move with the speed of these rollers until their front ends are gripped by the front rollers, the uniform motion is always the same, and variations in fibre distribution in the entering sliver are merely repeated in the drafted sliver, but are spread out over greater lengths in proportion to the draft. But some of the floating fibres may be dragged forward by the fibres already gripped by the front rollers and so reach the front rollers earlier than they would otherwise do. This forms a thick place in the drafted sliver and a thin place between the rollers. When this thin place begins to reach the front rollers, the number of floating fibres being dragged forward decreases because there are now fewer fast moving fibres to drag them forward. Consequently, a thick place forms between the rollers, which, in turn, reaches the front rollers and so the process repeats itself. We may, therefore, expect the drafting to cause a periodic variation in the thickness of the drafted material. This is called the 'drafting wave'.

Foster's experimental results [8] show that the 'drafting wave' is ^{revealed} manifested by the oscillatory character of the serial correlation

coefficient which is of the damped harmonic type for cotton slivers.

3.7. Theories of Drafting (Contributions of Cox and Foster):

All theories of drafting aim at studying the motion of fibres between the rollers. Since complete experimental evidence on the motion of fibres in the drafting zone is not available, all such theories can only be tested by comparing the properties of the drafted slivers derived from theory with those observed by experiment. Here we shall be concerned only with the theories, namely those of Cox and Foster, which seek to explain the phenomenon of drafting waves and the Cavaney and Foster's observations on the added irregularity due to drafting.

a) Cox's Theory of Drafting

In this theory, Cox [4] considers $n(x)$, the linear density of sliver in the cross-section at x , to be an ordinary function of x and proceeds to analyse the pattern of drafting waves by studying the behaviour of the power-spectrum ('spectrum' in his terminology).

His theory rests on the following assumptions:

- (1) The length of the fibres in the input sliver are all supposed to be small, and negligible in comparison to the distance between the rollers.

* Based on it without further twist. This method is very different from the intermittent, and in this differs from ring spinning.

- (2) The fibre-chords (the lines joining the left and right end of fibres) remain unchanged* during drafting.
- (3) The average pressure between the fibres in contact at a given point is independent of the thickness at that point.
- (4) The coefficient of friction between two fibres in contact is, on the average, proportional to their relative velocity*.

He obtains on the basis of the above assumptions, 'drafting equations' and studies the 'steady state solutions' of the drafting equations. He shows that in mule drafting**, the final coefficient

* The idea seems to be that the orientation of the fibre remains unchanged in drafting.

** Cotton yarns are spun by one or other of two methods which are generally known as 'ring spinning' and mule 'spinning' of which the latter is the older method. The general features of the mule are as follows.

At one end is a fixed frame on which are placed the bobbins of cotton roving ready for spinning. Also attached to this frame are numerous groups of three pairs of rollers in line and rotating at progressively increasing speeds just as they do in the draw frames. Quite separate are carriages which run on the floor to and from the fixed frame carrying the bobbins, the length of travel being about 60 to 66 in. On the head of each carriage are numerous fine spindles.

In spinning, the roving is drawn from the bobbins to pass between the drafting rollers and thence to be wound on the spindles of the carriage. This takes place in three stages. Firstly, the carriage moves away from the frame and draws the roving with it. The roving is drafted as it passes ^{drawn} between the drafting rollers. In the next operation the spindle is further rotated to insert additional twist, such as is required to make the yarn strong. Then follows the third operation in which the carriage moves back to the frame whilst the spindle again rotates in such a manner that the yarn just formed is wound on it without further twist. This method of mule spinning is thus intermittent, and in this differs from ring spinning.

of variation of the length drafted depends on the law connecting friction with velocity, ^{on} the law connecting the pressure between fibres with sliver thickness, and on the way the drafting force is applied. In the case of roller drafting, he shows that a drafting wave is generally formed and that the tendency to form a drafting wave appears to grow more pronounced with increasing draft. If the initial thickness, i.e., $m(x)$, is 'completely random' (according to him, $m(x)$ is completely random, means that $m(x)$ is not auto-correlated), the effect of drafting is to multiply the coefficient of variation of thickness by $D g(F_0)/g[F/D]$, where D is the draft, F_0 the initial mean thickness and $g[f]$ the average normal pressure at a point where the sliver thickness is f . The auto-correlation is unaltered.

Cox's treatment is not directly suitable for developing a general theory of drafting of slivers in which the arrangement of fibres is random. It does not even lead to an explanation for the irregularity added by drafting, so that Cavaney and Foster's experimental results (which were discovered only after Cox put forward his theory) can not be explained by this theory.

(b) Foster's Theory of Drafting:

The fact that the theories which seek to explain the motion of fibres under the forces acting on them have not revealed much about the mechanism of the formation of drafting waves has led Foster [6]

to approach the problem in a different way. He has put forward a theory in which the motion of fibres in the drafting zone is assumed to satisfy certain conditions which ultimately lead to an explanation of the drafting waves. The basic assumptions of his theory are:

- (1) that the input sliver is uniform, that is, the number of fibres in a cross-section is the same for all cross-sections, and is composed of fibres of the same length L , which are straight and parallel to the sliver axis,
- (2) that all fibres continue to move with the speed of the back rollers, until their mid-points reach a boundary between the rollers, on crossing which the speed of all of them changes to that of the front rollers,
- and
- (3) that the boundary is not stationary, but that its distance from the front rollers depends upon the number and distribution along the length of the sliver of the fast moving fibres.

From the first two assumptions he obtains an equation (which he calls the special equation of continuity) connecting $n(t)$, the number of fibres held by the front rollers at time t , $v(t)$ and $s(t)$, the velocity and the distance of the moving boundary and n , the number

of fibres in a cross-section of the input sliver. By choosing a specific form for the dependence of the motion of the moving boundary and the distribution of the fast moving fibres, he obtains another equation connecting the above quantities. The most general form of these equations are not known. By regarding the motion of the moving boundary to be oscillatory, he studies the character of $n(t)$ and shows how to obtain the 'wave-length' and the amplitude of the drafting wave thus caused. By identifying the square of the amplitude of the drafting wave with the variation in the thickness of the drafted sliver he studies the behaviour of the added irregularity due to drafting with changing draft.

Thus the most fundamental contribution of Foster's theory is that, in so far as the irregularity properties of slivers and the effect of drafting on these are concerned, it is not necessary to make detailed assumptions on the forces acting on the fibres in the drafting zone, but that it will be sufficient to make some broad assumptions on the type of motion of these fibres which can be verified by experiment. However, even Foster's approach has its own disadvantages. These are:

- (1) that even in the special case when the input sliver is uniform and consists of fibres of the same length, his theory does not lead directly to an expression for the

irregularity added by drafting,

and

- (2) that even when the input sliver is not uniform, but a random one, his theory can only lead to stochastic differential equations which are even more difficult to handle and which raise altogether different problems like the existence of stationary solutions etc.

Consequently, Foster's theory raises an important question as to whether it is possible to develop a theory of drafting for random slivers which can lead in a natural way to the derivation of the irregularity properties of the drafted sliver without making detailed assumptions about the forces acting on the various fibres. That it is possible to develop such a theory is established in this thesis. We shall see in chapters IV and VI how a systematic use of the theory of stationary stochastic processes of the continuous parameter forms a mathematical basis for unifying all the earlier work on the irregularity of slivers and deriving a theory of drafting which explains not only the mechanism of the drafting waves, but also leads to expressions which can, in principle, be verified by experiment.

CHAPTER IV

STATIONARY STOCHASTIC PROCESSES

We shall collect in this chapter the results in stationary stochastic processes that are used in subsequent chapters.

4.1. Stochastic Processes

Suppose that t is a real valued parameter (representing time, or distance along a fixed direction), and that $x(t)$, for each t belonging to a set T , is a random variable (real or complex valued) defined on ^acertain probability space. We say that $x(t)$ is a 'stochastic process' or a 'random process'. The range T of t is usually an infinite set. We also say that $x(t)$ is a discrete parameter process when T is denumerable, and a continuous parameter-process when T is an interval. $x(t)$ is spoken of sometimes also as a 'random function'. Here we shall be concerned with only continuous parameter processes.

A stochastic process can be regarded as a one-parameter family of random variables, or alternatively as an ensemble of functions. ^{Any member of this ensemble} is called a 'sample function' or a 'realization'. When the probability of all finite combinations of arbitrarily chosen interval sets is known in a consistent manner it has been established by

A. Kolmogoroff [35] that a probability measure can be uniquely defined on the Borel sets of the space of realizations. However, this measure becomes adequate, provided the functions admitted as elements in the space of realizations are restricted in a suitable way as for example, to functions which are continuous in the mean.

We assume that the process $x(t)$ is centred, so that we can, without loss of generality write $E x(t) = 0$ for all t . Further, we consider only those processes $x(t)$ for which $E |x(t)|^2 < \infty$ for all t in T . Two random variables X and Y (centred at their means) are regarded as the same if $E |X - Y|^2 = 0$.

Now suppose that $x(t)$ is a stochastic process, and that the range of t is $(-\infty, \infty)$. In view of the restrictions placed on $x(t)$ [that is, $E x(t) = 0$, and $E |x(t)|^2 < \infty$] $\phi(t, u) = E x(t) \overline{x(u)}$ exists for all t and u . The function $\phi(t, u)$ is called the co-variance function of the process, we then have $\phi(t, u) = \overline{\phi(u, t)}$ and $\phi(t, t) \geq 0$.

$x(t)$ is said to be 'continuous in the mean' if for every t , $E |x(t_n) - x(t)|^2 \rightarrow 0$ as $t_n \rightarrow t$. A necessary and sufficient condition for $x(t)$ to be continuous in the mean is that the co-variance function $\phi(t, u)$ should be continuous at every point of the line $t = u$.

provided the right-hand side exists.

A sequence of centred random variables $x_1, x_2, \dots, x_n$ is said to 'converge in the mean' to a random variable x if $E |x_n - x|^2 \rightarrow 0$ as $n \rightarrow \infty$ and x is called the limit in the mean (l.i.m) of the sequence x_n .

Consider the family $L(x)$ of all the random variables of the form

$$c_1 x(t_1) + c_2 x(t_2) + \dots + c_n x(t_n) \quad \dots \quad (4.1.1)$$

where the c 's are complex constants. Adding to $L(x)$ the limits in the mean of convergent sequences of the form (4.1.1) we get an extended set $L_2(x)$. If, now, we define for any two elements of $L_2(x)$, the inner product (y, z) by

$$(y, z) = E(y \bar{z}), \text{ then the space } L_2(x)$$

becomes a Hilbert space, and it is known as the Hilbert space of the process $x(t)$ [25], [33].

Integrals of the type

$J_1 = \int_a^b g(t) x(t) dt$, where $g(t)$ is a non-random function can be defined as the limit in the mean of sequences of the corresponding Riemann sums. Then these integrals are random variables, and

$$E |J_1|^2 = \int_a^b \int_a^b g(t) \overline{g(u)} \phi(t, u) dt du$$

provided the right-hand side exists.

4.2. Stationary Processes

Consider a stochastic process $x(t)$. $x(t)$ is said to be 'stationary in the wide sense' if

$$(i) \ E x(t) = \bar{x} \text{ (constant) for all } t,$$

(ii) $E[x(t) - \bar{x}][x(u) - \bar{x}] = \beta(t - u)$ for all t and u . The function $\beta(t - u)$ is called the auto-covariance function of the process. $x(t)$ is said to be 'stationary in the strict sense' if all the finite dimensional distributions are invariant under the translation on the time axis. Unless otherwise stated, we shall always use 'stationary' to stand for 'stationary in the wider sense only'.

If $x(t)$ is stationary and real-valued, then $\beta(\xi)$ is an even function of ξ . It follows from a theorem of Khintchine [35] that, if $x(t)$ is stationary and continuous in the mean, there exists a bounded, non-negative and non-decreasing function $G(w)$ called the 'Spectrum' of the process, such that

$$\beta(\xi) = \int_{-\infty}^{\infty} e^{i\xi w} dG(w) \quad (4.21)$$

If $\beta(\xi)$ is absolutely integrable over $(-\infty, \infty)$ then the spectrum will be absolutely continuous. The derivative $g(w)$ of $G(w)$ is called the 'spectral density of the process'. In that case

$$g(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi w} \beta(\xi) d\xi \quad (4.22)$$

Alternatively $g(w)$ can be obtained as

$$g(w) = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \frac{1}{\sqrt{2\pi}} \int_{-T}^T x(t) \cdot e^{iwt} dt \right|^2 \quad (4.23)$$

when the spectrum is absolutely continuous ^{and} $\beta(\xi)$ can be obtained from (4.2.1) and (4.2.3). This approach is useful when the direct calculation of $\beta(\xi)$ is complicated.

We shall now prove the following result on the calculation of the spectral density, which will be used in chapter VI.

Theorem (4.2.) Suppose that $x_T(t)$ [$0 \leq t < \infty$] is for each real T a real stationary process and that $E x_T(t) = m$ for all t and T . If as $T \rightarrow \infty$ $x_T(t)$ converges in the mean to a process $x(t)$, then $x(t)$ is itself stationary. If further, the covariance function $\beta_T(\xi)$ of $x_T(t)$ vanishes outside a finite interval not depending on T , and is bounded as $T \rightarrow \infty$, then $x(t)$ has an absolutely continuous spectrum and its spectral density $g(w)$ is given by

$$g(w) = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left| \frac{1}{\sqrt{2\pi}} \int_{-T}^T x_T(t) \cdot e^{iwt} dt \right|^2$$

Proof:- The stationarity of $x(t)$ follows immediately from the stationarity of $x_T(t)$ and the definition of convergence in the mean. As $\beta_T(\xi)$ vanishes outside a fixed interval the covariance function $\beta(\xi)$ of $x(t)$, also vanishes outside the same interval,

which implies that the spectrum of $x(t)$ is absolutely continuous.

Now,

$$\begin{aligned} \frac{1}{T} &= \frac{1}{\sqrt{2\pi}} \int_0^T x_T(t) e^{i\omega t} dt \\ &= \frac{1}{2\pi T} \int_0^T \int_0^T \beta_T(t-u) e^{i\omega(t-u)} dt du + \frac{T^2 |e^{i\omega T} - 1|^2}{2\pi T} \\ &= \frac{1}{2\pi} \int_{-T}^T \left[1 - \frac{|v|}{T}\right] \beta_T(v) e^{i\omega v} dv + \frac{T^2 |e^{i\omega T} - 1|^2}{2\pi T} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_T(v) e^{i\omega v} dv + \frac{T^2 |e^{i\omega T} - 1|^2}{2\pi T} \end{aligned}$$

where $\psi_T(v) = \left[1 - \frac{|v|}{T}\right] \beta_T(v)$ if $|v| \leq T$

= 0 otherwise.

Let B be such that $|\beta_T(v)| < B$ ^{for} in $(-A, A)$ ^{and} for all T . Putting

$$K(v) = B \text{ if } |v| \leq A$$

$$= 0 \text{ if } |v| > A$$

we find that

$$|\psi_T(v) e^{i\omega v}| \leq K(v)$$

and $K(v)$ is integrable over $(-\infty, \infty)$. Further, $\lim_{T \rightarrow \infty} \psi_T(v) = \beta(v)$

Therefore, by applying Lebesgue bounded convergence theorem to

$\psi_T(v) e^{i\omega v}$ we get the desired result.

4.3. Stationary processes: Bivariate Case

Let $x(t)$ and $y(t)$ be two continuous parameter stationary processes, where the range of t is taken, as usual, to be the entire real axis. We say that $x(t)$ and $y(t)$ are stationarily correlated if

$\text{Cov}[x(t), y(u)]$ is a function $\overset{\text{of}}{\wedge}(t-u)$ only. $[= \phi_{xy}(t-u)]$

If $\beta_x(t-u)$ and $\beta_y(t-u)$ be the covariance function of $x(t)$ and $y(t)$ respectively then it follows from a theorem due to Cramer [26] that there exist three functions $G_x(w)$, $G_y(w)$ and $G_{xy}(w)$ of the real argument w such that

(i) $G_x(w)$ and $G_y(w)$ are non-negative and never decreasing.

(ii) $G_{xy}(w)$ is a complex valued function of bounded variation satisfying the difference inequality

$$|\Delta G_{xy}(w)|^2 \leq \Delta G_x(w) \Delta G_y(w)$$

(where $\Delta G_x(w)$ stands for $\int_{\Delta} dG_x(w)$)

and (iii) $\beta_x(t-u) = \int_{-\infty}^{\infty} e^{iw(t-u)} dG_x(w)$

$$\beta_y(t-u) = \int_{-\infty}^{\infty} e^{iw(t-u)} dG_y(w)$$

and $\beta_{xy}(t-u) = \int_{-\infty}^{\infty} e^{iw(t-u)} dG_{xy}(w)$

Further where $\beta_{xy}(t-u)$ is absolutely integrable, then $G_{xy}(w)$ is absolutely continuous and its derivative $G'_{xy}(w)$ is given by

$$c_{xy}(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iv\tau} f_{xy}(\tau) d\tau.$$

4.4. Deterministic and non-deterministic stationary processes:

Let $x(t)$ be a stationary process of the continuous parameter and let the Hilbert space spanned by the random variables $x(t)$ of the process up to $t = a$ be denoted by $L_2(x; a)$. Following Karhunen [34] and Kanner [31], we say that a stationary process $x(t)$ is deterministic if $L_2(x; t+h) = L_2(x; t)$, $h > 0$, for all t and h . For such processes the random variable of the process at time $t+h$ can be expressed linearly in terms of the variates of previous instants and their limit in the mean. The analytic condition (Karhunen [34]) for $x(t)$ to be deterministic is

$$\int_{-\infty}^{\infty} \frac{|\log \theta'(v)|}{1+v^2} dv = \infty$$

where $\theta'(v)$ is the deviative of the absolutely continuous part of the spectrum of the process. If a stationary process is not deterministic it is said to be non-deterministic. The non-deterministic is here equivalent to the non-expressibility of the process variate at time t linearly in terms of the process variates of earlier instants or their limit in the mean. Such non-determinism may be called linear non-determinism or non-determinism in the sense

of Karhunen [34]. This may not rule out the possibility of some other types.

An extreme case of non-determinism occurs when

Let $L_2(x; a) = 0$. If $P_X^h(a)$ denotes the projection of $x(a; h) (h > 0)$ on $L_2(x; a)$ the above extreme situation can be stated alternatively as

$$\lim_{a \rightarrow -\infty} \|P_X^h(a)\| = 0$$

In this case we say that the process is 'completely non-deterministic'.

4.5. Central-limit Theorem for m -dependent variables:

We shall now consider two well-known results (due to Hoeffding and Robbins) concerning the central-limit property of dependent random variables. The proofs of these theorems are given in Fraser's book [].

Definition 4.51:- We say that a sequence of random variables $X_1, \dots, X_n, \dots$ is m -dependent if $(X_1, \dots, X_r)$ is always independent of $(X_s, X_{s+1}, \dots)$ provided $s - r > m$.

Theorem 4.51: (Hoeffding and Robbins). Let an m -dependent sequence $X_1, X_2, \dots$ of random variables satisfy $E X_1 = 0$, $E |X_1|^3 \leq R^3 < \infty$ for $i=1, 2, \dots$. Further let the limit

$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{k=1}^p A_{1+k} = A$$

where $A_1 = 2 \sum_{j=0}^{n-1} \text{Cov} [X_{1+j}, X_{1+n}] + \text{Var} [X_{1+n}]$, exist uniformly for all n . Then $\sum_{i=1}^n X_i$ is asymptotically normal with mean 0 and variance nA_1 . If the sequence $X_1, X_2, \dots, X_n, \dots$ stationary, then the theorem 4.51 simplifies to.

Theorem 4.52: If $X_1, \dots, X_n, \dots$ is a stationary n -dependent sequence of random variables with $EX_1 = \mu$ and $E|X_1|^3$ existing, then, as $n \rightarrow \infty$, the limiting distribution of $n^{-1/2} \sum_{i=1}^n X_i$ is normal with mean $n^{1/2}\mu$ and variance

$$A = \text{Var} (X_1^2) + 2 [\text{Cov} (X_1, X_2) + \dots + \text{Cov} (X_1, X_{n+1})].$$

The bivariate form of Theorem 4.52 is

Theorem 4.53: If $(X_1, Y_1), \dots, (X_n, Y_n), \dots$ is a stationary n -dependent sequence of random variables which has

$$EX_1 = 0 = EY_1 \text{ and } E|X_1|^3 \text{ and } E|Y_1|^3$$

existing as $n \rightarrow \infty$, the limiting distribution of

$$(n^{-1/2} \sum_{i=1}^n X_i, n^{-1/2} \sum_{i=1}^n Y_i)$$

is normal with mean 0 and covariance matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

where

$$A = \text{Var}(X_1) + 2 \sum_{j=1}^n \text{Cov}(X_1, X_{1+j})$$

$$B = \text{Cov}(X_1, Y_1) + \sum_{j=1}^n [\text{Cov}(X_1, Y_{1+j}) + \text{Cov}(X_{1+j}, Y_1)]$$

$$C = \text{Var}(Y_1) + 2 \sum_{j=1}^n \text{Cov}(Y_1, Y_{1+j}).$$

It is now necessary to consider some special

cases in which we shall detail the properties of

the process $X(t)$ and $Y(t)$ and $a(x)$ denote the

density of the process $X(t)$ in the case

of a stationary process. The types that are

(1) stationary, (2) non-stationary, (3) non-stationary

processes, (4) stationary type of silver and (5) non-

stationary type of silver.

In the case of stationary processes, most of the properties of

the process $X(t)$ have been established by the earlier workers in a

series of papers, and often by involving unnecessary manipu-

lations. It is now possible to establish a mathematical model for the process

of silver and to show that these results in a consistent manner

in which the process establishes the central-limit pro-

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CHAPTER V

SOME STOCHASTIC MODELS FOR YARN AND SLIVER IRREGULARITY

5.1. An out line of the chapter:

In this chapter we shall consider some special types of slivers and study in some detail the properties of the random functions $n(x)$ and $m(x)$ [$n(x)$ and $m(x)$ denote respectively the number of fibres and the linear density in the cross-section at x of a sliver] for these types. The types that are considered here are (1) the Ideal-Sliver of Martindale, (2) non-uniform sliver of Poisson type, (3) 'Kamach-Hodson' type of sliver and (4) the so-called 'Quasi-ideal' sliver.

As has already been pointed out, most of the properties of ideal slivers have been established by the earlier workers in a rather disjointed manner and often by invoking unnecessary assumptions. We shall first formulate a mathematical model for the ideal sliver and deduce therefrom these results in a consistent manner. We shall in addition establish the central-limit property of $n(x)$ also.

5.2. The concept of a random-sliver:

From what has been said in chapter III about the structure of a sliver, it is clear that a sliver may be looked upon as a collec-

tion of fibres of known properties (such as length-distribution, density distribution etc.,), which are arranged in some manner along the sliver-axis as a result of a series of technological processes. Suppose, now, this sliver is subjected to these processes under the same condition to yield once again a sliver comprising of the same collection of fibres. If it turns out that the arrangement of fibres in the latter sliver is the same as that of the original we say that the sliver is of the non-random type. In order to obtain from a given collection of fibres a non-random type of sliver a set of machinery is required which can place a fibre exactly in its previous position. This, as has already been pointed by Martindale [15], is impossible. Consequently, the slivers obtained from a given set of machinery and a given collection of fibres are random in nature. That is to say, any two possible arrangements (by the same set of machinery) of the same collection of fibres in the form of a sliver are not, in general, identical. We shall refer to each of these possible arrangements comprising as a 'realization' of a sliver/ of the same collection of fibres. Under the influence of a given set of machinery, some of the realizations are more frequently obtained than others. In other words, the part played by the machinery, is to induce a probability distribution in the set of all 'realizations'.

I thank Dr. M. A. J. VAN DER WERF (Belgium) for his interest in this work published application of Statistical Processes in Textiles. unified approach to the properties of ideal slivers. methods are different. References: Bull. Soc. Royale Sciences de Liège (1951) (1951), Appl. Stat. (1951).

5.3. Martindale's Model: The Ideal Sliver:*

It is pointed out in (5.2) that a sliver may be looked upon as a collection of fibres arranged along a fixed direction called the 'sliver axis' according to a definite probability law. Since we shall be exclusively interested in the cross-sectional properties, such as the number of fibres in a cross-section and the linear density, the aforesaid probability law need specify ^{only} the manner in which the fibre-ends (left or right) are distributed along the sliver axis and the ^{manner in which} the fibres are oriented with respect to it.

(a) Definition of an ideal sliver:

An 'ideal sliver' is one for which the fibre-arrangement obeys the following rules:

- (i) All fibres are straight and parallel to the sliver axis.
- (ii) The linear density 'a' of a fibre is constant along its length 'l'. Every fibre is characterized by its length and linear density. These are independently distributed in the population of fibres making the sliver.
- (iii) The number of right ends of fibres in an interval (a, b) on the sliver axis is a Poisson variate with mean $\lambda(b-a)$ and is distributed independently of the number of right ends in any interval disjoint with (a,b).

* I thank Dr. Breny (formerly of the International Wool Secretariat, Liege, Belgium) for acquainting me with his work published in French, on the application of Stochastic Processes to Textiles. He, too, uses a unified approach to the properties of ideal slivers (Th:5.31), but our methods are different.
References: Bull. Soc. Royale Science's Liege (1953). Annales Textiles (1957). Appl Sci Res (1958).

(b) Construction of the Fundamental Random Functions:

Choose a point O , on the sliver-axis, as origin. [See Figure 4] Let N_X be the number of fibres with right ends in $(-X, X)$ in a single realisation of the ideal sliver. Numbering these fibres as $1, 2, \dots, N_X$, let $x_1, x_2, \dots, x_{N_X}$, $l_1, l_2, \dots, l_{N_X}$ and $a_1, a_2, \dots, a_{N_X}$ be the co-ordinates of the right ends measured along the sliver axis lengths and linear densities of the fibres.

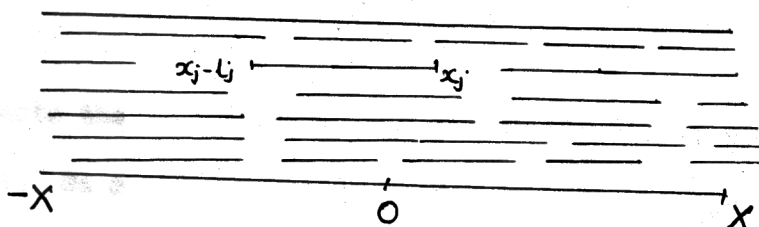


FIGURE 4

Define $w_j(x)$, $n_X(x)$ and $m_X(x)$ respectively by

$$w_j(x) = \begin{cases} 1 & \text{if the } j\text{th fibre lies in the cross-section at } x \\ 0, & \text{otherwise} \end{cases} \quad (5.31)$$

$$n_X(x) = \sum_{j=1}^{N_X} w_j(x) \quad (5.32)$$

$$\text{and } m_X(x) = \sum_{j=1}^{N_X} a_j w_j(x) \quad (5.33)$$

Since the rule of fibre arrangement in any sliver is, in

general, a probabilistic one, $n_X(x)$ and $n(x)$ are random functions of X and x . If $m(x)$ and $n(x)$ are the linear density and number of fibres respectively in the cross-section at x , then

$$n(x) = \lim_{X \rightarrow \infty} \frac{1}{X} n_X(x) \quad (5.34)$$

and

$$m(x) = \lim_{X \rightarrow \infty} \frac{1}{X} m_X(x) \quad (5.35)$$

Let us note that when the range ^{of ℓ} is finite the limits are actually attained. We shall, hereafter, refer to $m(x)$ and $n(x)$ as the fundamental random functions of the sliver.

(c) Properties of $m(x)$ and $n(x)$ for an ideal sliver:

From the definition of an 'ideal sliver', it follows that $N_1, N_2, \dots, N_{N_X}$ are, for a fixed N_X , independently and uniformly distributed in $(-X, X)$. To see this, we first note that for a fixed N_X , $N_1, N_2, \dots, N_{N_X}$ are independently and identically distributed in $(-X, X)$. Consider now the interval $(-X, x)$ contained in $(-X, X)$. Let A_1 denote the event that there are 1 right ends in $(-X, x)$ and B_{N_X} the event that there are N_X right ends in $(-X, X)$. Then $A_1 = \bigcup_{N_X=0}^{\infty} A_1 B_{N_X}$.

Further from (5.32):

$$\begin{aligned} P(A_1 | B_{N_X}) &= P(A_1 / B_{N_X}) P(B_{N_X}) \\ &= 0 \text{ for } N_X < 1 \end{aligned} \quad (5.36)$$

$$\therefore P(A_1) = \sum_{N_X=1}^{\infty} P(A_1 | B_{N_X}) \quad 1 = 0, 1, 2, \dots$$

But from the preceding considerations and the definition of an ideal silver we have.

$$P(A_1) = \frac{[\gamma(x+X)]^1}{1!} \cdot e^{-\gamma(x+X)}$$

$$P(A_1 / B_{N_X}) = \binom{N_X}{1} \cdot [P_X(x)]^1 [1 - P_X(x)]^{N_X-1}$$

where $P_X(x)$ is the common distribution function of the x_j 's

$$\text{and } P(B_{N_X}) = \frac{(2\gamma X)^{N_X}}{N_X!} \cdot e^{-2\gamma X}$$

Substituting these expressions in 5.36, we get

$$\frac{[\gamma(x+X)]^1}{1!} \cdot e^{-\gamma(x+X)} = \frac{[2\gamma X P_X(x)]^1}{1!} \cdot e^{-2\gamma X P_X(x)}$$

whence it follows that

$$P_X(x) = \frac{x+X}{2X} \text{ yielding the desired result.}$$

Theorem (5.11):

(i) $n(x)$ and $m(x)$ are stationary and have absolutely continuous spectra. Further we have

If $\rho_m(\cdot)$ and $g_m(v)$ are the auto-covariance function and the spectral density of $m(x)$ [$\rho_m(\cdot)$ and $g_m(v)$ refer to $m(x)$], then

$$\rho_m(\cdot) = \frac{\overline{a^2}}{2} \int_{|\cdot|}^{\infty} (1 - |t|) dP(t)$$

where $P(\cdot)$ is the distribution function of fibre-lengths and ∞ is the maximum fibre-length

[Spencer-Smith and Todd (17)]

$$(11) \quad g_m(v) = \frac{\overline{a^2}}{2\pi v^2} \int_0^\infty (1 - \cos w) dP(w)$$

[Fujino and Iabucata (12)]

(11) $n(x)$ is, for each x , a Poisson variate with mean

[Martindale (15)]

(iv) If $B_m(L)$ stands for the C.V.² of

$$\frac{1}{L} \int_0^{x+L} n(y) dy \quad \text{and} \quad V_m(\infty) = \frac{A_m(0)}{(\overline{a})^2}$$

then,

$$B_m(L) = \frac{V_m(\infty)}{L} - \frac{1}{3L^2} + \frac{1}{3L^2} \int_L^\infty (-t)^3 dP(t)$$

[Renyi (1)]

* It will assume in what follows that ∞ is finite.

(v) The characteristic function $\phi(t_1, t_2)$ of $n(x)$ and $n(y)$ [$y > x$] is given by

$$\phi(t_1, t_2) = \exp \left\{ \nu \bar{l} [(e^{it_1} - 1) + (e^{it_2} - 1)] + \nu (e^{it_1} - 1)(e^{it_2} - 1) \int_{y-x}^{l_{\max}} [l - (y-x)] dF(l) \right\}$$

Proof:

(i)(1) We shall see presently that all the above properties of $n(x)$ which essentially comprise the results obtained by the earlier workers, are more or less immediate consequences of the definition of an ideal slider.

We have from 5.51

$$U_j(x) = 1 \text{ if } x_j - l_j \leq x \leq x_j \\ = 0 \text{ otherwise}$$

We shall choose X so large that $(x - l_{\max}, x)$ is contained in $(-X, X)$. Since x_j 's are, for a fixed N_X identically, independently and uniformly distributed in $(-X, X)$, it follows that, for any j ,

$$\Pr[U_j(x) = 1] = \Pr[x_j - l_j \leq x \leq x_j] = \frac{l_j}{2X} = \frac{\bar{l}}{2X} \quad (5.37)$$

$$\begin{aligned} \Pr[U_j(x) = 1, U_j(x+\xi) = 1] &= \Pr[x_j - l_j \leq x \leq x+\xi \leq x_j] \\ &= \frac{1}{2X} \int_{\xi}^{l_{\max}} (l - \xi) dF(l) \quad (5.38) \\ &\quad (\xi \geq 0) \end{aligned}$$

From the condition (iii) of the definition of an ideal sliver, the definition of $n(x)$ and the fact that N_X is a Poisson variate with mean $\lambda \nu X$, it now follows that

$$E n(x) = \nu \lambda \bar{l}$$

$$\begin{aligned} \text{Cov} [n(x), n(x + \xi)] &= \nu \lambda^2 \int_{|\xi|}^{l_{\max}} (l - |\xi|) dP(l) \\ &= \beta_n(\xi), \text{ say,} \end{aligned}$$

since the integral depends only on the lag ξ the two relations show that $n(x)$ is a stationary process. $n(x)$ is only a special case of the process $n(x)$, viz., when the density of each fibre becomes unity. Hence $n(x)$ is also a stationary process. The absolute continuity of the spectrum follows from the fact that $\beta_n(\xi)$ vanishes for $|\xi| \geq l_{\max}$. The expression for the spectral density $e_n(\nu)$ for the $n(x)$ process is obtained from $\beta_n(\xi)$ by Khintchine's representation as follows:

$$\begin{aligned} e_n(\nu) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\nu\xi} \beta_n(\xi) d\xi \\ &= \frac{\nu \lambda^2}{2} \int_0^{\infty} \cos \nu \xi d\xi \int_0^{l_{\max}} (l - \xi) dP(l) \\ &= \frac{\nu \lambda^2}{2} \int_0^{l_{\max}} dP(l) \int_0^l (l - \xi) \cos \nu \xi d\xi \\ &= \frac{\nu \lambda^2}{2} \int_0^{l_{\max}} (1 - \cos \nu l) dP(l). \end{aligned}$$

(iii) That $n(x)$ is a Poisson variate with mean $\bar{n}$ follows from

$$\begin{aligned} \Pr [n(x) = r] &= \lim_{K \rightarrow \infty} \Pr [n_K(x) = r] \\ &= \lim_{K \rightarrow \infty} \sum_{N_K=r}^{\infty} \binom{N_K}{r} \left(\frac{\bar{n}}{2K}\right)^r \left(1 - \frac{\bar{n}}{2K}\right)^{N_K-r} \frac{(2K)^{N_K}}{N_K!} e^{-2K} \\ &= \frac{(\bar{n})^r}{r!} e^{-\bar{n}} \quad r = 0, 1, 2, \dots \end{aligned}$$

(iv) The expression for $n_m(l)$ can now be obtained using the expression for $\beta_m(\xi)$. In fact, we have

$$\begin{aligned} n_m(l) &= \frac{1}{m^2} \text{Var.} \left[\frac{1}{L} \int_0^{L+l} n(y) dy \right] \quad \text{where } N = N_m(x) \\ &= \frac{1}{m^2 L^2} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int_{-\frac{L}{2}}^{\frac{L}{2}} \beta_m(y-x) dy \\ &= \frac{2}{m^2 L^2} \int_0^L (L-\xi) \beta_m(\xi) d\xi \\ &= \frac{2 V_m(m)}{L^2} \int_0^L (L-\xi) d\xi \int_{\xi}^{l_{\max}} (l-\xi) dP(l) \\ &= \frac{V_m(m)}{L} \left\{ \frac{L^2}{L} - \frac{L^3}{3L^2} + \frac{1}{3L^2} \int_0^{l_{\max}} (l-L)^3 dP(l) \right\} \end{aligned}$$

by inverting the order of integration.

(v) Lastly, the characteristic function $\phi(t_1, t_2)$ of $n(x)$ and $n(y)$ [$y > x$] may be obtained as follows. To do this, we shall first choose X so large that $(-X, X)$ contains the interval $(x - l_{\max}, y)$. We then have

$$\Pr [u_j(x) = 1; u_j(y) = 1] = \int_{y-x}^{l_{\max}} \frac{[l - (y-x)]}{2X} dF(l),$$

$$\Pr [u_j(x) = 0; u_j(y) = 1]$$

$$= \Pr [u_j(y) = 1] - \Pr [u_j(x) = 1; u_j(y) = 1]$$

$$= \frac{\bar{l}}{2X} - \frac{1}{2X} \int_{y-x}^{l_{\max}} [l - (y-x)] dF(l)$$

$$= \Pr [u_j(x) = 1; u_j(y) = 0],$$

$$\text{and } \Pr [u_j(x) = 0; u_j(y) = 0]$$

$$= \Pr [u_j(x) = 0] - \Pr [u_j(x) = 0; u_j(y) = 1]$$

$$= 1 - \frac{\bar{l}}{2X} + \frac{1}{2X} \int_{y-x}^{l_{\max}} [l - (y-x)] dF(l)$$

$$\therefore E \exp [it_1 u_j(x) + it_2 u_j(y)]$$

$$= 1 - \frac{\bar{l}}{2X} + \frac{1}{2X} \int_{y-x}^{l_{\max}} [l - (y-x)] dF(l)$$

$$+ [e^{it_1} - 1] \frac{\bar{l}}{2X} - \frac{1}{2X} \int_{y-x}^{l_{\max}} [l - (y-x)] dF(l)$$

$$+ e^{i(t_1+t_2)} \frac{1}{2X} \int_{y-x}^{l_{\max}} [l - (y-x)] dF(l).$$

therefore,

$$\phi(t_1, t_2) = \lim_{X \rightarrow \infty} E \exp \left[i t_1 \sum_{j=1}^{N_X} v_j(x) + i t_2 \sum_{j=1}^{N_X} v_j(y) \right]$$

$$\text{(since } n(x) = \lim_{X \rightarrow \infty} \sum_{j=1}^{N_X} v_j(x) \text{)}$$

$$= \lim_{X \rightarrow \infty} E \left\{ E \exp [i t_1 v_j(x) + i t_2 v_j(y)] \right\}^{N_X}$$

(since $v_j(x)$ and $v_j(y)$ are independent for $j \neq k$)

$$= \lim_{X \rightarrow \infty} \exp \left\{ -2 \nu X (1 - E \exp [i t_1 v_j(x) + i t_2 v_j(y)]) \right\}$$

$$= \exp \left\{ -\nu \left(2\bar{L} - \int_{y=x}^{L_{\max}} [L - (y-x)] dF(y) \right. \right. \\ \left. \left. + [e^{i t_1} + e^{i t_2}] \left[\bar{L} - \int_{y=x}^{L_{\max}} [L - (y-x)] dF(y) \right] \right. \right. \\ \left. \left. + e^{i(t_1+t_2)} \int_{y=x}^{L_{\max}} [L - (y-x)] dF(y) \right) \right\}$$

$$= \exp \left\{ \nu \bar{L} [(e^{i t_1} - 1) + (e^{i t_2} - 1)] \right. \\ \left. + \nu (e^{i t_1} - 1)(e^{i t_2} - 1) \int_{y=x}^{L_{\max}} [L - (y-x)] dF(y) \right\}$$

Theorem 5.12.

$n(x)$ and $m(x)$ are continuous in the mean and are completely non-deterministic.

Proof:

From the integral for $\beta_m(\xi)$ it is seen that it is a continuous function of ξ . Hence $m(x)$ [as also $n(x)$] is continuous in the mean. Assuming that a P-measure is put on the infinite dimensional space of process variates of $m(x)$, subject only to the restrictions of $E[m(x)] = \sqrt{a} \bar{e}$ and $\beta_m(\xi) = \sqrt{a} \int_{|\xi|}^{l_{\max}} (1 - |\xi|) dP(l)$, let $L_2(m)$ be the Hilbert space of the process $m(x)$, and $L_2(m; a)$ be the closed linear manifold spanned by the set $\{m(x); x \leq a\}$. For $a \leq -l_{\max}$, then $m(0)$ is orthogonal (uncorrelated) to every process variate for which $x \leq -l_{\max}$ and hence orthogonal to $L_2(m; a)$, $x \leq -l_{\max}$ — that the norm of the projection of $m(0)$ on $L_2(m; a)$ is zero. That is, $m(x)$ is completely non-deterministic.

The implication of the completely non-deterministic nature of $n(x)$ and $m(x)$ is that they are evolutionary in the sense that the closed linear manifolds $L_2(n; a)$ and $L_2(m; a)$ are continuously developing as 'a' increases starting initially with the element zero. It generally happens that an operation in a stationary process leaves the features of determinism or otherwise unaffected. We come across the same situation here also. We shall see in chapter [5],

that the operation of drafting, does not change the completely non-deterministic nature of the random function $n(x)$ of the input sliver.

Theorem 5.33: (Central limit property for $n(x)$)

Consider the k equi-distant cross-sections at $x, x+x_0, x+2x_0, \dots$ and $x+(k-1)x_0$ respectively ($x_0 > 0$). As $k \rightarrow \infty$, the limiting distribution of $\sqrt{k} \left(\frac{1}{k} \sum_{r=1}^k n(x_r) - \bar{n} \right)$ ($x_r = x + (r-1)x_0$) is asymptotically normal with zero mean and variance equal to

$$\frac{\sum_{r=1}^{\infty} \frac{1}{r} \int_{x_r/x_0}^{\infty} -\ln(x_0) dP(\cdot)}{r = - \left[\frac{x}{x_0} \right] \ln(x_0)}$$

where $[\quad]$ stands for the integral part of the number within it.

Further, as $X \rightarrow \infty$ the limiting distribution of

$$\sqrt{2X} \left(\frac{1}{2X} \int_{-X}^X n(x) dx - \bar{n} \right) \text{ is normal with zero mean and variance } (= E) .$$

Proof: From the form of the characteristic function of $n(x)$ and $n(y)$, as given in Theorem 5.31, it is clear that $n(x)$ and $n(y)$ are independently distributed for if $|x-y| > \dots$ Putting

$$x_r = r(x_0) \text{ and } k_0 = \left[\frac{x}{x_0} \right] \text{ we find that}$$

the sequence $X_1, X_2, \dots$ is k_0 -dependent stationary, and

$$\text{Cov}(X_1, X_{1+l}) = \beta_n(r_{k_0}) = \int_{r_{k_0}}^{l_{\max}} (l - r_{k_0}) dF(l)$$

where $\beta_n(\xi)$ is the covariance function of $n(x)$.

Therefore, we have in view of the theorem 4.52, that the limiting distribution of

$$\sqrt{k} \left(\frac{1}{k} \sum_{r=1}^k X_r - \overline{v} \right) \text{ as } k \rightarrow \infty \text{ is normal with zero mean}$$

$$\text{and variance} = \text{Var } X_1 + 2 \sum_{r=1}^{k_0} \text{Cov}(X_1, X_{1+r})$$

$$= \sum_{r=-k_0}^{k_0} \int_{|r|_{k_0}}^{l_{\max}} (l - |r|_{k_0}) dF(l).$$

For the other part, we put

$$Y_r = \frac{1}{-X_r + (r-1)l_{\max}} \int_{-X_r + (r-1)l_{\max}}^{-X_r + rl_{\max}} n(x) dx \quad (r = 1, 2, \dots, k)$$

$$\text{where } k = \left\lceil \frac{2X}{l_{\max}} \right\rceil$$

$$\text{and } X = \int_{-X_{k_0}}^X n(x) dx$$

Then, the sequence (as $k \rightarrow \infty$) $Y_1, Y_2, \dots$ is stationary and 1-dependent. Further, it is easy to verify that

$\frac{1}{\sqrt{2k}} \left(Z - \sqrt{l} \left(1 - \frac{k l_{\max}}{2k} \right) \right)$ Converges in probability to zero as $k \rightarrow \infty$. Therefore, from theorem 4.52 it follows that as $k \rightarrow \infty$ the limiting distribution of

$$\frac{1}{\sqrt{k}} \left[\frac{1}{k} \sum_{j=1}^k Y_j - \sqrt{l} l_{\max} \right] \text{ is normal with zero mean and}$$

and variances equal to $\text{Var}(Y_1) + 2 \text{Cov}(Y_1, Y_2)$.

$$= k l_{\max}$$

$$\text{But } \text{Var}(Y_1) = \text{Var} \left(\int_{-X}^X n(x) dx \right)$$

$$= \text{Var} \left(\int_0^{l_{\max}} n(x) dx \right) \text{ (because of the stationarity of } n(x) \text{)}$$

$$= \int_0^{l_{\max}} \int_0^{l_{\max}} \rho_n(y-x) dx dy$$

$$= \int_{-l_{\max}}^{l_{\max}} [l_{\max} - |z|] \rho_n(z) dz$$

$$\text{Cov}(Y_1, Y_2) = \text{Cov} \left[\int_{-X}^X n(x) dx, \int_{l_{\max}-X}^{2l_{\max}-X} n(y) dy \right]$$

$$= \text{Cov} \left[\int_0^{l_{\max}} n(x) dx, \int_{l_{\max}}^{2l_{\max}} n(y) dy \right]$$

$$= \int_0^{l_{\max}} dx \int_{l_{\max}}^{2l_{\max}} \rho_n(y-x) dy = \int_0^{l_{\max}} z \rho_n(z) dz$$

$$\text{Therefore } \text{Var}(Y_1) + 2 \text{Cov}(Y_1, Y_2)$$

$$= 2 l_{\max} \int_0^{l_{\max}} z \rho_n(z) dz$$

$$= 2 \sqrt{l_{\max}} \int_0^{l_{\max}} z \rho_n(z) dz = 2 \sqrt{l_{\max}} l^2$$

Now

$$\frac{1}{\sqrt{2K}} \left[\frac{1}{2K} \int_{-X}^X n(x) dx - \sqrt{L} \right]$$

$$= \frac{\sqrt{k}}{\sqrt{2K}} \cdot \sqrt{k} \left(\frac{X_1 + \dots + X_k}{k} - \sqrt{L_{\max}} \sqrt{L} \right) + \frac{1 - \left(1 - \frac{k L_{\max}}{2K}\right) \sqrt{L}}{\sqrt{2K}}$$

From the preceding considerations and the fact that

$$\frac{\sqrt{k}}{\sqrt{2K}} \rightarrow \frac{1}{\sqrt{L_{\max}}} \quad \text{as } K \rightarrow \infty$$

it follows that the limiting distributions (as $K \rightarrow \infty$) of the first term of the right hand side of the foregoing equation is normal with zero mean and variance

The second term converges in probability to zero. This completes the proof of the assertion.

Remarks:- Even when the sections $x_1, x_2, \dots$, are not equi-spaced, the limiting distribution of $\frac{1}{k} \sum_{r=1}^k n(x_r) - \sqrt{L}$ is normal provided $x_{i+1} - x_i \geq \delta$ for all i and some $\delta > 0$.

This can be proved by using theorem 4.51. Further, it is clear from theorem 5.33 that in order to estimate $\int n(x) dx$ it is preferable to choose sections which are further apart than

It can be verified in a similar manner that $n(x)$ and $n(y)$ are independently distributed if $|x - y| > L_{\max}$. Thus, all the above results have analogous extension to $n(x)$ also.

5.4. Poisson Model for a Non-Uniform Sliver:

Foster, Gregory and Wommsely (10) studied the properties of slivers in which the number of right end of fibres in $(x, x+dx)$ is $\lambda(x)dx$. In this section we shall consider only briefly the random analogue of the Foster et al model obtained by regarding the distribution of fibre right ends as a generalized Poisson law. The formal aspects of the generalized Poisson law on the real line have been studied in detail by [14] Copeland and Francis [24] and Kac and Bartwits [32].

Definition: 'Non-uniform Sliver of the Poisson Type'

We say that a sliver is 'non-uniform sliver of the Poisson type' if in addition to (i) and (ii) of (5.3a), there exists an measurable function $k(x)$ such that the number $N(E)$ of fibre-right ends in a bounded Borel sub-set E of the real line is a Poisson variate with mean $\int_E k(x)dx$ and if $N(E)$ is distributed independently of $N(E_1)$ where E_1 is a bounded Borel set disjoint with E .

Here also we adopt the same notation and definitions as given in the preceding sections. Thus if we denote K_X as the integral $\int_{-X}^X k(x)dx$, then it is clear that $N_1, N_2, \dots, N_{N_X}$ are, for a fixed N_X , independently distributed in $(-X, X)$, each with the probability density $\frac{k(x)}{K_X}$.

Theorem 5.4. For a non-uniform sliver of the Poisson type, we have,

$$(i) \quad E n(x) = \frac{1}{a} \int_0^{l_{\max}} [K(x+l) - K(x)] dP(l)$$

$$(ii) \quad \text{Cov}(n(x), n(y)) = \frac{1}{a^2} \int_{y-x}^{l_{\max}} [K(x+l) - K(y)] dP(l)$$

$$\text{where } K(x) = \int_0^x h(x) dx \quad (x \leq y)$$

Proof:

It is clear that N_X is a Poisson variate with mean K_X and that

$$E N_j(x) = \frac{1}{K_X} \int_0^{l_{\max}} [K(x+l) - K(x)] dP(l)$$

and

$$E N_j(x) N_j(y) = \frac{1}{K_X} \int_{y-x}^{l_{\max}} [K(x+l) - K(y)] dP(l) \quad \text{for } x \leq y$$

These facts lead to the desired results.

Corollary: For a non-uniform sliver of the Poisson type, $n(x)$ and $n(x)$ are not, in general, stationary. For such a sliver $n(x)$ and $n(x)$ will be stationary if and only if the sliver under consideration is ideal.

If all the fibres have the same length, then it follows from Kac and Karvitz's analysis (32) that the distributions of the

Fourier coefficients of $n_X(x)$ and $m_X(x)$ are asymptotically normal as $K_X \rightarrow \infty$.

3.3. The Hannah-Rodden Model:

Hannah and Rodden (13) have considered a yarn model in which, the placing of fibres is more restricted than the arrangement in an ideal sliver. Brzdy [2] subsequently pointed out that their definition was incomplete and gave an alternative formulation of the model. He showed that even the calculation of the serial co-efficient for $n(x)$ was incorrect; but he has not given the correct expression in all cases. In this section, we shall adopt Brzdy's formulation of Hannah-Rodden model and derive from it, the second order properties of $n(x)$ and $m(x)$ for such a sliver.

Definition: 'Hannah-Rodden Type of Sliver'.

Following Brzdy we say that a sliver is of the Hannah-Rodden type if, in addition to the assumption (1) of (4.1.1) the following also hold good.

Let h and p be any two positive numbers and α a random variable uniformly distributed in $(0, p)$. For every value of α , consider the partitioning of the real line into the intervals $I_j = (\alpha + jp - h, \alpha + jp)$ [$j = 0, \pm 1, \pm 2, \pm 3 \dots$]. We assume that the sliver is so formed that each of the intervals I_j contains

exactly one right end. If x_j is the axial-co-ordinate of the right end in I_j , then x_j for each j and x , is uniformly distributed in $(a + jp - h, a + jp)$ independently of the rest of the x 's. We further assume that the lengths $l_1, l_2, \dots$ of the fibres are distributed independently of the variables x, x_j ($j = 0, \pm 1, \pm 2, \dots$).

Let l_j be the length of the j th fibre (by the j th fibre we mean that fibre whose right end lies in I_j). Defining $w_j(x)$ as in (5.36) and $n(x)$ as $n(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=-N}^N w_j(x)$, we may sum up Bravy's modified results for the present case as follows.

Theorem 5.5.

$$E n(x) = \frac{1}{hp} \int_0^p dx \int_0^{l_{\max}} dP(l) \sum_{j=-\infty}^{\infty} \{F(x-a-jp+h) - F(x-a-jp)\}$$

and $\text{Cov. } [n(x), n(y)]$

$$= \frac{1}{hp} \int_0^p dx \int_{y-x}^{l_{\max}} dP(l) \sum_{j=-\infty}^{\infty} \{F(x+l+h-a-jp) - F(y+h-a-jp)\} - \sum_{j=-\infty}^{\infty} E w_j(x) E w_j(y) \quad \text{for } x \leq y,$$

where $F(x) = 0$

for $x \leq 0$

$= \frac{x}{h}$

for $0 \leq x \leq h$

$= 1$

for $x \geq h$

$$E w_j(x) = \frac{1}{hp} \int_0^p dx \int_0^{l_{\max}} dP(l) \sum_{j=-\infty}^{\infty} \{F(x+h-l-a-jp) - F(x+h-a-jp)\}$$

Proof:

By direct computation.

This result shows that for a Hammah-Rodden type of sliver, $n(x)$ is, in general, non-stationary. Consequently the spectral analysis and the concept of determinism do not have any relevance in this case.

5.6. Quasi-Ideal Slivers:

In this section we shall consider briefly another sliver model which is closely related to the 'ideal sliver'. The author et al (21) have studied this model in connection with the study of the extent of fibre disorder in slivers. [c.f. Lindsay (14) also]. Hence we do not assume that all the fibres in the sliver are straight and parallel, but having a shape as shown below.

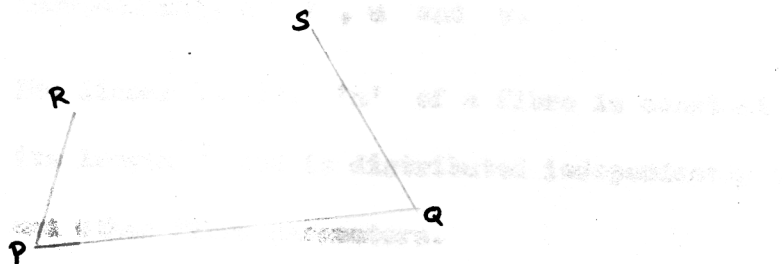


Fig. 5

To call the portions PR , QS and PQ respectively by 'the trailing hook', 'leading hook' and 'non-hooked portion'. The hooked portions, themselves, are straight and all the portions of the fibre may be randomly oriented with respect to the sliver axis.

Definition of a Quasi-Ideal Sliver:

A quasi-ideal sliver is one for which the following assumptions hold good:

- (i) If, l , l_u , and l_v respectively stand for the length, the length of the trailing hook and the length of the leading hook of a fibre, then u and v vary from fibre to fibre independently of l .
- (ii) Similarly, the inclinations θ_0 , θ_1 and θ_2 respectively of the non-hooked portion, trailing hook and the leading hook with the sliver axis are distributed independently of l , u and v .
- (iii) The linear density ' μ ' of a fibre is constant along its length l and is distributed independently of l and other fibre parameters.
- (iv) All the fibre parameters like l , μ , v ... etc., are distributed independently of the position of the fibre along the sliver axis.

- (v) The right ends of the non-hooked portions are distributed along the sliver axis according to the Poisson law, with an average density of ν ends per unit length.

Consider, now, the interval $(-X, X)$ and let N_X be the number of the right ends of the non-hooked portions of the fibres in $(-X, X)$.

Let $l_j, u_j, v_j, a_j, \theta_{0j}, \theta_{1j}$ and θ_{2j} be the parameters and x_j the axial co-ordinate of the right end of the non-hooked portion of the j th fibre ($j = 1, 2, \dots, N_X$) and the numbering is arbitrary). x_j 's are, then, independently and uniformly distributed in $(-X, X)$. Further, defining for any x in $(-X, X)$,

$$v_{0j}(x) = 1 \text{ if } x_j - l_j(1-u_j-v_j)\cos\theta_{0j} \leq x \leq x_j \\ = 0 \text{ otherwise.}$$

$$v_{1j}(x) = 1 \text{ if } x_j - l_j(1-u_j-v_j)\cos\theta_{0j} \leq x \leq$$

$$x_j - l_j(1-u_j-v_j)\cos\theta_{0j} + l_j u_j \cos\theta_{1j}$$

$$= 0 \text{ otherwise.}$$

$$v_{2j}(x) = 1 \text{ if } x_j - l_j v_j \cos\theta_{2j} \leq x \leq x_j$$

$$= 0 \text{ otherwise.}$$

$$w_j(x) = \frac{w_{0j}(x)}{\cos \theta_{0j}} + \frac{w_{1j}(x)}{\cos \theta_{1j}} + \frac{w_{2j}(x)}{\cos \theta_{2j}}$$

and

$$n_X(x) = \sum_{j=1}^{N_X} a_j w_j(x)$$

We find that the linear density $n(x)$ of the sliver in the cross-section at x , may be represented as:

$$n(x) = \lim_{I \rightarrow \infty} \frac{1}{I} n_X(x).$$

Theorem 5.6. For a quasi-ideal sliver, $n(x)$ has the following properties:

- (i) $n(x)$ is stationary and completely non-deterministic
- (ii) the spectrum of $n(x)$ is absolutely continuous and its spectral density $g_n(v)$ is given by

$$g_n(v) = \frac{\pi \bar{a}^2}{2\pi v^2} \sum_{u,v,\theta_0,\theta_1,\theta_2} \int_0^{l_{\max}} dp(l) |A(l, u, v, \theta_0, \theta_1, \theta_2; v)|^2$$

where

$$\begin{aligned} A(l, u, v, \theta_0, \theta_1, \theta_2; v) = & \frac{1 - \exp \{ i v l (1 - u - v) \cos \theta_0 \}}{\cos \theta_0} \\ & + \frac{[\exp(-i v l u \cos \theta_1) - 1] \exp \{ i v l (1 - u - v) \cos \theta_0 \}}{\cos \theta_1} \\ & + \frac{1 - \exp \{ i v l v \cos \theta_2 \}}{\cos \theta_2} \end{aligned}$$

and $E_{u,v,\theta_0,\theta_1,\theta_2}$ stands for the operation of taking expectations with respect to u, v, θ_0, θ_1 , and θ_2 .

Proof: Since we are interested in calculating $E n(x)$ and $\text{cov. } [n(x), n(x + \xi)]$, we fix x and $\xi (\geq 0)$ and choose X so large that the interval $(-X, X)$ contains $(x - l_{\max}, x + \xi + l_{\max})$. Then it is easy to verify that

$$E [v_{\theta_j}(x) | \theta_{\theta_j}] = \frac{\bar{l} \cos \theta_{\theta_j} (1 - u - v)}{2X} \text{ etc., so that}$$

$$E v_j(x) = \bar{l}/2X$$

$$\therefore E n(x) = v \bar{l}$$

Since $v_j(x)$ and $v_k(x)$ are independently distributed for $j \neq k$ and $E v_{k,j}(x) v_{k',j}(x + \xi)$ ($k, k' = 0, 1, 2$) function of alone, show that $n(x)$ is stationary. If $\beta_n(\cdot)$ is the autocovariance function of $n(x)$, then it follows from the fact that $E v_j(x) v_j(x + \xi) = 0$ for $|\xi| \geq l_{\max}$ ^{that $q_m(\xi) = 0$, for $|\xi| \geq l_{\max}$} . Consequently, the spectrum of $n(x)$ is absolutely continuous. The proof of the theorem 5.32 leads to the completely non-deterministic nature of $n(x)$, in this case also.

The direct calculation of $\beta_n(\xi)$ is complicated and so we first calculate $g_n(v)$ from the formula

$$g_n(w) = \lim_{X \rightarrow \infty} \frac{1}{2X} \mathbb{E} \left| \frac{1}{\sqrt{2X}} \int_{-X}^X a_n(x) e^{-iwx} dx \right|^2$$

$\beta_n(\zeta)$ can then be obtained from $g_n(w)$.

We have

$$\int_{-X}^X a_j(x) e^{-iwx} dx = \frac{\Lambda(\ell_{j_1, j_2, \dots, j_n}^0, \ell_{j_1, j_2, \dots, j_n}^1, \dots, \ell_{j_1, j_2, \dots, j_n}^{n-1}, w)}{-iw} \exp. (-iwx_{j_1})$$

$$\left| \frac{1}{\sqrt{2X}} \int_{-X}^X a_n(x) e^{-iwx} dx \right|^2 = \frac{\Lambda_j(w)}{-iw} \exp. (-iwx_{j_1}) \quad \text{say}$$

$$= \frac{1}{2w^2} \sum_{j=1}^n a_j^2 |A_j(w)|^2 + \sum_{j \neq k} a_j a_k \exp. -iw(x_{j_1} - x_{k_1}) A_j(w) A_k^*(w)$$

$$\text{But } \mathbb{E} \exp. \{-iw(x_{j_1} - x_{k_1})\} = \frac{\sin^2 wx}{w^2 x^2}.$$

From these facts the desired result follows after some further simplification.

CHAPTER VI

A STOCHASTIC MODEL FOR DRAFTING

6.1. An outline of the chapter

In this chapter an attempt has been made to describe how the fibre-arrangement in an ideal sliver is ^a affected by drafting. More precisely, it has been shown that if $n_0(t)$ be the number of fibres held by the back rollers t and $n_1(t)$ the number held by the front rollers at time t then the stationarity of $n_1(t)$ and its other properties can be derived from the following assumptions.

- 1) The input sliver is ideal and is composed of fibres of the same length L .
- 2) Each fibre after leaving the back rollers continues to move the speed (v_1) of the back rollers until its speed abruptly changes to that (v_2) of the back rollers and moves there after with the latter speed.
- 3) The transitions times τ_1 and τ_2 of two fibres one of which leaves the back rollers at time t_1 and the other at t_2 have an absolutely continuous distribution and their joint density is given by

$$\rho(t_1, t_2) / t_1 t_2$$

$$= f(\tau_1) f(\tau_2) \left\{ 1 + \frac{\rho(t_1 - t_2)}{\sigma^2} (\tau_1 - \bar{\tau})(\tau_2 - \bar{\tau}) \right\}$$

where $f(\tau_1)$ is the density of τ_1 , and is independent of t_1 , and

$$\rho(t_1 - t_2) = 0 \text{ if } |t_1 - t_2| > \frac{h}{v_1} - \frac{L(h-L)}{v_1^2}$$

$$(4) \quad f(\tau_1) = \frac{\lambda v_1 \cosh \lambda v_1 \tau_1}{\sinh \lambda(h-L)}, 0 \leq \tau_1 \leq \frac{h-L}{v_1}$$

with $\bar{\tau} = E \tau$ and $\sigma^2 = \text{var}(\tau)$.

A rationale for these assumptions has also been given.

The concept of ideal drafting has been introduced in 6.5. It has been shown that drafting waves are in general a consequence of the correlations, $\rho(|t_1 - t_2|)$, between transition times and that the Cavanagh-Foster's formula holds when the drafting 'asymptotically ideal' and $\rho(|t_1 - t_2|)$ is linear in $|t_1 - t_2|$.

In 6.8 these considerations have been extended to the case when all the fibres in the input are not of the same length. In this case it turns out that correlation coefficient between the transition times of two fibres depends not only on their times, t_1 and t_2 , of entry into the drafting zone but also on their length l_1 and l_2 . This should depend on t_1 and t_2 only through

$t_1 - t_2$ is order that $n_1(t)$ may be stationary. It has been shown that this condition is satisfied by taking the above correlation coefficient to be a function of the maximum time for which the fibres are simultaneously in the drafting zone. Lastly, a central-limit property for $[n_0(t), n_1(t)]$ is established.

6.2. Introduction:

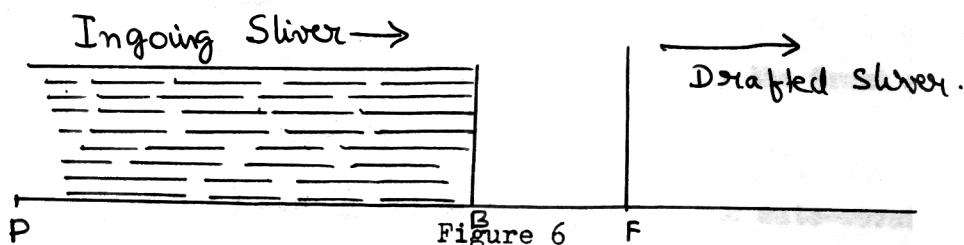
In the last chapter, we have studied some stochastic models for slivers of which the ideal sliver is the simplest structure from the mathematical standpoint. In this chapter we shall consider, the effect of roller drafting on fibre arrangement of the input sliver. To do this, we naturally have to make some assumptions on the motion of floating fibres (that is, fibres in the drafting zone which are not gripped either by the front or back rollers.) As these assumptions are statements about distributions of certain families of random variables, the theory put forward in this chapter is called 'A Stochastic Model For Drafting'. Further we shall be concerned in the present chapter exclusively with the drafting of 'ideal slivers' although these considerations can be generalized to the other types of slivers as well.

Part I.

DRAFTING OF IDEAL SLIVERS WHOSE CONSTITUENT
FIBRES ARE ALL OF THE SAME LENGTH 'L'

6.3. Notations:

Let B be the nip of the back rollers and F that of the front rollers. Consider a portion BP (to the left of the back rollers, with P arbitrarily chosen) of the sliver that is about to enter the drafting zone BF . (See Figure 6).



Let $BF = h$ ($> L$, the fibre length) and $BP = X$. 'h' is known as the 'roller setting'. Choose B as the origin of measurement of the distances along the sliver axis and regard the direction FB as positive. Time will be measured from the instant when the input sliver is about to enter the drafting zone BF .

Let N_X be the number of fibre right ends in BP and $x_1, x_2, \dots, x_{N_X}$ (numbered arbitrarily) the axial co-ordinates of these N_X right ends.

Further, let

v_1 be the surface speed of the back rollers

v_2 be the surface speed of the front rollers

$$D = \frac{v_2}{v_1}$$

$t_j = \frac{L - x_1}{v_1}$ be the time that the j th fibre is released from the back rollers.

$$T = \frac{L}{v_1}$$

$n_0(t)$: the number of fibres held by the back rollers at time t

$n_1(t)$: the number of fibres held by the front rollers at time t

$\beta_0(\xi)$ and $g_0(w)$ be respectively the auto-covariance function and the spectral density of $n_0(t)$

$\beta_1(\xi)$ and $g_1(w)$ respectively be the auto-covariance function and the spectral density of $n_1(t)$.

6.4. Some general considerations:

Since the input sliver is ideal, it follows that N_X is a Poisson Variate with mean $\lambda v_1 T$ and $t_1, t_2, \dots, t_{N_X}$ for a fixed N_X are independently and uniformly distributed in

$$\left(\frac{L}{v_1}, T + \frac{L}{v_1}\right).$$

Further

$$n_0(t) = 1.1.m \sum_{j=1}^{N_X} u_{0j}(t) \quad (6.41)$$

where $u_{0j}(t) = 1$ if $t_j - \frac{L}{v_1} \leq t \leq t_j$
 $= 0$ otherwise.

The process $n_0(t)$ [$0 \leq t < \infty$] is merely another way of representing the process $n(x)$ [x being measured from B along the sliver axis]. The number of fibres in the cross-section at x of the input, $n_0(t)$ and $n(x)$ are related by

$$n(x) = n_0\left(\frac{-x}{v_1}\right)$$

$n_1(t)$ is similarly related to the number of fibres in a cross-section of the drafted sliver. Then, the problem of drafting theory is to predict the properties of $n_1(t)$. Since we have assumed the input to be ideal, it follows from Theorem 1 that $n_0(t)$ is stationary.

$$\text{Also } \rho_0(\xi) = \begin{cases} \nu(L - v_1 |\xi|) & \text{if } |\xi| \leq \frac{L}{v_1} \\ 0 & \text{otherwise} \end{cases}$$

$$g_0(v) = \frac{\nu v_1}{\pi v^2} \left(1 - \cos \frac{vL}{v_1}\right).$$

In order to compare these properties of $n_0(t)$ with those of $n_1(t)$, the theory must in the first place ensure that $n_1(t)$ will indeed be stationary.

Following Foster (6) we shall assume that a fibre after leaving the back rollers continues to move with the speed v_1 of the back rollers until it suddenly attains the speed v_2 of the front rollers and there after continues to move with the latter

speed. This assumption is basic to what is called a 'two velocity model' of drafting and we shall adhere to it throughout the present chapter.

The time at which a fibre changes its speed from v_1 to v_2 varies from fibre to fibre and is a random variable, since the arrangement of fibres in the input is random. Suppose now a fibre leaves the back rollers at time t and after leaving travels for a time $\tau(t)$ ^{which will be} [also called the transition time] with speed v_1 before it attains the speed v_2 . Then $[\tau(t); 0 \leq t < \infty]$ is a stochastic process. From the physical considerations we anticipate that unless $\tau(t)$ is stationary (strict sense) $n_1(t)$ will not, in general, be stationary. As we are mainly interested in the second order properties of $n_1(t)$, we merely stipulate that the distribution of $\tau(t)$ shall be independent of t and that the distribution of $\tau(t)$ and $\tau(t')$ shall depend on t and t' only through $|t - t'|$. Before proceeding to make further assumptions on the joint distribution of $\tau(t)$ and $\tau(t')$ we shall first obtain an expression for the spectral density of $n_1(t)$ [assuming that $n_1(t)$ has an absolutely continuous spectrum] and study what we shall call 'Ideal Drafting'.

6.5. Ideal Drafting and the spectral density of $n_1(t)$

Let $x_j(t)$ be the distance at time t of the front end of the j th fibre from B . Denote $\tau(t_j)$ [$j = 1, 2, \dots, N_X$] by τ_j . Defining

$$u_{1j}(t) = \begin{cases} 1 & \text{if } h \leq x_j(t) \leq h + L \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } n_1(T, t) = \sum_{j=1}^{N_X} u_{1j}(t)$$

we find that

$$n_1(t) = \lim_{T \rightarrow \infty} n_1(T, t).$$

It is clear that the j th fibre continues to move in the direction BF with the speed v_1 till a time $(t_j + \tau_j)$ has elapsed when its speed abruptly changes from v_1 to v_2 . Thereafter, it moves with the speed v_2 and does not retard or accelerate in its motion. Hence $x_j(t) = x_j + v_1 t$ if $t \leq t_j + \tau_j$.

$$= x_j + v_1(t_j + \tau_j) + v_2(t - t_j - \tau_j) \\ \text{if } t \geq t_j + \tau_j$$

Now $u_{1j}(t) = 1$ holds only if $t \geq t_j + \tau_j$ since all fibres attain the speed v_2 before reaching the front rollers. Therefore the event $u_{1j}(t) = 1$ is equivalent to the event $h \leq x_j(t) \leq h + L$

and $t \geq t_j + \tau_j$

that is, equivalent to

$$\frac{h-1}{v_1} \leq D(t - t_j) - (D-1)\tau_j \leq \frac{h}{v_1} \quad (6.51)$$

Use to

$$t_j + \frac{1}{D} [(D-1)\tau_j + \frac{h-1}{v_1}] \leq t \leq t_j + \frac{1}{D} [(D-1)\tau_j + \frac{h}{v_1}]$$

Hence,

$$\int_0^{T + \frac{1}{v_1}} u_{1j}(t) e^{-i\omega t} dt$$

$$= \frac{[1 - \exp(-\frac{i\omega h}{v_1 D})]}{i\omega} \exp\left[-\frac{i\omega}{D} \left\{ D t_j + (D-1)\tau_j + \frac{h-1}{v_1} \right\}\right]$$

Now we know that $S_1(\omega)$, the spectral density of $u_1(t)$, can be computed from the formula

$$S_1(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} E \left| \frac{1}{\sqrt{2\pi}} \int_0^T u_1(\tau, t) e^{-i\omega t} dt \right|^2$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} E \left| \frac{1}{\sqrt{2\pi}} \int_0^{T + \frac{1}{v_1}} u_1(\tau, t) e^{-i\omega t} dt \right|^2$$

$$\therefore S_1(\omega) = \lim_{T \rightarrow \infty} \frac{(1 - \cos \frac{\omega h}{v_1 D})}{\pi \omega^2 T} E \left[N_T + \sum_{j/k} \exp \frac{-i\omega}{D} \right.$$

$$\left. \left\{ D(t_j - t_k) + (D-1)(\tau_j - \tau_k) \right\} \right]$$

$$= \frac{(1 - \cos \frac{\omega h}{v_1 D})}{\pi \omega^2} \left\{ v v_1 + v^2 v_1^2 \lim_{T \rightarrow \infty} \frac{1}{T} E_T(\omega) \right\} \quad (6.52)$$

where

$$F_T(w) = \operatorname{Re} E \exp. \frac{-i\omega}{D} [D(t_j - t_k) + (D-1)(\tau_j - \tau_k)]$$

and Re stands for 'real part of' stated-in provided $n_1(T, t)$ satisfies the conditions stated in Theorem 4.2 of chapter iv.

Consequently, we have to make sure of this every time we use (6.52) to calculate $g_1(w)$.

We shall now define 'ideal drafting' and use (6.52) to obtain a criterion for ideal drafting. Suppose, now, that each fibre attains the speed v_2 only after reaching the front rollers. In this case

$$\tau_j = \frac{h-l}{v_1} \quad \text{with probability one for each } j$$

$$\therefore F_T(w) = \operatorname{Re} E \exp. i\omega(t_j - t_k)$$

$$= \frac{|e^{i\omega T} - 1|^2}{T^2}$$

$$\text{Let } T \rightarrow \infty \quad F_T(w) = 0 \quad \text{and} \quad g_1(w) = \frac{v_1}{E w^2} (1 - \cos \frac{w l}{v_1 D})$$

Comparison of $g_1(w)$ with $g_0(w)$ shows that the form of the spectral density is preserved but the parameter L is changed to $\frac{l}{D}$.

Further, $n_1(t)$ has an absolutely continuous spectrum and $E n_1(t) = \frac{v l}{D}$.

Consequently when drafting ensures that $\tau_j = \frac{h-l}{v_1}$ with probability one, the mean and the variance in the number of fibres in a cross-section of the output are deflated by the factor $\frac{l}{D}$ and the form of the auto covariance function of $n_0(t)$ is preserved during drafting. From these considerations we obtain the following:

Definition 1. We say that drafting is 'ideal' when

$$(i) \quad E n_1(t) = \frac{v_1 L}{D}$$

and (ii) $n_1(t)$ has an absolutely continuous spectrum and its spectral density $g_1(\omega)$ is given by

$$g_1(\omega) = \frac{v_1^2}{\omega^2} \left(1 - \cos \frac{\omega L}{v_1 D} \right).$$

It can be demonstrated (vide) that the relation $E n_1(t) = \frac{v_1 L}{D}$ holds good if the distribution of $\gamma(t)$ is independent of t . [cf. Theorem 6.41]. Consequently the condition for ideal drafting is that $n_1(t)$ must have an absolutely continuous spectrum and

$$\lim_{T \rightarrow \infty} T P_T(\omega) = 0 \quad (6.55)$$

It is, of course, implicit in this criterion that before applying it, we must ^{make} assure that drafting in the first place is 'stationary'. That is, it is such that $n_1(t)$ will be stationary.

$$\gamma_j = \frac{h-k}{v_1} \quad \text{for all } j \text{ with unit probability}$$

is not the only situation in which drafting is ideal. Consider, for example, the situation in which each fibre moves in the drafting zone independently of the rest of the fibres so that γ_j are now independently and identically distributed random variables. Further γ_j is distributed independently of t_j .

$$\begin{aligned}
 \therefore F_T(w) &= E \exp \left[\frac{-i w}{D} \left\{ D(t_j - t_k) + (D-1)(\eta_j - \eta_k) \right\} \right] \\
 &= \left| E e^{-i w t_j} \right|^2 \left| E \exp \frac{-i w (D-1) \eta_j}{D} \right|^2 \\
 &= \frac{\left| \int_0^T e^{-i w t} dt \right|^2}{T^2} \left| E \exp \frac{-i w (D-1) \eta_j}{D} \right|^2
 \end{aligned}$$

Since the distribution of η_j does not depend on T , it follows from the foregoing expression for $F_T(w)$ that $\lim_{T \rightarrow \infty} T F_T(w) = 0$.

In this case, it is easily verified that the spectrum of $n_1(t)$ is absolutely continuous. Thus, this case also corresponds to ideal drafting.

In what follows, we shall denote by $g_{1d}(w)$ the expression for $g_1(w)$ in the case of ideal drafting

$$\text{That is, } g_{1d}(w) = \frac{v_1}{\pi w^2} \left(1 - \cos \frac{w L}{v_1 D} \right) \quad (6.54)$$

From the foregoing considerations it is clear that when drafting is ideal $\beta_1(\xi)$, the auto covariance of $n_1(t)$ has the same form as that of $n_0(t)$ showing that drafting waves are not produced. Further, the increase in irregularity due to ideal drafting does not explain Cavaney and Foster's (5) observations. In other words, we should look for a suitable hypothesis on the distributions of η_j which would lead to a satisfactory explanation of the drafting waves and Cavaney and Foster's observations.

6.6. Certain assumptions concerning the distributions of transition times and their consequences.

We have seen that so long as each fibre attains the higher speed v_2 independently of the rest of the fibres, we cannot obtain a satisfactory picture of drafting which explains the formation of drafting waves. Thus, the problem is to formulate the inter-dependence of τ 's in a suitable way. To do this, we first assume the distribution of $\tau(t)$ is independent of t and is absolutely continuous. If $f(\tau)$ is the probability density of $\tau(t)$, then it is clear that $f(\tau)$ vanishes outside the interval $0 \leq \tau \leq \frac{h-l}{v_1}$.

Consider now two fibres one of which leaves the back rollers at time t and the other at time s . If t and s are such that if one of the fibres will surely leave the front rollers before the other is released by the back rollers, i.e., when

$|t - s| \geq \frac{h-l}{v_1} + \frac{l}{v_1 D}$, then these two cannot influence each other.

That is, if

$|t - s| \geq \frac{h-l}{v_1} + \frac{l}{v_1 D}$, then $\tau(t)$ and $\tau(s)$ will be independently distributed. If $h(\tau_1, \tau_2; t, s)$ is the joint density of $\tau(t)$ and $\tau(s)$, then $h(\tau_1, \tau_2; t, s) = f(\tau_1) f(\tau_2)$ if

$|t - s| \geq \frac{h-l}{v_1} + \frac{l}{v_1 D}$. Under some fairly general conditions.

$h(\tau_1, \tau_2; t, s)$ can be put in the form

$$h(\tau_1, \tau_2; t, s) = f(\tau_1) f(\tau_2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_m(\tau_1) a_{m,n} (it-s) P_n(\tau_2)$$

where $P_m(\tau)$ [$m = 0, 1, 2, \dots$] are orthonormal polynomials with respect to the density function $f(\tau)$. Here we shall consider only the first two terms of the expansion. That is, we shall assume that

$$h(\tau_1, \tau_2; t, s) = f(\tau_1) f(\tau_2) \left\{ 1 + \frac{\varrho(it-s)}{\sigma^2} (\tau_1 - \bar{\tau})(\tau_2 - \bar{\tau}) \right\}$$

where $\varrho(it-s) = 0$ if $|it-s| \geq \frac{h-1}{v_1} + \frac{1}{v_1 D}$ and σ^2 are respectively the mean and the variance of $\tau(t)$.

From the above expression for $h(\tau_1, \tau_2; t, s)$ it follows that $\varrho(it-s)$ is the correlation coefficient of $\tau(t)$ and $\tau(s)$. If we wish to take into consideration the higher degree terms in τ_1 and τ_2 then we must have complete information regarding all the higher order moments of the distribution of $\tau(t)$ and $\tau(s)$. We will presently show that for the purposes already outlined it will be sufficient to consider only the first two terms.

To choose a suitable form for $f(\tau)$, consider the following example. Consider two fibres OA and A_1B_1 (see figure 7), each other. Let us suppose that A_1B_1 moves uniformly in the direction

OA with a constant speed v_2 say and that it exerts on OA a forward force proportional to its overlap with OA.



FIGURE 7

If OA were to move only under the influence of $A_1 B_1$ then it is easy to see that the overlap $f(t)$, at time t will be of the form $A e^{at} + B e^{-at}$. A similar result can be obtained when OA is subjected to the influence of several fibres simultaneously, some of which accelerate the motion of OA and the rest retard its motion without any fibre of either group (i.e., one moving with the speed v_1 or v_2) changing its own course of motion. In such a case, we consider the total 'effective overlap', defined as, the sum total of overlaps with the fast moving fibres minus the sum total of overlaps with the slow moving fibres instead of a single overlap, and find that the total effective overlap is still of the form $A e^{at} + B e^{-at}$.

Accordingly, if assuming that the probability of a fibre attaining the speed v_2 in $(\eta, \eta + d\eta)$ is proportional to the total effective overlap, is in effect equivalent to taking $f(\eta)$ to be of the form $A e^{a\eta} + B e^{-a\eta}$. In what follows, we shall assume $f(\eta) = A \cosh \lambda v_1 \eta$, for the sake of simplicity. Using the fact that $f(\eta)$ is a density function, we find that

$$f(\eta) = \frac{\lambda v_1 \cosh \lambda v_1 \eta}{\sinh \lambda L u} \quad , \quad 0 \leq \eta \leq \frac{L u}{v_1}$$

where $u = \frac{h - l}{L}$.

It is clear from the above expression for $f(\tau)$ that when $\lambda \rightarrow \infty$, $z(t)$ converges in probability to $\frac{L u}{v_1}$; that is, when $\lambda \rightarrow \infty$, drafting will be ideal.

We may now summarise the foregoing considerations into the following two assumptions on the distributions of $z_j = z(t_j)$

$$A_1: f(z_j) = \frac{\lambda v_1 \cosh \lambda v_1 z_j}{\sinh \lambda L u} \quad 0 \leq z_j \leq \frac{L u}{v_1}$$

A_2 : The joint density $h(z_j, z_k/t_j, t_k)$ of z_j and z_k for fixed t_j and t_k is given by

$$h(z_j, z_k/t_j, t_k) = f(z_j)f(z_k) \cdot$$

$$\left[1 + \frac{g(t_j - t_k)}{2} (z_j - \bar{z})(z_k - \bar{z}) \right]$$

where $g(t_j - t_k) = 0$ if $|t_j - t_k| \geq \frac{L u}{v_1} + \frac{L}{v_1 D}$ and $u = \frac{h-L}{L}$.

Consequences of the assumptions A_1 and A_2

Theorem (6.61):

If the input aliver is ideal and composed of fibres of equal length L and if the assumption A_2 holds, then we have:

(i) $n_1(t)$ is stationary, completely non-deterministic and its spectrum is absolutely continuous*

$$(ii) S_1(\omega) = S_{1d}(\omega) \left[1 + 2 \gamma v_1 S_2(\omega) \int_0^{\frac{L}{v_1 D} (D u + 1)} g(x) \cos \omega x dx \right]$$

* In what follows, the range of the parameter t of the process $n_1(t)$ will be restricted only to the interval

$t \geq \frac{h}{v_1} + \frac{L}{v_1 D}$. This implies no severe restriction because,

$$\text{where } G_2(u) = \left| \int_0^{\frac{Lu}{v_1}} \frac{(\eta - \bar{\eta})}{\sigma} f(\eta) \exp. \frac{i u (D-1) \eta}{D} d\eta \right|^2$$

Proof:

$$\text{Since } n_1(t) = \lim_{T \rightarrow \infty} \sum_{j=1}^T u_{1j}(t)$$

$$\text{we have, } E n_1(t) = \lim_{T \rightarrow \infty} E \sum_{j=1}^T P_T$$

$$= \lim_{T \rightarrow \infty} \sum_{j=1}^T P_T$$

$$\text{where } P_T = \frac{1}{T} \int_{\frac{L}{v_1}}^{T + \frac{L}{v_1}} p(t_j) dt_j$$

$$\text{and } p(t_j) = \Pr [u_{1j}(t) = 1/t_j]$$

$$= \Pr \left[\frac{L \cdot N}{v_1} \leq D(t - t_j) - (D-1) \eta_j \leq \frac{L \cdot N}{v_1} \right] \quad [\text{cf. 6.51}]$$

$$= \Pr \left[\frac{D(t_0 - t_1)}{(D-1)} \leq \eta_j \leq \frac{D}{(D-1)} (t_1 - t_j) \right]$$

$$\text{where } t_0 = t - \frac{L}{v_1 D} (1 + u)$$

$$\text{and } t_1 = t - \frac{L \cdot N}{v_1 D}$$

$$\text{Therefore, } P(t_j) = F \left[\frac{D(t_1 - t_j)}{(D-1)} \right] - F \left[\frac{D(t_0 - t_j)}{D-1} \right]$$

where $F(\eta_j)$ is the distribution function of η_j . Since the range of η_j is $(0, \frac{L \cdot N}{v_1})$ it follows that $F(\eta) = 0$ for $\eta \leq 0$

and $F(\eta) = 1$ for $\eta \geq \frac{L \cdot N}{v_1}$.

the process $n_1(t)$ starts only from

$t = \frac{L}{v_1}$ i.e. $n_1(t) = 0$ for $t < \frac{L}{v_1}$. Thus, all that the above

restriction means is that we should neglect an initial portion of length L of the output sliver]

Setting $t_0' = t - \frac{1}{v_1 D} (1 + D\alpha)$

and $t_0'' = t_1 - \frac{1 + D(1-\alpha)}{v_1 D}$ and using the above we get when $t_0'' > t_0$

$$\begin{aligned} E n_1 &= \int_{t_0}^{t_0'} p(t_j) dt_j + \int_{t_0'}^{t_0''} p(t_j) dt_j + \int_{t_0''}^{t_1} p(t_j) dt_j \\ &= \int_{t_0}^{t_0'} \left[1 - F \left(\frac{D(t_0 - t_j)}{(D-1)} \right) \right] dt_j + \int_{t_0'}^{t_0''} dt_j \\ &\quad + \int_{t_0''}^{t_1} F \left[\frac{D(t_1 - t_j)}{(D-1)} \right] dt_j \\ &= (t_0'' - t_0) = \int_0^{t_0 - t_0'} F \left(\frac{Dx}{D-1} \right) dx + \int_0^{t_1 - t_0''} F \left(\frac{Dx}{D-1} \right) dx \end{aligned}$$

But $t_0 - t_0' = t_1 - t_0''$ and $t_0'' - t_0 = t_1 - t_0 = \frac{1}{v_1 D}$

Hence $E n_1(t) = \frac{1}{D}$

(6.61)

From the relation that

$$\begin{aligned} h(r_j, r_k / t_j, t_k) &= f(r_j) f(r_k) + \\ &\quad + \frac{D(t_1 - t_k)}{D^2} f(r_j) f(r_k) (r_j - \bar{r})(r_k - \bar{r}) \end{aligned}$$

it can be proved in a similar way that $E u_j(t) u_k(t)$ [j, k] depends only on $|t - t'|$ and that $E u_j(t) u_k(t)$ is also a function of $|t - t'|$. Thus $n_1(t)$ is stationary.

Since $E u_j(t) u_j(t') = 0$ if $|t - t'| > \frac{1}{v_1}$

$$E u_j(t) u_k(t') = E u_j(t) E u_k(t')$$

$$\text{if } |t - t'| \geq \frac{1}{v_1} + \frac{1}{v_1 D}$$

it follows that the co-variance function of $n_1(T, t)$ vanishes for $|\xi| \geq \text{Max.} \left[\frac{L}{v_1}, \frac{L u}{v_1} + \frac{L}{v_1 D} \right]$ for all T . Therefore, the spectrum of $n_1(t)$ is absolutely continuous and $n_1(t)$ is linearly completely non-deterministic. Since the direct computation of $\beta_1(\xi)$ is complicated and since the conditions stated in Theorem (4.2) are satisfied we shall first compute $S_1(u)$ from (6.5.2) and obtain $\beta_1(\xi)$ from it.

Now from (6.52) we have

$$S_1(u) = S_{1d}(u) \left[1 + \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F_T(u) dt \right]$$

where $F_T(u) = \text{Re. E exp.} \frac{-i u}{D} [D(t_j - t_k) + (D-1)(r_j - r_k)]$

$$\therefore F_T(u) = \text{Re.} \frac{1}{T^2} \int_{\frac{L}{v_1}}^{T + \frac{L}{v_1}} \int_{\frac{L}{v_1}}^{T + \frac{L}{v_1}} F_1(t_j, t_k, u) e^{i u(t_j - t_k)} dt_j dt_k$$

where

$$\begin{aligned} F_1(t_j, t_k, u) &= \int_0^{\frac{L u}{v_1}} \int_0^{\frac{L u}{v_1}} h(r_j, r_k / t_j, t_k) \exp \left[\frac{-i u(D-1)(r_j - r_k)}{D} \right] dr_j dr_k \\ &= \left| \int_0^{\frac{L u}{v_1}} \exp \left[\frac{-i u(D-1)r_j}{D} \right] f(r_j) dr_j \right|^2 \\ &\quad + \frac{g(t_j - t_k)}{D^2} \left| \int_0^{\frac{L u}{v_1}} \exp \left[\frac{-i u(D-1)r_j}{D} \right] (r_j - \bar{r}) f(r_j) dr_j \right|^2 \end{aligned}$$

$$= G_1(w) + \varphi(|t_j - t_k|) G_2(w), \quad \text{say.}$$

$$\begin{aligned} \therefore P_T(w) &= G_1(w) \frac{|1 - e^{-i w T}|^2}{w^2 T^2} \\ &+ G_2(w) \iint_{|t_j - t_k| \leq \frac{L}{v_1} \left(u + \frac{1}{D}\right)} \varphi(|t_j - t_k|) e^{i w(t_j - t_k)} dt_j dt_k \\ &= \frac{G_1(w)}{w^2 T^2} |1 - e^{-i w T}|^2 \\ &+ 2 \frac{G_2(w)}{T^2} \int_0^{\frac{L}{v_1} \left(u + \frac{1}{D}\right)} (T-x) \cos \varphi(x) \cos wx \, dx \\ \therefore S_1(w) &= S_{1d}(w) \left\{ 1 + 2 \frac{1}{v_1} G_2(w) \int_0^{\frac{L}{v_1} \left(u + \frac{1}{D}\right)} \varphi(x) \cos wx \, dx \right\} \end{aligned}$$

This expression shows that the evaluation of $\beta_1(\xi)$ is quite complicated. However, if we use the assumption A_2 also, then it is possible to obtain an asymptotic expression for $\beta_1(\xi)$ which will lead to a simple explanation of drifting waves. This is the content of Theorem 6.63, Theorem 6.62 gives the relationship between $n_0(t)$ and $n_1(t)$.

Theorem 6.62: The processes $n_0(t)$ and $n_1(t)$ are stationary correlated and if

$$\beta_{01}(\xi) = \text{cov.} [n_0(t), n_1[t + \xi]]$$

then

$$\rho_{01}(\xi) = 0 \text{ if } \xi \leq 0$$

$$\begin{aligned} & \xi - \frac{h}{v_1 D} \\ & - v_1 \left[\int_{\xi - \frac{h}{v_1 D}}^{\xi - \frac{h}{v_1 D} + \frac{h}{v_1}} F\left(\frac{D-x}{D-1}\right) dx - \right. \\ & \left. \int_{\xi - \frac{h}{v_1 D} - \frac{h}{v_1}}^{\xi - \frac{h}{v_1 D}} F\left(\frac{D-x}{D-1}\right) dx \right], \xi \geq 0 \\ & \xi - \frac{h}{v_1 D} - \frac{h}{v_1} \end{aligned}$$

where $F(x)$ is the distribution function of η_j .

Proof: From the reasons pointed out in the proof for the stationarity of $n_1(t)$ [cf. Theorem 6.61], we will restrict our considerations to the interval $t \geq \frac{h}{v_1} + \frac{h}{v_1 D}$.

We, then have

$$\begin{aligned} E[u_{0j}(t)] &= \frac{h}{v_1 D} \\ \text{and} \quad E[u_{1j}(t)] &= \frac{h}{v_1 D} \end{aligned}$$

Further, if $j \neq k$, $u_{0j}(t)$ and $u_{1k}(t + \xi)$ are independent.

$$\therefore E[u_{0j}(t) u_{1k}(t + \xi)] = \frac{h^2}{v_1^2 D^2}$$

If $j \neq k$ we have

$$E u_{0j}(t) u_{1j}(t + \tau) = 0 \quad \text{if } \tau \leq 0$$

$$= P \left\{ t \leq t_j \leq t + \frac{h}{v_1} + \frac{D}{D-1} [t + \tau - t_j - \frac{h}{v_1 D}] \leq \tau_j \right.$$

$$\left. \leq \frac{D}{D-1} [t + \tau - t_j - \frac{h}{v_1 D}] \right\} \quad \text{if } \tau \geq 0$$

$$= \frac{1}{\tau} \int_{\tau - \frac{h}{v_1 D} - \frac{1}{v_1}}^{\tau - \frac{h}{v_1 D}} \left\{ F \left[\frac{D(x + \frac{h}{v_1 D})}{D-1} \right] - F \left[\frac{Dx}{D-1} \right] \right\} dx$$

Therefore, $\therefore \text{Cov.} [u_0(t), u_1(t + \tau)] = \beta_{01}(\tau)$

$$= \lim_{T \rightarrow \infty} \text{Cov.} \left[\sum_{j=1}^{N_X} u_{0j}(t), \sum_{j=1}^{N_X} u_{1j}(t + \tau) \right]$$

$$= 0 \quad \text{if } \tau \leq 0$$

$$= \nu v_1 \int_{\tau - \frac{h}{v_1 D} - \frac{1}{v_1}}^{\tau - \frac{h}{v_1 D}} \left\{ F \left[\frac{D(x + \frac{h}{v_1 D})}{D-1} \right] - F \left[\frac{Dx}{D-1} \right] \right\} dx \quad \text{if } \tau \geq 0$$

From Theorems 6.61 and 6.62, it is clear that from $\beta_{01}(\tau)$ and $\beta_1(\tau)$ we can obtain $f(\tau)$ and $g(\tau)$. $f(\tau)$ and $g(\tau)$ together

summarise in a statistical way the movement of the fibres in the drafting zone and thus lead to a clear picture of what happens during drafting. We shall now obtain an asymptotic expression for $\beta_1(\xi)$

Theorem 6.62: If A_1 and A_2 hold, and if drafting is 'asymptotically ideal', that is when λ is large, we have

$$\beta_1(\xi) = \beta_{1d}(\xi) + \frac{v^2(D-1)^2}{\lambda^2 D^2} [2g(\xi) - g(\xi + \frac{L}{v_1 D}) - g(\xi - \frac{L}{v_1 D})]$$

where $\beta_{1d}(\xi) = 2 \int_0^\infty \cos v\xi \quad c_{1d}(v) \, dv$

$$= \left[\frac{L}{D} - v_1 |\xi| \right] \text{ if } |\xi| \leq \frac{L}{v_1 D}$$

$$= 0 \text{ otherwise.}$$

Proof:

Using (6.5) we find that

$$\bar{\eta} = \frac{L M}{v_1} = \frac{\cosh \lambda L M - 1}{\lambda v_1 \sinh \lambda L u}$$

and $\sigma^2 = \frac{1}{\lambda^2 v_1} \left\{ 2 - \frac{(\cosh \lambda L M - 1)^2}{\sinh^2 \lambda L u} - \frac{2 \lambda L M}{\sinh \lambda L u} \right\}$

Hence $\lambda v_1 \bar{\eta} \rightarrow 1$ and $\lambda v_1 (\frac{L M}{v_1} - \bar{\eta}) \rightarrow 1$ as $\lambda \rightarrow \infty$.

We shall now show that

$$\lim_{\lambda \rightarrow \infty} \lambda^2 v_1^2 c_2(v) = \frac{v^2(D-1)^2}{D^2}.$$

$$\text{Let } \bar{f}(s) = \int_0^{\frac{L}{v_1}} \frac{(\gamma - \bar{\gamma})}{s\gamma} f(\gamma) e^{it\gamma} d\gamma$$

$$\therefore G_2(w) = \left| \bar{f} \left[-\frac{w(D-1)}{D} \right] \right|^2$$

$$\text{Since } f(\gamma) = \frac{\lambda v_1 \cosh \lambda v_1 \gamma}{\sinh \lambda L u}$$

we then have

$$\lim_{\lambda \rightarrow \infty} \lambda v_1 \bar{f}(s) = \lim_{\lambda \rightarrow \infty} \frac{(\lambda v_1)^2 \exp(-\lambda L u + \frac{it}{v_1} \frac{L}{u})}{s\gamma (e^{\lambda L u} - e^{-\lambda L u}) (\lambda v_1 + it)}$$

$$= \left\{ \left(\frac{L}{v_1} - \bar{\gamma} \right) - \frac{1}{\lambda v_1 + it} \right\}$$

[omitting the terms which tend to zero exponentially]

$$= it e^{itL/v_1}$$

$$\therefore \lim_{\lambda \rightarrow \infty} \lambda^2 v_1^2 G_2(w) = \frac{v^2(D-1)^2}{D^2}$$

This means that when λ is large, we may replace $G_2(w)$ by its asymptotic value. Substituting this approximation in the expression for $G_1(w)$ obtained in Theorem 6.61.

We get

$$\begin{aligned} G_1(w) &= G_{1d}(w) \left\{ 1 + 2v_1 \frac{v^2(D-1)^2}{\lambda^2 v_1^2 D^2} \int_0^{L/v_1} \varphi(x) \cos wx \cdot dx \right. \\ &\quad \left. + \frac{v v_1 (1 - \cos \frac{wL}{v_1 D})}{\pi w^2} + 2 \frac{v^2(D-1)^2}{\pi \lambda^2 D^2} \left\{ (1 - \cos \frac{wL}{v_1 D}) \right. \right. \\ &\quad \left. \left. \int_0^{L/v_1} \varphi(x) \cos wx \cdot dx \right\} \right\} \end{aligned}$$

Evaluating $\int_0^{\infty} \cos w \xi g_2(w) dw$ we get the derived result.

and

6.7. Explanation of Drafting waves Cavanagh-Foster's formula:

From Theorem 6.63 it is clear that the behaviour of $\beta_1(\xi)$ depends very much on that of the correlation function $g(x)$ of the transition times. If $g(x)$ is of oscillatory character, then $\beta_1(\xi)$ also will be oscillatory. But we have seen that the co-variance function of the input is a monotonic decreasing function and is never negative.

To explain drafting waves all we have to do is to show that $\beta_1(\xi)$ in general is oscillatory and can even become negative for certain values of ξ . Suppose that $g(x)$ decreases linearly to zero as x varies from 0 to $\frac{L}{v_1 D} (D+1)$ and that $D \geq 1$. Then the expression for $\beta_1(\xi)$ becomes

$$\begin{aligned} \beta_1(\xi) &= \gamma \left[1 + \frac{2\gamma(D-1)^2}{\lambda^2 L D (D+1)} \right] \left[\frac{L}{D} - v_1 \xi \right] \quad \text{if } 0 \leq \xi \leq \frac{L}{v_1 D} \\ &= 0 \quad \text{if } \frac{L}{v_1} \leq \xi \leq \frac{L}{v_1} + \frac{L}{v_1 D} \\ &= \frac{\gamma v_1 (D-1)^2}{\lambda^2 L D (D+1)} \left[\frac{L}{v_1} - \xi \right] \quad \text{if } \frac{L}{v_1} \leq \xi \leq \frac{L}{v_1} + \frac{L}{v_1 D} \\ &= -\frac{\gamma^2 (D-1)^2}{\lambda^2 D^2} \left[\frac{L(D+2) - v_1 D \xi}{L(D+1)} \right] \\ &\quad \text{if } \frac{L(D+1)}{v_1 D} \leq \xi \leq \frac{L(D+2)}{v_1 D} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

This expression shows that $\beta_1(\xi)$ is not monotonic and that it is negative in the interval $\frac{1}{v_1} \leq \xi \leq \frac{1}{v_1} + \frac{2}{v_1 D}$. The negativity of $\beta_1(\xi)$ gives rise to alternations of maxima and minima of $n_1(t)$, giving the signal a wave like appearance.

$\Rightarrow$ Taking as in the previous example $g(x)$ to be linear we get

$$\text{Var. } n_1(t) = \beta_1(t) = \frac{v_1}{\lambda} \left[1 + \frac{2 \sqrt{(D-1)^2}}{\lambda^2 L D (D+1)} \right] \quad u \rightarrow 0,$$

$\lambda \rightarrow \infty$ is given by $\lambda = \frac{2}{L D}$, where λ_0 does not depend on D

$$\text{and } \therefore c.v.^2 \cdot n_1(t) = \frac{D-1}{v_1} \left[1 + \frac{2 \sqrt{(D-1)^2}}{\lambda^2 L D (D+1)} \right]$$

Making use of this fact, we find that the foregoing

$$\begin{aligned} c.v.^2 \cdot n_1(t) - c.v.^2 \cdot n_0(t) &= \frac{D-1}{v_1} + \frac{2 \sqrt{(D-1)^2}}{\lambda^2 L^2 (D+1)} \\ (c.v.)^2 n_1(t) - (c.v.)^2 n_0(t) &= \frac{D-1}{v_1} + \frac{2 \sqrt{(D-1)^2}}{\lambda^2 L^2 (D+1)} \end{aligned}$$

If D is large, we may replace $D+1$ by D and $\frac{D-1}{D}$ by unity. Making these approximations we find that

formula of Cavaney and Foster [6].

$$c.v.^2 \cdot n_1(t) - c.v.^2 \cdot n_0(t) = \frac{D-1}{v_1} + \frac{2 \sqrt{(D-1)^2}}{\lambda^2 L^2}$$

$$6.98 \text{ DRAFTING OF IDEAL SIGNALS (GENERALIZATION TO THE CASE WHERE ALL THE FIBRES ARE NOT OF THE SAME LENGTH)}$$

Thus the expression for $[c.v.^2 \cdot n_1(t) - c.v.^2 \cdot n_0(t)]$ for large values of D , is similar to the empirical formula of Cavaney and Foster [6]. The similarity becomes even more prominent, if we take into account the dependence of λ on u and D . Since λ is related only to the distribution of the time of transition of

of a fibre from the velocity state v_1 to that of v_2 we may assume that λ is not very sensitive to changes in D . But we have already seen that the case $\lambda \rightarrow \infty$ corresponds to ideal drafting. In a similar manner, it can be easily verified that $u \rightarrow 0$ also corresponds to ideal drafting. Thus, the simplest form of dependence of λ on u which ensures that as $u \rightarrow 0$, $\lambda \rightarrow \infty$ is given by $\lambda = \frac{\lambda_0}{Lu}$, where λ_0 does not depend on D and u .

Making use of this fact, we find that the foregoing expression for 'added irregularity' simplifies to

$$(C.V)^2 n_1(t) - (C.V)^2 n_0(t) = \frac{D-1}{L} \left[1 + \frac{2L}{\lambda_0^2} u \right] \text{ which is the empirical}$$

formula of Cavaney and Foster [6].

Part II

6.98 DRAFTING OF IDEAL SLIVERS. [GENERALIZATION TO THE CASE WHERE ALL THE FIBRES ARE NOT OF THE SAME LENGTH]

In this section we shall extend considerations of Part I to the case where all the fibres in the input are not of the same length. Most of the notations used in this section are the same as those used earlier, and the additional ones will be

explained at the appropriate places.

Let l_j be the length of the j th fibre. The distribution of τ_j depends on l_j which has ^{same} the distribution function $P(\cdot)$ for all j . Further, we have for $n_0(t)$

$$n_0(w) = \frac{v v_1}{\pi w^2} \int_0^{l_{\max}} (1 - \cos \frac{w l}{v_1}) dP(l)$$

We can extend the considerations given in 6.5 to the general case also and show that here also drafting becomes ideal as $\lambda \rightarrow \infty$.

Further, we have

$$n_{1d}(w) = \frac{v v_1}{\pi w^2} \int_0^{l_{\max}} (1 - \cos \frac{w l}{v_1}) dP(l) \quad (6.81)$$

$$\rho_{1d}(\cdot) = v \int_0^{l_{\max}} \left(\frac{l}{D} - v_1 + 3 \right) dP(l) \quad (6.82)$$

$$n_2(w) = n_{1d}(w) + \frac{v^2 v_1^2}{2\pi w^2} \operatorname{Re} \cdot E \left[\left(\exp \cdot \frac{i w l_j}{v_1 D} - 1 \right) \left(\exp \cdot \left(\frac{-i w l_k}{v_1 D} \right) - 1 \right) P(l_j, l_k, w) \right] \quad (6.83)$$

where

$$P(l_j, l_k, w) = \lim_{T \rightarrow \infty} \frac{1}{T} E \left[P_T(l_j, l_k, w) \right] \quad (6.84)$$

and

$$P_T(l_j, l_k, w) = E \left[\exp \cdot \frac{-i w}{D} \left\{ D(t_j - t_k) + (D - 1)(\tau_j - \tau_k) \right\} / l_j, l_k \right] \quad (6.85)$$

$$f(z_j / l_j) = \frac{\lambda v_1 \cosh \lambda v_1 z_j}{\sinh \lambda(h - l_j)} \quad (6.86)$$

$$0 \leq z_j \leq \frac{h - l_j}{v_1}$$

$$h(z_j, t_j, t_k, l_j, l_k) = f(z_j / l_j) f(z_k / l_k) \quad (6.87)$$

$$[1 + (a_j z_j - b_j)(a_k z_k - b_k) \rho(t_j, t_k, l_j, l_k)]$$

where

$$a_j = \frac{1}{\sigma_j}$$

$$b_j = \bar{z}_j / \sigma_j$$

$$\bar{z}_j = E[z_j / l_j]$$

and

$$\sigma_j^2 = \text{Var.}[z_j / l_j].$$

Up to this point the expansion is simple enough. The problem is to choose $\rho(t_j, t_k, l_j, l_k)$, the correlation coefficient between the 'transition times' of two fibres one of which (of length l_j) leaves the back rollers at time t_j and the other (of length l_k) at time t_k , as a function of t_j, t_k, l_j and l_k . When all the l_j are equal the problem was simply resolved by the requirement that $\rho(t_j, t_k)$ depend on t_j and t_k only through their difference. In the present case, even after imposing the above restrictions ρ would still depend upon three variables.

To simplify further the dependence of ρ on t_j, t_k , we proceed as follows. Let $u(t_j, t_k, l_j, l_k) [u(j, k)]$ for

brevity] denote the maximum time for which the j th and k th fibres are simultaneously in the drafting zone. As soon as we refer to a fibre as the j th fibre, it means that its time of entry into the drafting zone (or equivalently the time at which released by the back rollers) and its length are given, so that $u(j, k)$ has to be construed as the maximum over τ_j and τ_k only. Now, we say that a fibre is in the drafting zone if its left end lies between the back and the front rollers. The time for which the j th fibre will be in the drafting zone is $\frac{h}{v_1 D} + \tau_j \frac{D-1}{D}$. Therefore, if the j th fibre were to enter the drafting zone before the k th ^{one} does, then the time for which the j th and k th will be simultaneously in the drafting zone is $\frac{h}{v_1 D} + \tau_j \left(\frac{D-1}{D} \right) - (t_k - t_j)$

$$\begin{aligned}
 \text{Therefore } u(j, k) &= \max_{\substack{h - l_j \\ 0 \leq \tau_j \leq \frac{h - l_j}{v_1}}} \left[\frac{h}{v_1 D} + \tau_j \frac{D-1}{D} + t_j - t_k \right] \\
 &= t_j + \frac{h - l_j}{v_1} + \frac{l_j}{v_1 D} - t_k, \quad t_j \leq t_k
 \end{aligned}$$

If $t_k \geq t_j + \frac{h - l_j \left(\frac{D-1}{D} \right)}{v_1}$, then the k th fibre can not enter the drafting zone until the j th leaves it in which case $u(j, k) = 0$ and hence $\mathcal{V}[u(j, k)] = 0$. Interchanging j and k in the above argument, we get the expression for the case $t_j \geq t_k$.

Thus we have,

$$\begin{aligned}
 u(j, k) &= 0 \quad \text{if } t_k \geq t_j + d_j \\
 &= t_j + d_j - t_k \\
 &\quad \text{if } t_j \leq t_k \leq t_j + d_j \\
 &= t_k + d_k - t_j \\
 &\quad \text{if } t_k \leq t_j \leq t_k + d_k \\
 &= 0 \quad \text{if } t_j \geq t_k + d_k
 \end{aligned} \tag{6.88}$$

where $d_j = \frac{1}{v_1} \left[h - l_j \left(\frac{D-1}{D} \right) \right]$

From (6.88) it is clear that $u(j, k)$ depends on t_j and t_k only through their difference and that when all the fibres are of the same length L , it reduces to $\frac{1}{v_1} \left[h - \frac{L(D-1)}{D} \right] - |t_j - t_k|$. Consequently we assume that $\varphi(t_j, t_k, l_j, l_k)$ depends on t_j, t_k, l_j and l_k only through $u(j, k)$. That is,

$$\varphi(t_j, t_k, l_j, l_k) = \varphi_1[u(j, k)] \tag{6.89}$$

where $|\varphi_1(x)| \leq 1$

and $0 \leq u(j, k) \leq \frac{h}{v_1}$ and $\varphi_1(0) = 0$.

We shall now deduce from these considerations the results corresponding to Theorems (6.61), (6.62) and (6.63). Of these, the result concerning the covariance function of $n_0(t)$ and $n_1(t)$ is an obvious one and hence we shall ^{not} consider it here in detail.

Theorem 6.81:

If the joint distribution of z_j, z_k ($j \neq k$) is given by (6.86), (6.87) and (6.89) then $n_1(t)$ has an absolutely continuous spectrum and its spectral density $g_1(\omega)$ is given by

$$g_1(\omega) = g_{1d}(\omega) + \frac{\gamma v_1^2}{\pi \omega^2} \operatorname{Re} \cdot H_1(\omega) \overline{H_2(\omega)}$$

where

$$H_1(\omega) = \int_0^{l_{\max}} \left(\exp. \frac{i\omega l}{v_1 D} - 1 \right) c_2(l, \omega) dP(l) \cdot \frac{1}{v_1} \left[h - \frac{l(D-1)}{D} \right] \int_0^{\frac{1}{v_1} \left(h - \frac{l(D-1)}{D} \right) - \pi} f_1(x) \exp. i\omega x dx$$

$$H_2(\omega) = \int_0^{l_{\max}} \left[\exp. \frac{i\omega l}{v_1 D} - 1 \right] c_2(l, \omega) dP(l)$$

$$c_2(l, \omega) = \int_0^{\frac{h-l}{v_1}} [a(l) z - b(l)] \exp. \left[\frac{-i\omega(D-1)z}{D} \right] f(z) dz$$

$$a(l) = \frac{1}{\sqrt{\operatorname{Var}(z|l)}}$$

$$b(l) = E(z|l) / \sqrt{\operatorname{Var}(z|l)}$$

and $\bar{a}$ denotes the complex conjugate.

Further, as in Theorem 6.62, $n_1(t)$ is stationarily correlated with $n_0(t)$.

Proof:

The stationarity of $n_1(t)$ and the absolute continuity of its spectrum and the validity of [6.93] can be established in exactly the same way as for Theorem 6.61. Hence it remains only to calculate $G_1(v)$

From 6.93, we have

$$G_1(v) = G_{1A}(v) + \frac{v^2 v_1^2}{2 v^2} \operatorname{Re} E \left[\left(\exp. \frac{1 + l_j}{v_1 D} - 1 \right) \right. \\ \left. \left(\exp. \left(\frac{-1 + l_k}{v_1 D} \right) - 1 \right) F(l_j, l_k, v) \right]$$

where

$$F(l_j, l_k, v) = \lim_{T \rightarrow \infty} \frac{1}{T} F_T(l_j, l_k, v)$$

and

$$F_T(l_j, l_k, v) = E \left\{ \exp. \frac{-1 + l_j}{D} \left\{ D(t_j - t_k) + (D-1)(\tau_j - \tau_k) \right\} / l_j, l_k \right\}$$

Now

$$F_T(l_j, l_k, v) = \frac{1}{T^2} \int_{\frac{l_j}{v_1}}^{T + \frac{l_j}{v_1}} \int_{\frac{l_k}{v_1}}^{T + \frac{l_k}{v_1}} \exp. -1 + l_j(t_j - t_k) F(t_j, t_k, l_j, l_k, v) dt_j dt_k$$

where

$$F(t_j, t_k, l_j, l_k, v) = E \left\{ \exp. \frac{-1 + l_j(D-1)}{D} (\tau_j - \tau_k) / t_j, t_k, l_j, l_k \right\}$$

$$= \int_0^{c_j} \int_0^{c_k} r(z_j/l_j) r(z_k/l_k) dz_j dz_k \\ [1 + \varphi_1(u(j, k)) (a_j z_j - b_j)(a_k z_k - b_k)] \times \\ \exp. \left[\frac{-4v(D-1)}{D} (z_j - z_k) \right]$$

[Using 6.97 and 6.99]

$$= \phi_1(l_j, v) \phi_1(l_k, v) + \varphi_1(u(j, k)) \phi_2(l_j, v) \phi_2(l_k, v)$$

where

$$\phi_1(l_j, v) = \int_0^{c_j} r(z_j/l_j) \exp. \left[\frac{-4v(D-1)z_j}{D} \right] dz_j$$

$$\phi_2(l_j, v) = \int_0^{c_j} (a_j z_j - b_j) r(z_j/l_j) \\ \cdot \exp. \left[\frac{-4v(D-1)z_j}{D} \right] dz_j$$

$$\text{and } c_j = \frac{l_j - l_k}{v_1}$$

But

$$\int_{\frac{l_j}{v_1}}^{T + \frac{l_j}{v_1}} dt_j \int_{\frac{l_k}{v_1}}^{T + \frac{l_k}{v_1}} dt_k \exp. [-4v(t_j - t_k)] \\ = \exp. [-4v(l_j - l_k)] \frac{|e^{-4vT} - 1|^2}{v^2}$$

and

$$\int_{\frac{l_k}{v_1}}^{\frac{l_j}{v_1}} \int_{\frac{l_k}{v_1}}^{\frac{l_j}{v_1}} \varphi_1(u(j, k)) \exp. [-4v(t_j - t_k)] dt_j dt_k \\ \frac{l_j}{v_1} \leq t_j \leq T + \frac{l_j}{v_1} \\ \frac{l_k}{v_1} \leq t_k \leq T + \frac{l_k}{v_1} \\ = \int_{\frac{l_j}{v_1}}^{T + \frac{l_j}{v_1}} dt_j \int_{\frac{l_k}{v_1}}^{T + \frac{l_k}{v_1}} \varphi_1(u(t_k - t_j, l_j, l_k)) \\ \exp. [-4v(t_j - t_k)] dt_k$$

[Since $u(j, k)$ depends only on t_j and t_k through $t_j - t_k$]

$$= \int_{t_j - t_k - T}^{T + t_j - t_k} g_1[u(x, t_j, t_k)] \exp. (iux) dx.$$

$$\text{Min.} \left[T + \frac{t_j}{v_1}, T - t_j - \frac{t_k}{v_1} \right]$$

$$\int_{\text{Max.} \left[\frac{t_j}{v_1}, \frac{t_k}{v_1} - x \right]} dt_j$$

[by putting $x = t_k - t_j$ and inverting the order of integration]

$$= \int_{t_j - t_k - T}^{T + t_j - t_k} [T - |x + t_k - t_j|] g_1[u(x, t_j, t_k)] e^{iux} dx.$$

Therefore

$$\begin{aligned} F(t_j, t_k, u) &= g_1(t_j, u) \overline{g_1(t_k, u)} \frac{|\exp.(-iut) - 1|^2}{u^2 t^2} \exp. \frac{-iut(t_j - t_k)}{v_1} \\ &+ \frac{g_2(t_j, u) \overline{g_2(t_k, u)}}{t^2} \int_{t_j - t_k - T}^{T + t_j - t_k} g_1[u(x, t_j, t_k)] \cdot \\ &\quad [T - |x + t_k - t_j|] e^{iux} dx. \end{aligned}$$

Therefore, $F(t_j, t_k, u)$

$$= g_2(t_j, u) \overline{g_2(t_k, u)} \int_{-\infty}^{\infty} g_1[u(x, t_j, t_k)] e^{iux} dx.$$

Substituting for $u(j, k)$ from 5.3.8 in the above expression for $F(l_j, l_k, w)$, we get after simplification

$$\operatorname{Re} \sum_{l_j, l_k} \left\{ F(l_j, l_k, w) \left[\exp. \frac{2\pi l_j}{v_1 D} - 1 \right] \left[\exp. \left(\frac{-2\pi l_k}{v_1 D} \right) - 1 \right] \right\}$$

$= 2 \operatorname{Re} H_1(w) H_2(w)$. The proof the second ^{statement} is exactly similar to that of Theorem 6.72. This completes the proof of the Theorem.

The above expression for $g_1(w)$ is quite complicated and does not lead to a simple and workable expression for $\beta_1(\zeta)$. But, as in Theorem 6.63, we can obtain an asymptotic expression for $\beta_1(\zeta)$ and this is given in Theorem 6.82.

Theorem 6.82: When drafting is asymptotically ideal, (i.e., when λ is large) and the conditions stated in Theorem 6.91 are satisfied, then for $\zeta \geq 0$

$$\begin{aligned} \beta_1(\zeta) \sim & \beta_{1d}(\zeta) + \frac{\tilde{v} (p-1)^2}{2\lambda^2 p^2} \int_0^{l_{\max}} \int_0^{l_{\max}} [\alpha(l_1, \zeta + \zeta_0) \\ & + \alpha(l_1, \zeta - \zeta_0) - \alpha(l_1, \zeta + \zeta_1) - \alpha(l_1, \zeta - \zeta_1) \\ & - \alpha(l_1, \zeta + \zeta_2) - \alpha(l_1, \zeta - \zeta_2) + \alpha(l_1, \zeta + \frac{p-1}{p} \zeta_0) \\ & + \alpha(\zeta, \zeta - \frac{p-1}{p} \zeta_0)] \alpha(l_1) \alpha(l_2) \end{aligned}$$

where

$$\xi_0 = c_2 - c_1$$

$$\xi_1 = d_2 - c_1$$

$$\xi_2 = c_2 - d_1$$

$$\alpha(l, x) = g_1 [d_1 - |x|] \text{ if } |x| \leq d_1 \text{ ———}$$

$$= 0 \text{ otherwise.}$$

Proof: We have seen in the proof of Theorem 6.63 that

as $\lambda \rightarrow \infty$

$$\lambda v_1 \int_0^{\frac{h-l}{v_1}} \frac{\gamma - \bar{\gamma}}{\sigma_\gamma} f(\gamma) e^{i\lambda \gamma} d\gamma \rightarrow i \exp. \frac{i\lambda(h-l)}{v_1}$$

From this it follows that

$$\lim_{\lambda \rightarrow \infty} \lambda v_1 g_2(l_j, v) = \frac{-i\lambda(B-1)}{B} \exp. \frac{-i\lambda(B-1)C_j}{B}$$

That is, for large λ , we may use the approximation

$$g_2(l_j, v) = \frac{-i\lambda(B-1)}{\lambda v_1 B} \exp. \frac{-i\lambda(B-1)(h-l_j)}{v_1 B}$$

using the above approximation for $g_2(l_j, v)$ we find that the expression for $g_1(v)$ obtained in Theorem 6.81 simplifies to

$$g_1(v) = \frac{1}{v} \int_0^v \left(\frac{1}{v} \int_0^v \alpha(l, x) dx \right) dl$$

Using this we obtain the desired result.

$$\begin{aligned}
\epsilon_1(w) &= \frac{v_1}{\pi w^2} \int_0^{l_{\max}} \left[1 - \cos \frac{\pi l}{v_1 D} \right] dP(l) \\
&+ \frac{v^2 (D-1)^2}{\pi \lambda^2 D^2} \text{Re.} \int_0^{l_{\max}} dP(l_1) \int_0^{l_{\max}} dP(l_2) \cdot \\
&\quad \left(\exp. + \frac{i w l_1}{v_1 D} - 1 \right) \exp. \left(\frac{-i w l_2}{v_1 D} - 1 \right) \cdot \\
&\quad \left[\exp. \frac{i w (D-1)(l_1 - l_2)}{v_1 D} \int_0^\infty \alpha(l_1, x) e^{i w x} dx \right] \\
&= \epsilon_{1A}(w) + \frac{v^2 (D-1)^2}{\pi \lambda^2 D^2} \int_0^{l_{\max}} \int_0^{l_{\max}} dP(l_1) dP(l_2) \\
&\quad \int_0^\infty A(l_1, l_2, x, w) \alpha(l_1, x) dx
\end{aligned}$$

where

$$\begin{aligned}
A(l_1, l_2, x, w) &= \cos w(x + \xi_0) - \cos w[x + \xi_1) \\
&- \cos w(x + \xi_2) + \cos w\left[x + \frac{(D-1)}{D} \xi_0\right)
\end{aligned}$$

Now $\beta_1(\xi)$ can be obtained by evaluating $2 \int_0^\infty \epsilon_1(w) \cos w \xi dw$.

Evaluation of this integral leads to integrals of the type

$$\int_0^\infty \cos w \xi dw \int_0^\infty \alpha(l_1, x) \cos w(x + \beta) dx dw$$

which can be evaluated by the well known methods of Fourier Series and integrals. The value of the last integral is easily seen to be $\frac{\pi}{2} [\alpha(l_1, \xi + \beta) + \alpha(l_1, \xi - \beta)]$.

Using this we obtain the desired result.

From Theorem 6.82 we can obtain the expression for $V[\bar{n}_1(t)] = \beta_1(0)$. Using this and assuming $\varphi_1(x)$ to be linear, we obtain after some simplification

$$\begin{aligned} C.v^2 \bar{n}_1(t) - C.v^2 \bar{n}_0(t) \\ = \frac{D-1}{\sqrt{\ell}} + \frac{(D-1)^2}{\lambda^2 \bar{\ell}^2 h D} \int_0^{l_{\max}} \int_0^{l_{\max}} dP(l_1) dP(l_2) \\ \times [|D l_1 - (D-1) l_2| + |(D-1) l_1 - D l_2| - (2D-1) |l_1 - l_2|] \end{aligned}$$

When D is large we may replace $|D l_1 - (D-1) l_2|$, $|(D-1) l_1 - D l_2|$ by $D |l_1 - l_2|$ and $\frac{D-1}{D}$ by unity. Making these approximations we find that the expression for $C.v^2 \bar{n}_1(t) - C.v^2 \bar{n}_0(t)$ reduces to

$$\frac{D-1}{\sqrt{\ell}} + \frac{(D-1)}{\lambda^2 \bar{\ell}^2 h} \int_0^{l_{\max}} \int_0^{l_{\max}} |l_1 - l_2| dP(l_1) dP(l_2)$$

which is formally similar to Coveney and Foster's result. The explanation of the drafting in the present case is the same as that given earlier.

We shall now establish the following central-limit property of $\bar{n}_0(t)$ and $\bar{n}_1(t)$, which holds also for the density processes $n_0(t)$ and $n_1(t)$.

Theorem 6.93: For any $t_0 > 0$ and any positive integer k

let

$$X_k = \sqrt{k} \left(\frac{1}{k} \sum_{r=0}^{k-1} n_0(t + r t_0) - \bar{v} \right)$$

$$Y_k = \sqrt{k} \left(\frac{1}{k} \sum_{r=0}^{k-1} n_1(t + r t_0) - \frac{\bar{v} l}{D} \right)$$

Then the (X_k, Y_k) ($k = 1, 2, \dots$) is (as $k \rightarrow \infty$) asymptotically normal with zero mean and covariance matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

where $A = \beta_0(0) + 2 \sum_{r=1}^{k_0} \beta_0(r t_0)$

$$B = \beta_{01} + \sum_{r=1}^{k_0} \beta_{01}(r t_0)$$

$$C = \beta_1(0) + 2 \sum_{r=1}^{k_0} \beta_1(r t_0)$$

and k_0 is the integral part of $\frac{h}{v_1 t_0} \left[h + \frac{l_{\max}}{D} \right]$

Proof: Let us first note that if

$$|t - u| > \frac{h}{v_1}, \{n_0(t), n_1(t)\} \text{ and } n_0(u), n_1(u)$$

are independent. The rest follows by applying Theorem 4.53.

6.9. Concluding Remarks:

In chapter V we have seen that when the fibres in a sliver are all straight and parallel to the sliver axis and the distribution of fibre-right-ends along the sliver axis is a Poisson process, the

* This is so because, $\beta_{01}(\xi) = \text{cov.}[n_0(t), n_1(t + \xi)]$ vanishes for $\xi < 0$.

processes $n(x)$ and $m(x)$ are stationary. It will be of interest to study the converse of this proposition, namely, that of finding out ^{of} _^ ~~nodes~~ arrangements of fibre-ends which will make $m(x)$ and $n(x)$ stationary, particularly because for other known yarn models like that of Hannah-Rodden [19] $m(x)$ and $n(x)$ are not stationary.

In the present chapter we have sought to explain the effect of drafting on the irregularity of the ingoing sliver without making any detailed assumptions on the forces controlling the movement of fibres in the drafting zone, but by making some reasonable assumptions on the type of motion of ~~fletting~~ fibres and on the distributions of the transition times of these fibres. In so doing, we have looked for assumptions which make $n_1(t)$ stationary since when drafting is steady and free from mechanical faults like roller eccentricity, we expect the fluctuations caused by drafting to be of stationary type. Another important feature that we noticed about drafting is that it causes only local disturbances in the fibre-arrangement in the input sliver. This is manifested partly by the fact that the input $n_0(t)$ and the output $n_1(t)$ are both linearly and completely non-deterministic.

Martindale's [15] arguments show that the ideal sliver characterizes the pattern of inherent variations in the properties of a sliver. Consequently, we can take the ideal sliver as a basis

against which an actual sliver may be compared. On the other hand, if the sliver to be compared is known to have been drafted, then it is not the ideal sliver against which comparison must be made, but the one obtained by drafting it. Consequently, it is important to generalize the consideration and results of this chapter to the case where the speeds of the floating fibres are not restricted to v_1 and v_2 only but can have all the speeds in that range. The fact that very little experimental data is available which reveals the manner in which fibres change their speeds from v_1 to v_2 makes the problem of generalization difficult. In the two-velocity model we could describe the effect of drafting by means of two functions, which can be deduced from the second-order properties of $n_1(t)$ and $n_0(t)$. But in the general case it is not clear whether one can characterize the behaviour of a drafting mechanism by only two functions which reveal to a certain extent the motion of floating fibres and which can be calculated from the second-order properties of $n_1(t)$ and $n_0(t)$. Another, interesting problem of drafting is to find out if $n_1(t)$ can be obtained from $n_0(t)$ by some simple linear transformation.

REFERENCES

A. TECHNOLOGICAL

1. Brany, H. 'Calculation of Variance Length Curve from the length distribution of fibres' - I and II. Jour. Text. Inst. (Proceedings) Vol.41, 1953 pp. 1 and 10.
2. ---- 'Variance Length Relations in a Yarn'. Letter to the editor Ibid (Transactions) Vol.49, 1958 pp. 48.
3. Cavaney, B. and Foster, G.A.R. 'Irregularity of materials drafted on Gotton Spinning Machines and its dependence on draft, doubling and roller setting - Part I. The Speed Frames' Ibid (Transactions) Vol. 46, 1955, pp. 529.
4. Cox, D. R. 'Theory of drafting of wool fibres - I' Proc. Roy. Soc. Vol. 197(A), 1949, pp. 28.
5. Cox, D. R. and Townsend, M. W. 'The use of correlogram or measuring yarn irregularity'. Jour. Text. Inst. (Proceedings) Vol. 42, 1951, pp. 145.
6. Foster, G.A.R. 'Fibre motion in roller drafting' Ibid (Transactions) Vol. 42, 1951, pp. 335.
7. ---- 'The causes of irregularity of cotton yarns' Ibid (Proceedings) Vol. 41, 1950, pp. 357.
8. ---- 'Some instruments for the analysis of time-series and their applications to Textile Research'. Suppl. Jour. Roy. Stat. Soc. Vol. 8, 1946, pp. 43.
9. Foster, G.A.R. and Martindale, J.G. 'The form and length of drafting waves in cotton rovings'. Shirley. Inst. Mex. Vol. XIX 1944-1945, pp. 210.
10. Gregory, J., and Womersley, J.R. 'Some fundamental calculations relating to the arrangements of fibres in slivers and rovings' Ibid, Vol. XIX, 1944-1945, pp. 197.

11. Grosberg, P. 'The medium and long-term variations of a yarn - Part I' Jour. Text. Inst. (Transactions) Vol. 46, 1955, pp. 301.
12. Fujino, K. and Kabavata, S. 'Theoretical analysis of the spectral density of random slivers'. Journal of the Textile Machinery Society, Japan, Vol. 5 No.1.
13. Hannah, H. and Redden, S. 'Variances - Length relations in a yarn with restricted fibre position'. Jour. Text. Inst. (Transactions) Vol. 47, 1956, pp. 402.
14. Lindsley, C.H. 'Measurement of fibre orientation'. Text. Res. Jour. Vol. 21, 1951, pp. 39.
15. Martindale, J. G. 'A new method of observing the irregularity of yarns with some observations on the origin of irregularities in worsted slivers and yarns'. Jour. Text. Inst. (Transactions) Vol. 36, 1945, pp. 35.
16. ----- 'A review on the causes of yarn Irregularity' Ibid (Transactions) Vol. 41, 1950, pp. 340.
17. Spencer-Smith, J.L. and Todd, H.A.C. 'A time series met with in Textile Research' Jour. Roy. Stat. Soc. Suppl. (B), Vol.VII, No.2, 1941, pp. 131.
18. Sudarsana Rao, J. 'A Mathematical Model for the Ideal Sliver and its applications to the theory of Roller Drafting'. Jour. Text. Inst. (Transactions) Vol. 52, 1961, pp. 571-601.
19. -- ----- 'Theory of Roller Drafting - II' To appear in the Transactions of the Jour. Text. Inst.
20. ----- and Garde, A. R. 'Calculation of the Combing Ratios for Ideal and Quasi-Ideal slivers with 'Applications to the Study of Fibre configuration slivers', Submitted for publication in the Journal for the publication in the Journal of the Textile Institute.
21. Townsend, H.V. and Cox, D. R. 'Analysis of yarn Irregularity' Ibid (Proceedings) Vol. 42, 1951, pp. 107.

22. Vanden Abeele, A.M.

'Contribution to the study of irregularity of yarns, revings and slivers'. Ibid (Proceedings) Vol. 42, 1951, pp. 162.

B. STOCHASTIC PROCESSES

23. Bartlett, M.S.

'An Introduction to stochastic processes with special reference to methods and Applications'. (Camb. Univ. 1956).

24. Copeland, A.M., and Francis, R.

'A postulational treatment of the Poisson law'. Ann. Math. Vol. 37, 1936, pp. 357.

25. Cramer, H.

'A contribution to the theory of stochastic processes'. Proceedings of the Second Berkeley Symposium, pp. 329.

26. -----

'On the theory of stationary Random Processes'. Ann. Math. Vol. 41, 1940.

27. Doob, J. L.

'Stochastic Processes' John Wiley, 1953.

28. Fortet, R.

'Random functions from a Poisson Process'. Proceedings of the Second Berkeley Symposium, pp. 329.

29. Fraser, D.A.S.

'Non-parametric methods in Statistics' (John Wiley, 1957), Chapter 6, pp. 215.

30. Grenander, U., and Rosenblatt, M.

'Statistical Analysis of stationary time series'. (John Wiley, 1956).

31. Hannar, O.

'Deterministic and non-deterministic stationary processes'. Arkiv-för Matematik Band 1, No.14, 1950, pp. 161.

32. Kac, M., and Karvits, W.

'Statistical Analysis of Certain types of random functions'. Ann. Math. Stat. Vol. 15, 1944, pp. 173.

33. Karhunen, K.

'Über lineare Methoden in der Wahrscheinlichkeitrechnung' Ann. Ac. Sci. Fennica. A, I, 37 (1947).

34. Karhunen, K. 'Über die Struktur Stationärer Zufälliger Funktionen'
Arkiv for Matematik Band 1, nr. 13, 1949,
pp. 141.
35. Khintchine, A. 'Korrelation Theori der stationären stochastischen Prozesse'.
Math. Annale , 109, 1934.
36. Kolmogoroff, A.N. 'Foundations of the theory of Probability'
Chelsea Publishing Co. (1950).

