(No. of Printed Pages: 2)

Indian Statistical Institute

M.Tech. in Computer Science (2025-26)

End Semester Examination 2025

Paper: **Linear Algebra** — Date: **17/11/2025**

Time : **3 Hrs.** — Full Marks : **50**

Note: Symbols used have their usual meaning. *The figures in the margin indicate full marks.*

Answer any FIVE from the following questions. $5 \times 10 = 50$

1. (a) Let A be a 5×3 matrix with columns a_1, a_2, a_3. If $b = a_1 + a_2 = a_2 + a_3$ then what can you conclude about the number of solutions of the linear system Ax = b? Explain.

   (b) Show that $W = \{f(x) \in \mathbb{P}_n[x]\,;\, f(1) = 0\}$ is a subspace of $\mathbb{P}_n[x]$. Find its dimension and a basis.

   (c) Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be linearly independent. If vector u is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_r$, while vector v is not, show that the vectors $tu + v, \alpha_1, \alpha_2, \ldots, \alpha_r$ are linearly independent for any scalar t.

   [3 + 4 + 4]

2. (a) Consider $\mathbb{P}_n[x]$ over $\mathbb{R}$. Define

   $$T(p(x)) = xp'(x) - p(x), p(x) \in \mathbb{P}_n[x].$$

   Show that T is a linear transformation on $\mathbb{P}_n[x]$. Find null of T and range of T and prove that

   $$N(T) \bigoplus R(T) = \mathbb{P}_n[x].$$

   (b) Let A be an $m \times n$ matrix of rank n and let $P = A(A^TA)^{-1}A^T$. Show that $Pb = b$ for every $b \in R(A)$ and if $b \in R(A)^{\perp}$ then $Pb = 0$.

   [6 + 4]

3. (a) Let A be an $n \times n$ matrix. Show that a vector x in $\mathbb{R}^n$ is an eigenvector belonging to A if and only if the subspace S of $\mathbb{R}^n$ spanned by x and Ax has dimension 1.

   (b) Let A be a diagonalizable matrix whose eigenvalues are all either 1 or -1. Show that $A^{-1} = A$.

   (c) Let A be an $n \times n$ matrix with an eigenvalue λ of multiplicity n. Show that A is diagonalizable if and only if $A = \lambda\mathbb{I}$.

   [3 + 3 + 4]

4. (a) Let A be a real $n \times n$ matrix with eigenvalue $\lambda_1 = a + bi$, where $a, b \in \mathbb{R}$ and $b \neq 0$. Let $z_1 = x + iy$, where $x, y \in \mathbb{R}^n$, be an eigenvector corresponding to λ_1. If $S = \text{Span}(x, y)$, then dim $S = 2$ and S is invariant under A.

(b) Let A be an $m \times n$ matrix with singular value decomposition $U\Sigma V^T$. Show that, for any vector $x \in \mathbb{R}^n$,

$$\sigma_n\|x\|_2 \leq \|Ax\|_2 \leq \sigma_1\|x\|_2,$$

where σ_1 and σ_2 denote the largest and smallest singular values of A, respectively.

[5 + 5]

5. (a) Let $p_0, p_1, \ldots, p_n$ be a sequence of orthogonal monic polynomials. Then show that

$$p_{n+1}(x) = (x - \beta_{n+1})p_n(x) - \alpha\gamma_n p_{n-1}(x)(n \geq 0),$$

where $p_{-1}(x)$ denotes the zero polynomial, $\gamma_0 = 1$, $\beta_n = \dfrac{\langle p_{n-1}, xp_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle}$ and $\gamma_n = \dfrac{\langle p_n, p_n\rangle}{\langle p_{n-1}, p_{n-1}\rangle}$ for $n \in \mathbb{N}$.

(b) Let $p_0(x), p_1(x)$, and $p_2(x)$ be orthogonal monic polynomials with respect to the inner product

$$\langle p(x), q(x)\rangle = \int_{-1}^{1} \frac{p(x)q(x)}{1+x^2}\, dx.$$

Using (a) calculate $p_1(x)$ and $p_2(x)$.

[5 + 5]

6. (a) Let A be an $m \times n$ matrix with singular value decomposition $U\Sigma V^T$, and suppose that A has rank r , where $r < n$. Show that $\{v_1, v_2, \ldots, v_r\}$ is an orthonormal basis for $R(A^T)$.

(b) Let A be an $n \times n$ matrix with singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$ and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that $|\lambda_1\lambda_2 \ldots \lambda_n| = \sigma_1\sigma_2 \ldots \sigma_n$.

[5 + 5]

7. (a) Let $A = Q_1R_1 = Q_2R_2$, where Q_1 and Q_2 are orthogonal and R_1 and R_2 are both upper triangular and nonsingular.

i. Show that $Q_1^TQ_2$ is diagonal.

ii. How do R_1 and R_2 compare? Explain.

(b) Let $A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $u_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

i. Compute u_1, u_2, u_3, and u_4, using the power method.

ii. Explain why the power method will fail to converge in this case.

[(2 + 2) + (4 + 2)]