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An Uncertainty Principle for Eigenfunction Expansions

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ABSTRACT. In this paper, we extend a theorem of Hardy's on Fourier transform pairs to: (a) a noncompact-type Riemannian symmetric space of rank one, with respect to the eigenfunction expansion of the invariant Laplacian; (b) a compact Riemannian manifold with respect to the eigenfunction expansion of a positive elliptic operator; and (c) $\mathbb{R}^n$ with respect to Hermite and Laguerre expansions.

1. Introduction

In the opinion of many experts uncertainty principles can be associated to general eigenfunction expansions. The purpose of this paper is to give concrete evidence of such a "folklore theorem." To explain what we mean, consider a periodic signal of the kind in Figure 1, that is, a signal "highly concentrated" around multiples of 2π (or, what is the same, if one wants to think of it as a function on S^1, then it is highly concentrated around the identity). Then, using rather elementary arguments, one can show that the Fourier coefficients $\hat{f}(n)$ of the signal cannot decay very rapidly as $|n| \to \infty$; for instance, one cannot have an estimate of the kind $|\hat{f}(n)| \leq Ce^{-|n|}$. Another way of saying this is

> If f is a nontrivial function that is highly concentrated in a neighbourhood of a point $x_o \in S^1$, then $\hat{f}(n)$ cannot decay "very rapidly." (UP)

This can be considered as a very simple kind of Uncertainty Principle (i.e., both f and $\hat{f}$ cannot be simultaneously highly concentrated). Observe that if one considers the elliptic operator $\Delta = \frac{d^2}{dx^2}$ on S^1 (or in this special case, even $\frac{d}{dx}$ would do), then $\{e^{in(.)}\}_{n\in \boldsymbol{Z}}$ are precisely the eigenfunctions for Δ and $f(x) \approx \Sigma \hat{f}(n)e^{inx}$ is the eigenfunction expansion for the elliptic operator Δ. As mentioned above, we wish to provide evidence of something that is accepted in the folklore. Let L be an elliptic differential operator on a manifold M. Then an analogue of (UP) will hold for f and its Fourier coefficients with respect to the eigenfunction expansion associated with L. (Of course, the eigenfunction expansion could be discrete, continuous, or a mixture of both.)

The preceding discussion is admittedly a little vague! However, we hope the following four sections will make it clear. Finally, we would like to add that certain forms of the uncertainty principle seem to be valid in very abstract situations. For instance in [2] De Jeu established that the uncertainty principle due to Donoho and Stark [3] is valid whenever one has an integral operator for which a Plancherel theorem holds. However, for the kind of uncertainty principle presented in this paper, one *may* have to restrict oneself to eigenfunction expansions; at least the proofs given here use certain detailed analytical facts that seem to be tied up with the eigenfunction expansion in question.

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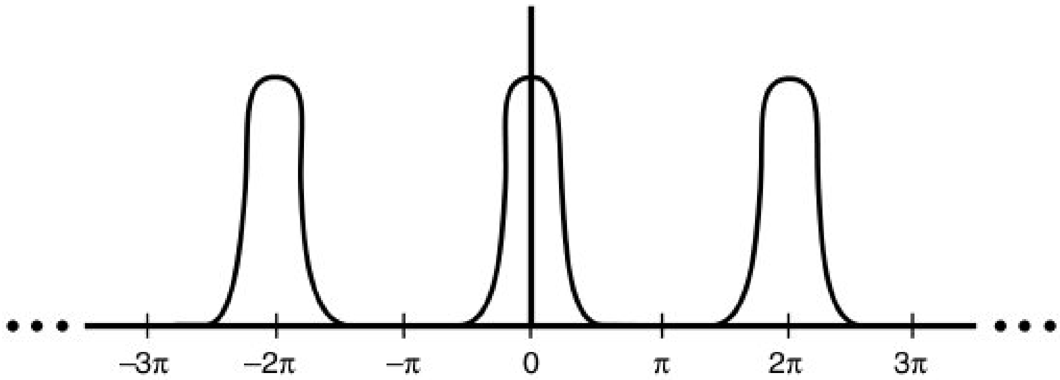

FIGURE 1.

2. Hardy's Theorem for $\mathbb{R}^n$

If f is a sufficiently nice function on $\mathbb{R}^n$ one has the Fourier inversion formula $f(x) = \int_{\mathbb{R}^n} \hat{f}(\lambda)e^{i\lambda.x}\,d\lambda$, where $\hat{f}(\lambda)$ is the usual Fourier transform on $\mathbb{R}^n$. Since $x \mapsto e^{i\lambda.x}$ are eigenfunctions of the usual Laplacian Δ on $\mathbb{R}^n$, the Fourier inversion formula can be viewed as the eigenfunction expansion for the pair $(\mathbb{R}^n, \Delta)$. In this case, concentration around a point $x_o \in \mathbb{R}^n$ could be taken as a "very rapid decrease" of f at infinity. Hardy's theorem is an illustration of (UP) for the pair $(\mathbb{R}^n, \Delta)$.

Theorem 1.

Suppose f is a measurable function on $\mathbb{R}^n$ such that

$$|f(x)| \leq Ce^{-\alpha|x|^2}, \quad |\hat{f}(\lambda)| \leq Ce^{-\beta|\lambda|^2}, \qquad x, \lambda \in \mathbb{R}^n, \tag{2.1}$$

where α, β are positive constants. If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e. (If $\alpha\beta < \frac{1}{4}$, there are infinitely many linearly independent functions satisfying (2.1) *and if $\alpha\beta = \frac{1}{4}$, then $f(x) = Ae^{-\alpha|x|^2}$.)*

For a proof of the above theorem, in the case $n = 1$, see [4]. The n-dimensional case can be reduced to the one-dimensional case via the Radon transform (see [15]). For $n = 1$, there is a more general result due to Beurling from which Hardy's theorem can be deduced [12].

Remark 2. Actually, instead of demanding a pointwise estimate on f as in the theorem, one can replace it by a weaker condition like $(\int_{\mathbb{R}^n \setminus B(0,R)} |f(x)|^2\,dx)^{\frac{1}{2}} \leq Ce^{-\alpha R^2}$. □

3. Symmetric Spaces

In this section we will consider an important subclass of noncompact Riemannian manifolds, that is, symmetric spaces of the noncompact type. For any unexplained notation and terminology in this section, refer to [10]. Let X be a symmetric space of the noncompact type. Then X is of the form G/K where G is the connected component of the group of isometries of X and K is a maximal compact subgroup of G. It turns out that G is a connected noncompact semisimple Lie group with finite center. Let $d(.,.)$ be the canonical distance on X induced by its Riemannian structure, dx the canonical G-invariant volume form on X, and Δ the associated Laplace–Beltrami operator on X. Then Δ is a positive selfadjoint elliptic operator on X, with real-analytic coefficients. (For example, if X = upper half plane with the Poincaré metric $= \{x_1 + ix_2 : x_2 > 0\}$, the metric ds is given by $ds^2 = (dx_1^2 + dx_2^2)/x_2^2$, the volume form $dx = dx_1dx_2/x_2^2$, and $\Delta = -x_2^2(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2)$.) To keep the exposition simple, we will make the two assumptions that (i) Rank $X = 1$ and (ii) the functions f that we consider are all spherical, that is, K-invariant, that is, $f(k.x) = f(x), \forall k \in K$, $x \in X$. Strictly speaking, neither of these assumptions is necessary, but we will not elaborate here on why this is so.

We can now describe the eigenfunction expansion for spherical f with respect to Δ. For each $\lambda \in \mathbb{C}$ there is the associated elementary spherical function φ_λ (see [7] for details); each φ_λ is spherical, real-analytic, and an eigenfunction of Δ. Further $\varphi_\lambda = \varphi_{-\lambda}$, and this is the only identification among the $\{\varphi_\lambda\}_{\lambda \in \mathbb{C}}$. For sufficiently nice spherical f and for $\lambda \in \mathbb{R}$, one can define $\hat{f}(\lambda)$ by $\hat{f}(\lambda) = \int_X f(x)\varphi_\lambda(x)\,dx$ and has the associated eigenfunction expansion, $f(x) = \int_{\mathbb{R}} \hat{f}(\lambda)\varphi_{-\lambda}(x)\;d\mu(\lambda)$, for an explicitly computable even density $d\mu(\lambda)$ on $\mathbb{R}$, which is of at most polynomial growth. (For example, if X is the Poincaré upper half plane, the eigenvalue corresponding to φ_λ, $\lambda \in \mathbb{C}$, is $\lambda^2 + 1$ and $d\mu(\lambda) = C\lambda \tanh \pi\lambda\, d\lambda$ on $\mathbb{R}$.) Let x_o be some fixed point in X, say the point that corresponds to the coset eK under the correspondence $X \leftrightarrow G/K$.

With the above notation, we have another illustration of (UP), for the pair (X, Δ), that is, the following analogue of Hardy's theorem.

Theorem 3.

Let f be a measurable spherical function on X such that $|f(x)| \le Ce^{-\alpha\, d(x,x_o)^2}$, $x \in X$, and $|\hat{f}(\lambda)| \le Ce^{-\beta\lambda^2}$, $\lambda \in \mathbb{R}$. If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e.

Remark 4. To get the constant $\frac{1}{4}$, one will have to normalize the Riemannian structure on X suitably. □

This theorem will follow from more general results that will appear in a forthcoming paper [14]. However, for the sake of completeness we indicate how the proof proceeds in this special case.

Given the very rapid decay of f, one shows $\hat{f}$ that is initially defined only for $\lambda \in \mathbb{R}$ extends to an even entire function on $\mathbb{C}$. This will follow from the fact that (a) for each x, $\lambda \mapsto \varphi_\lambda(x)$ is entire in λ and (b) one has the following very rough estimate on φ_λ : $|\varphi_\lambda(x)| \le Ae^{|\mathrm{Im}\,\lambda| d(x,x_o)}$, $\lambda \in \mathbb{C}$ (see [8]). By writing down the integral defining $\hat{f}(\lambda)$, then using polar co-ordinates on X and using the above estimate for φ_λ, $\lambda \in \mathbb{C}$, one estimates $\hat{f}(\lambda)$, $\lambda \in \mathbb{C}$. After this, the proof is exactly as in the case of $\mathbb{R}$. Thus, one gets $|\hat{f}(\lambda)| \le \mathrm{Const}\, e^{|\lambda|^2/4\alpha'}$, $\lambda \in \mathbb{C}$, where α' is a constant slightly smaller than α such that $\alpha'\beta$ continues to be greater than $\frac{1}{4}$. On the other hand, one is given that $|\hat{f}(\lambda)| \le Ce^{-\beta\lambda^2}$, $\lambda \in \mathbb{R}$. But $-\beta < -\frac{1}{4\alpha'}$ and so $|\hat{f}(\lambda)| \le Ce^{-1/4\alpha'\lambda^2}$, $\lambda \in \mathbb{R}$. We now use the following lemma from complex analysis (see [18]).

If h is an even entire function such that $h(z) = O(e^{a|z|^2})$, $z \in \mathbb{C}$ and $h(t) = O(e^{-at^2})$, $t \in \mathbb{R}$, then $h(z) = \mathrm{Const}\; e^{-az^2}$.

From the above, it therefore follows that $\hat{f}(\lambda) = \mathrm{Const}\, e^{-\lambda^2/4\alpha'}$. On the other hand, $|\hat{f}(\lambda)| \le Ce^{-\beta\lambda^2}$ for $\lambda \in \mathbb{R}$; this along with the fact $\beta - \frac{1}{4\alpha'} > 0$ easily give $\hat{f} \equiv 0$, which in turn implies $f \equiv 0$, since $f \mapsto \hat{f}$ is one-to-one.

(Another approach is to consider the so-called Harish transform F_f of f. F_f is an even function on $\mathbb{R}$ (see [7]). One then proves that F_f has similar estimates of decay as the original f. One also knows that the ordinary (i.e., Euclidean) Fourier transform of F_f is equal to $\hat{f}$, where $\hat{f}$ is as defined in this section. One then concludes from Hardy's theorem for $\mathbb{R}$ that $F_f \equiv 0$. Since $f \mapsto F_f$ is one-to-one, the theorem would follow. In spirit, this approach is similar to using the Radon transform in proving Hardy's theorem for $\mathbb{R}^n$, given the result for $\mathbb{R}$.)

4. Eigenfunction Expansions on Compact Manifolds

In this section M denotes a compact, connected, real-analytic Riemannian manifold of dimension n. Let L be an elliptic, positive, selfadjoint differential operator on M of order l and with real-analytic coefficients. We quickly recall some basic facts about eigenfunction expansions for the pair (M, L). Let dx denote the canonical volume element on M. One knows that there exists an infinite subset $\Lambda \subseteq \mathbb{R}^+ \cup \{0\}$, $\Lambda = \{0 \leq \lambda_1 < \lambda_2 < \cdots\}$, $\lambda_i \to \infty$, such that $L^2(M, dx) = \bigoplus^{\perp} V_{\lambda_i}$, where $0 < \dim V_{\lambda_i} = n_i < \infty$ and $V_{\lambda_i} = \{f : Lf = \lambda_i f\}$, $\bigoplus^{\perp}$ denotes orthogonal direct sum, and each function in V_{λ_i} is real-analytic, $i = 1, 2, \ldots$. If $\lambda \in \Lambda$, let P_λ denote the orthogonal projection of $L^2(M)$ onto the finite-dimensional space V_λ. Thus the eigenfunction expansion for the pair (M, L) is given by

$$f \overset{L^2}{=} \sum_{\lambda \in \Lambda} P_\lambda f, \qquad f \in L^2(M).$$

(For details of the above, see [9].)

Before stating and proving an analogue of (UP) in this set up, we introduce the notion of a function being concentrated in a neighbourhood of $x_o \in M$. Let d denote the distance on M induced by the Riemannian structure. Since M is compact, we may assume by renormalizing the metric if necessary that $\sup_{x \in M} d(x, x_o)$ is 1. Let f be in $L^2(M)$. We will think of f as being concentrated around x_o, if, for some positive α, it satisfies

$$\left(\int_{M \setminus B(x_o, r)} |f|^2 \right)^{\frac{1}{2}} \leq C\, e^{-1/(1-r)^\alpha} \tag{4.2}$$

for some constant C.

Remark 5. The motivation for this notion of concentration is provided by the remark at the end of §2. If $r \to 1$, notice that the $RHS \to 0$ very fast. In particular, if f is supported in $B(x_o, r_o)$, $r_o < 1$, then the above condition is guaranteed. □

To prove an analogue of (UP) for the pair (M, L) we need the following result.

Theorem 6.

Let $f \in L^2(M)$ satisfy the condition $\|P_\lambda f\|_2 \leq A e^{-\lambda^{1/l}}$, $\lambda \in \Lambda$, for some positive constant A. Then f is real-analytic.

Proof. For $\lambda \in \mathbb{R}^+$, let $N(\lambda) = \sum_{\lambda_i \leq \lambda} n_i$ and $M(\lambda) =$ Cardinality of $(\Lambda \cap [0, \lambda])$. Then clearly $M(\lambda) \leq N(\lambda)$. By (a weak form of) Weyl's estimate (see [9]), one has $N(\lambda) \leq C\lambda^{n/l}$, where C is a constant. Thus, also, $M(\lambda) \leq C\lambda^{n/l}$. First we observe that the rapid decay of $\|P_\lambda f\|_2$ easily implies via standard Sobolev theory that $f \in C^\infty(M)$. (Actually one should say $f = g$ a.e., where $g \in C^\infty(M)$.) We now prove that f is actually real-analytic on M. From the fact that

$$f \overset{L^2}{=} \sum_{\lambda \in \Lambda} P_\lambda f \quad \text{and} \quad P_\lambda f \in V_\lambda$$

we have, for any $k \in \mathbf{Z}^+$,

$$L^k f \overset{L^2}{=} \sum_{\lambda \in \Lambda} \lambda^k\, P_\lambda f,$$

from which it follows that

$$\|L^k f\|_2^2 = \sum_{\lambda \in \Lambda} \lambda^{2k} \|P_\lambda f\|_2^2 \leq A^2 \sum_{\lambda \in \Lambda} \lambda^{2k}\, e^{-2\lambda^{1/l}}.$$

This last expression is just $A^2 \int_0^\infty \lambda^{2k} e^{-2\lambda^{1/l}}\, dM(\lambda)$, where $M(\lambda)$ is the function introduced earlier. An integration by parts yields that this is just $A^2 \int_0^\infty \frac{d}{d\lambda}(\lambda^{2k} e^{-2\lambda^{1/l}}) M(\lambda)\, d\lambda$, which is $\leq C \int_0^\infty \frac{d}{d\lambda}(\lambda^{2k} e^{-2\lambda^{1/l}}) \lambda^{n/l}\, d\lambda$. This integral can be easily estimated, and it can be shown that

$(\int_0^\infty \frac{d}{d\lambda}(\lambda^{2k} e^{-2\lambda^{1/l}})\lambda^{n/l}\, d\lambda)^{\frac{1}{2}} \leq C^{k+1}(kl)!$ for some positive constant C. Thus, $\| L^k f \|_2 \leq C^{k+1}(kl)!$. From the Kotake–Narasimhan theorem (see [13, Theorem 3.8.9]) it follows that f is real-analytic on M. □

The following is an analogue of (UP) for the pair (M, L).

Corollary 7.

Suppose $f \in L^2(M)$ *satisfies the hypothesis of Theorem 6 and condition* (4.2). *Then* $f \equiv 0$.

Proof. Since f satisfies the hypothesis of the above theorem, we know that f is real-analytic on M. Now let y_o be a point on M such that $d(x_o, y_o) = 1$. Since M is compact and $\sup_{x \in M} d(x_o, x) = 1$, there exists such a point y_o. The condition (4.2) easily translates to

$$\left(\int_{B(y_o, r)} |f|^2 \right)^{\frac{1}{2}} \leq A\, e^{-1/r^\alpha}. \tag{4.3}$$

On the other hand, if we consider a local coordinate system around y_o, and if some derivative $(D^\alpha f)(y_o) \neq 0$, then one can easily show by using the Taylor expansion of f around y_o that an estimate like (4.3) cannot hold. Thus $(D^\alpha f)(y_o) = 0,\ \forall \alpha$. However, since f is real-analytic and M is a connected manifold, this easily implies $f \equiv 0$. □

Notice that the proof uses the ellipticity of L explicitly in several places, such as the eigenfunction expansion, Weyl's estimate, and the Kotake–Narasimhan theorem.

Remark 8. The above may not necessarily be the "best possible" result. However we are really only interested in establishing that a reasonable version of (UP) holds in this rather general set up. □

For different generalizations of the uncertainty principles see [6]. For a more quantitative version of an uncertainty principle for the eigenfunction expansion on a sphere, see [16].

5. Eigenfunction Expansions for Certain Elliptic Operators on $\mathbb{R}^n$

In this section we consider a couple of differential operators on $\mathbb{R}^n$ that are closely related to the Laplacian and prove that the analogue of (UP) holds in both cases. In both cases that we are about to consider, the eigenfunction expansions are discrete.

5.1. The Hermite Operator

Consider the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$, where Δ is the usual Laplacian. For each multi-index $\mu = (\mu_1, \dots, \mu_n)$, let ϕ_μ be the normalized Hermite function on $\mathbb{R}^n$ (see [17] for details). Then one knows that $H\phi_\mu = (2|\mu| + n)\phi_\mu$; and the associated eigenfunction expansion for the pair $(\mathbb{R}^n, H)$ is given by

$$f \overset{L^2}{=} \sum_\mu (f, \phi_\mu)\phi_\mu,$$

where $(\cdot, \cdot)$ denotes the usual inner product in $L^2(\mathbb{R}^n)$. With this notation one can prove the following result.

Theorem 9.

Let f *be a measurable function on* $\mathbb{R}^n$ *such that* $|f(x)| \leq Ce^{-\alpha|x|^2}$, $x \in \mathbb{R}^n$, *and* $|(f, \phi_\mu)| \leq Ce^{-\beta(2|\mu|+n)}$, $\mu \in \mathbb{N}^n$, *for some positive constants* α *and* β. *If* $\alpha \tanh \beta > \frac{1}{2}$, *then* $f = 0$ *a.e.* (*In the above* $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ *but for a multi-index* μ, $|\mu| = \mu_1 + \cdots + \mu_n$.)

Proof. The Hermite functions satisfy the generating function identity

$$\sum_{\mu} r^{|\mu|}\phi_\mu(x)\,\phi_\mu(y) = \pi^{-\frac{n}{2}}(1 - r^2)^{-\frac{n}{2}}\, e^{-\frac{1}{2}\frac{1+r^2}{1-r^2}(|x|^2+|y|^2)+\frac{2r}{1-r^2}x\cdot y}.$$

Also, one knows that the Hermite functions are eigenfunctions of the usual Fourier transform, that is, $\hat{\phi}_\mu = (-i)^{|\mu|}\phi_\mu$.

Thus, if

$$f \overset{L^2}{=} \sum (f, \phi_\mu)\phi_\mu, \tag{5.4}$$

one has

$$\hat{f} \overset{L^2}{=} \sum (-i)^{|\mu|}(f, \phi_\mu)\,\phi_\mu. \tag{5.5}$$

The very rapid decay of (f, ϕ_μ) and sup norm estimates on ϕ_μ easily show that (5.4) actually converges uniformly and so we may assume f is continuous and that (5.4) holds pointwise. The same holds for (5.5). Thus $\hat{f}(x) = \sum(-i)^{|\mu|}(f, \phi_\mu)\phi_\mu(x)$, where the series converges absolutely and uniformly.

An application of Cauchy–Schwarz yields

$$|\hat{f}(x)| \le \left(\sum_\mu |(f, \phi_\mu)|\right)^{\frac{1}{2}} \left(\sum_\mu |(f, \phi_\mu)||\phi_\mu(x)|^2\right)^{\frac{1}{2}}.$$

The hypothesis on (f, ϕ_μ) gives

$$|\hat{f}(x)| \le A_\beta \left(\sum_\mu e^{-\beta(2|\mu|+n)}|\phi_\mu(x)|^2\right)^{\frac{1}{2}}.$$

Taking $r = e^{-2\beta}$ in the generating function identity, one easily obtains the estimate

$$|\hat{f}(x)| \le C_\beta\, e^{-\frac{1}{2}(\tanh\beta)|x|^2}.$$

The theorem now follows from Hardy's theorem for $\mathbb{R}^n$. □

5.2. The Special Hermite Operator on $\mathbb{C}^n$

We now consider the Special Hermite operator L on $\mathbb{C}^n(=\mathbb{R}^{2n})$ given by $L = -\Delta + \frac{1}{4}|z|^2 - i\sum_{j=1}^n (x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j})$, where Δ is the usual Laplacian on $\mathbb{C}^n(=\mathbb{R}^{2n})$ and $z = (x, y) \in \mathbb{R}^{2n} = \mathbb{C}^n$. Define the function $\phi_k(z)$ on $\mathbb{C}^n$ by

$$\phi_k(z) = \left(\frac{2^{1-n}\,k!}{(k+n-1)!}\right)^{\frac{1}{2}} L_k^{n-1}\left(\frac{1}{2}r^2\right) e^{\frac{-1}{4}r^2}, \qquad r = |z|,$$

where $L_k^{n-1}(r)$ are the Laguerre polynomials of type $(n-1)$. Then one knows that $L\phi_k = (2k+n)\phi_k$. If $L^2_{\text{rad}}(\mathbb{C}^n)$ denotes radial functions in $L^2(\mathbb{C}^n)(= L^2(\mathbb{R}^{2n}))$, then for $f \in L^2_{\text{rad}}(\mathbb{C}^n)$ one has the eigenfunction expansion $f \overset{L^2}{=} \sum_k (f, \phi_k)\,\phi_k$. The ϕ_k's are also eigenfunctions of the symplectic Fourier transform $\mathcal{F}_s$ with eigenvalue $(-1)^k$ and one knows that $(\mathcal{F}_s f)(z) = \hat{f}(\frac{-y}{2}, \frac{x}{2})$, where $z = (x, y) \in \mathbb{R}^{2n}$ and $\hat{f}$ denotes the usual Fourier transform on $\mathbb{R}^{2n}$. Using this and the generating function identity for ϕ_k one can argue as in §5.1 and obtain the following theorem.

Theorem 10.

Let $f \in L^2_{\text{rad}}(\mathbb{C}^n)$. Suppose

$$|f(z)| \le Ce^{-\alpha|z|^2}, \qquad z \in \mathbb{C}^n,$$

and

$$|(f, \phi_k)| \leq Ce^{-\beta(2k+n)}, \qquad k \in \mathbb{N},$$

for some positive constants α *and* β. *If* $\alpha \tanh \frac{\beta}{2} > \frac{1}{4}$, *then* $f = 0$ *a.e.*

6. Concluding Remarks

Thus, the preceding discussion gives concrete evidence of the folklore theorem that Uncertainty Principles are a general feature of eigenfunction expansions. Elsewhere, many authors (for example, [1, 5, 11, 15]) have established various uncertainty principles for the group-theoretic Fourier transform on certain Lie groups. This is not altogether surprising in view of what has been said in this paper because matrix elements of a unitary representation of a Lie group G are generally eigenfunctions of certain natural differential operators attached to these Lie groups and are elliptic when restricted to suitable subclass of functions. (For instance, the Casimir operator on a semisimple Lie group G, when restricted to K-invariant functions, where K is a maximal compact subgroup, can be thought of as an elliptic operator on G/K.)

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