Mazur's Intersection Property and its Variants

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Mazur's Intersection Property and its Variants

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Dedicated to My Family and Prof. P. Bandyopadhyay

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Chapter 1

Introduction

The two quintessential objects to understand the geometry of a Banach space are the hyperplanes and the unit ball. One basic but crucial factor that every Banach space enjoys is that every closed, convex and bounded set in the space is the intersection of closed hyperplanes containing the set (Thanks to the Hahn-Banach theorem!). If, in the above property, we replace hyperplanes by closed balls, then Euclidean spaces have that property. That is, in Euclidean spaces, every closed, convex, and bounded set is an intersection of closed balls. This motivated S. Mazur [36] to introduce what he called the Property (I), and is now called the Mazur Intersection Property (MIP). Defined as

> *A Banach space X is said to have MIP if every closed, bounded, and convex set in X is the intersection of the closed balls containing it;*

it has been a fascinating property from the geometric point of view. It's easy to observe that not all Banach spaces have MIP. Since its introduction, attempts have been made to link this property with other convexity or smoothness properties. For instance, S. Mazur [36] himself proved that if X is Fréchet smooth, then X has MIP. Phelps in [39], discussed a sufficient condition for MIP in terms of the density of w*-strongly exposed points of the dual unit ball $B(X^*)$ in the unit sphere $S(X^*)$. This led to an important question concerning the converse of his necessary condition. That is, does MIP imply that w*-strongly exposed points of $B(X^*)$ are dense in $S(X^*)$? This question is still open. In fact, existence of a w*-strongly exposed point of $B(X^*)$ in a space with MIP is yet to be answered. Phelps also discussed a necessary condition for MIP in terms of a behaviour of the duality map. Phelps [39] showed that if X has MIP, then any selection of the duality map takes a dense subset of $S(X)$ to a dense subset of $S(X^*)$. He also characterised MIP for finite dimensional spaces in terms of the density of the extreme points of $B(X^*)$ in $S(X^*)$.

But it was the year 1978, which saw a breakthrough in this context. Giles, Gregory and Sims [21] characterised MIP. Following was their remarkable result:

Theorem 1.0.1. *[21, Theorem 2.1]*

(a) X has MIP.

(b) The set of w^*-denting points of $B(X^*)$ is dense in $S(X^*)$.

(c) The duality map is quasi-continuous. That is, for every $\varepsilon > 0$ and $f \in S(X^*)$, there exists $\delta > 0$ and $x \in S(X)$ such that $\|x - y\| < \delta$ implies $\|f - f_y\| < \varepsilon$ for all $f_y \in D(y)$.

(d) Any selection of the duality map takes a dense subset of $S(X)$ to a dense subset of $S(X^*)$.

(e) For every $\varepsilon > 0$ and $f \in S(X^*)$, there exists $x \in M_\varepsilon$ such that

$$\|f - f_x\| < \varepsilon$$

for all $f_x \in D(x)$, where M_ε is the set of points of ε-Fréchet differentiability.

This characterisation helped mathematicians to observe a lot of connections between MIP and other nice properties. For instance, from condition (e) above (and the Bishop-Phelps Theorem), it is easy to obtain Mazur's result that if X is Fréchet smooth, then X has MIP.

Now, separable Asplund spaces have dual LUR renorming which implies Fréchet differentiability. So, X being separable and Asplund space implies X has an equivalent MIP renorming. On the other hand, condition (d) above suggests that if X is separable and has MIP, then X^* is separable and therefore, X is Asplund.

The beautiful connection between MIP, the Asplund property and the Fréchet smooth renorming is only available in the case of separable Banach spaces. Jiménez-Sevilla and Moreno in [29] show that the example of Haydon [25] of an Asplund space without any Fréchet smooth renorming has a renorming with MIP. So, MIP does not imply Fréchet smooth renorming in general.

The same authors in [30] disproved the connection between the Asplund property and MIP in case of non-separable Banach spaces. They prove that any Banach space can be isometrically embedded in a Banach space with MIP. So, any non-Asplund space can also be embedded in a space with MIP. This bigger space cannot be Asplund, since being Asplund is a hereditary property. Also, in the same paper, they show that the Asplund space $C(K)$ of continuous functions over a Kunen compact set K cannot be renormed to have MIP. On the other hand, Bačák and Hájek [7] show that under ZFC axiom with an additional Martin's Maximum axiom, any Asplund space with density character ω_1 has an MIP renorming.

Now, it's a usual trend in the Banach space theory to uniformise properties and observe the impact. Since separable Banach spaces with MIP have dual LUR renorming, it makes sense to ask if this connection respects uniformity. That is, does the uniform version of MIP imply uniformly rotund renorming? Whitfield and Zizler in [47] introduced a uniform version of MIP called the Uniform Mazur Intersection Property (UMIP). Since then, the problem "does UMIP imply uniformly rotund renorming?" remains unsolved. Whitfield and Zizler obtained characterisations similar to the ones obtained in [21], in the same article, except for the w*-denting condition. This probably hindered the study of UMIP for a long time and there was no paper on UMIP after [47], until 2021. And thus the (still unsolved!) problem saw no progress. A breakthrough was achieved by Chen and Lin when they introduced the idea of semidenting points in [10] and showed that X has MIP if and only if all points of $S(X^*)$ are w*-semidenting points of $B(X^*)$. Giles in [20] gave a different proof of the same characterisation. These helped the authors in [3] to obtain the missing piece in the characterisation of UMIP in terms of uniform w*-semidenting points of $B(X^*)$.

Chapter 2 of the current thesis presents this breakthrough result from [3], namely, X has UMIP if and only if all points of $S(X^*)$ are uniformly w*-semidenting points. In the process, we present simpler proofs of some characterisations in [47]. We also introduce a w*-version of UMIP in the spirit of [21] and obtain similar characterisations.

A close strengthening of MIP is what we call the "Hyperplane Mazur Intersection Property" (H-MIP) (the property "$\mathcal{P} = \mathcal{H}$" discussed in [20]) where we consider separation of a closed convex bounded sets from a hyperplane by a closed ball.

Interestingly, a Banach space X with H-MIP can be characterised as the space in which every element of $S(X^*)$ is a w*-denting point of $B(X^*)$ [10, Proposition 5.3]. Now, it follows from [40, Theorem 1] that this, in turn, implies that X^* has a dual LUR norm. Notice that this result does not require any separability assumption. It also follows from [26, Theorem 3.1] and [27, Theorem 4] that this implies X is Fréchet differentiable. So, H-MIP implies Fréchet smoothness. This relationship motivates us to look at the hyperplane version of UMIP in connection with the uniform smoothness in X. In Chapter 3, we prove that the uniform version of H-MIP (H-UMIP) indeed characterises uniform smoothness of X.

Dutta and Lin [17] introduced a modulus characterising a denting point. Their work inspired us to define a similar modulus which characterises a semidenting point. This leads us to a modulus for UMIP. We hope that this would provide us with some new insights in dealing with UMIP.

Also, we introduce various notions of uniformly denting points which are equivalent to the notion discussed by Dutta and Lin [17]. This helps us in characterising the H-UMIP in terms of uniformly w*-denting points and eventually uniform smoothness. This work has appeared in [24].

Another property we study in Chapter 3 is the asymptotic notion of uniform smoothness. Asymptotic uniform convexity and smoothness have received a lot of attention in the literature of Banach space theory. [31] is a good source of information on these properties where the authors' concern is the existence of points of ε-Fréchet differentiability of Lipschitz maps. Authors in [23] and [32] also spend a good deal of time on the property UKK* (readers may refer to the mentioned source for the original definition) which is equivalent to w*-AUC. Their focus is mainly concerned with renormings of Banach spaces with UKK* in connection with Szlenk index. Raja in [41] deals with the existence of an AUS norm using an ordinal type of index. Other than these, [16, 37, 35, 38] are some of the important works in the study of AUS.

In Chapter 3, we characterise AUS in terms of asymptotic notions of uniform w*-denting points and a ball separation property similar to H-UMIP.

Soon after [21], there appeared several papers dealing with generalisations of MIP, basically studying when members of a given family of closed bounded convex sets are intersection of balls. For example,

(a) $\mathcal{K} = \{$all compact convex sets in $X\}$,

(b) $\mathcal{W} = \{$all weakly compact convex sets in $X\}$,

(c) $\mathcal{F} = \{$all compact convex sets in X with finite affine dimension$\}$.

For a family $\mathcal{C}$ of closed bounded convex sets, let us say that X has $\mathcal{C}$-MIP if every $C \in \mathcal{C}$ is the intersection of closed balls containing it.

Whitfield and Zizler in [46] prove that X has $\mathcal{K}$-MIP (they called it property CI) if the cone generated by the set of extreme points of $B(X^*)$ is dense in X^* in the topology of uniform convergence over elements of $\mathcal{K}$. In [42], Sersouri proves that $\mathcal{K}$-MIP is, in fact, equivalent to the above condition. In another article [43], Sersouri showed that X has $\mathcal{F}$-MIP if and only if the cone generated by the set of extreme points of $B(X^*)$ is w*-dense in X^*. Zizler in [48] studied $\mathcal{W}$-MIP.

Bandyopadhyay in [2] attempts to characterise $\mathcal{C}$-MIP for a generalised class $\mathcal{C}$. He obtains some sufficient and some necessary conditions analogous to Theorem 1.0.1. See Theorem 4.2.3 in Chapter 4. Next in line is [11] where Chen and Lin characterise $\mathcal{C}$-denting points (Definition 1.0.27) for a suitable family $\mathcal{C}$ of subsets of X as

Theorem 1.0.2. *[11, Theorem 1.3] $f \in S(X^*)$ is a $\mathcal{C}$-denting point if and only if whenever for any $C \in \mathcal{C}$, $\inf f(C) > 0$, then there exists a closed ball B containing C such that $\inf f(B) > 0$.*

And use it to obtain

Theorem 1.0.3. *[11, Theorem 1.10] For a "compatible" class $\mathcal{C}$ of subsets of X, consider the following statements:*

(a) The cone generated by the set of $\mathcal{C}$-denting points is dense in X^* in the topology τ_C of uniform convergence on elements of $\mathcal{C}$.

(b) X has $\mathcal{C}$-MIP.

Then (a) $\implies$ (b). And if every w-slice of $B(X^*)$ contains a $\mathcal{C}$-denting point, then the two statements are equivalent.*

More recently, Cheng and Dong in [12] introduced the notion of $\mathcal{C}$-semidenting points [12, Definition 1.1] to characterise $\mathcal{C}$-MIP [12, Theorem 2.3]. But, their notion of a $\mathcal{C}$-semidenting point doesn't fit naturally as a generalisation of either w*-semidenting points or $\mathcal{C}$-denting points. We propose an alternative definition (Definition 1.0.29). To avoid conflict of notions, we will call this strong $\mathcal{C}$-semidenting point. However, we could not characterise $\mathcal{C}$-MIP using strong $\mathcal{C}$-semidenting points.

It was Vanderwerff [45] who discussed a variant of $\mathcal{C}$-MIP which suits our purpose. Instead of considering just a family $\mathcal{C}$, he discussed representation of sets of the form $\overline{A + \lambda B(X)}$, for $A \in \mathcal{C}$ and $\lambda \geq 0$, as intersection of closed balls. It is easy to see that this is stronger than $\mathcal{C}$-MIP. We call this the strong $\mathcal{C}$-MIP and prove that all points of $S(X^*)$ being strong $\mathcal{C}$-semidenting characterises strong $\mathcal{C}$-MIP. Chapter 4 deals with this relationship. We will also discuss the uniform version of $\mathcal{C}$-MIP in the same light and obtain analogous results.

We have already mentioned the connection between MIP and the Asplund property for separable Banach spaces. Also, using notions discussed by Bourgin [9, pg. 120], we define $\mathcal{C}$-Asplund property for a compatible class $\mathcal{C}$. We obtain a desired connection between $\mathcal{C}$-MIP and $\mathcal{C}$-Asplund property. For instance, in Chapter 4, we prove that for a compatible class $\mathcal{C}$ if X is separable, then X having $\mathcal{C}$-MIP implies X is $\mathcal{C}$-Asplund.

In geometry of Banach spaces, one often encounters properties that are isometric, and not isomorphic. That is, they depend on the specific norm and are not shared by all equivalent norms. However, among these isometric properties, there are some that satisfy the following: If a Banach space X has one norm with the property P, then "almost every" (residual with respect to some suitable metric) equivalent norm on X also has property P. For example, Fabian et al. in [18], proved that rotundity, locally uniform rotundity and uniform rotundity as well as having duals with these properties are such "residual" properties. Georgiev in [19] proved the residuality of MIP norms. Chen and Lin in [10] uses the ball separation criterion for w*-denting points to prove residuality of normalised w*-denting points, from which it can be shown that H-MIP also is a residual property. We will take this study forward in Chapter 5 and obtain residuality of AUS, uniformly smooth and UMIP norms. The main highlight in proving the residuality is the ball separation linked to these properties.

The residuality of AUS norms is shown to hold in [16, Remark 4.5] but only for reflexive spaces. We prove this for general Banach spaces.

Whitfield and Zizler in [47, Proposition 1] show that uniform convexity of X^* implies UMIP in X and in finite dimensional spaces, MIP and UMIP are the same. They also gave an example [47, Proposition 2] of an infinite dimensional separable Banach space which has MIP but not UMIP. In this example, they consider ℓ_2-sum of $\ell^2_{p_n}$ where $p_n > 1$ and $p_n \to 1$. Since the dual of an ℓ^2_p space for $p > 1$ is uniformly convex, it has UMIP. So, this example shows that for $1 < p < \infty$, UMIP is not in general stable under ℓ_p-sum contrary to MIP behaviour. In Chapter 6, we analyse conditions for the stability of UMIP under ℓ_p-sums for $1 < p < \infty$. We show, in particular, that $\ell_p(X)$ has the UMIP if and only if X has the UMIP.

As already noted, Fréchet smoothness implies MIP. And since Fréchet smoothness is hereditary, it actually implies hereditary MIP, that is, all subspaces of X also has MIP. How about the converse? It is easy to show that hereditary MIP implies Gâteaux smoothness. Here we show that it doesn't imply Fréchet smoothness by producing a counterexample building on an interesting example by Borwein and Fabian [8].

In the end, we discuss some questions that are related to our study and still not solved. We hope our work contributes in developing a better understanding towards these questions.

LET'S BEGIN OUR JOURNEY!

Notations, Terminologies and Basic Definitions

We consider real Banach spaces only. Let X be a Banach space. For $x \in X$ and $r > 0$, we denote by $B(x, r)$ *the open ball* $\{y \in X : \|x - y\| < r\}$ and by $B[x, r]$ *the closed ball* $\{y \in X : \|x - y\| \leq r\}$. We denote by $B(X)$ the *closed unit ball* $\{x \in X : \|x\| \leq 1\}$, by $B(X)^o$ the *open unit ball* and by $S(X)$ the *unit sphere* $\{x \in X : \|x\| = 1\}$.

For a bounded set $C \subseteq X$ and $x \in X$, let

(a) $d(x, C) = \inf\{\|x - z\| : z \in C\}$, the distance function; and

(b) $diam(C) = \sup\{\|x - y\| : x, y \in C\}$, the diameter of C.

For a bounded subset $C \subseteq X$, define a seminorm $\|\cdot\|_C$ on X^* as

$$\|f\|_C := \sup\{|f(c)| : c \in C\}, f \in X^*.$$

For $r > 0$, $f \in X^*$, we can analogously define an open $\|\cdot\|_C$-ball as

$$B_C(f, r) := \{g \in X^* : \|f - g\|_C < r\}.$$

And for a bounded subset $A \subseteq X^*$, we can define

$$diam_C(A) := \sup\{\|f - g\|_C : f, g \in A\}.$$

We denote by $\text{co}(A)$ (resp. $\text{aco}(A)$) the convex (resp. absolutely convex) hull of A.

For $A \subseteq X$, the cone generated by A is $\text{cone}(A) = \{\lambda a : \lambda \geq 0, a \in A\}$.

Geometry of a Banach space is mainly concerned with the study of the shape of its unit ball and the unit sphere in terms of convexity and smoothness. In the definitions below, we capture some geometric notions to study the shape of the unit ball. We start with the most basic notion, that is, extreme points.

Definition 1.0.4. For a convex set $C \subseteq X$, $x \in C$ is an *extreme point* of C if $y, z \in C$ and $x = \dfrac{y+z}{2}$ implies $y = z = x$.

Definition 1.0.5. A Banach space X is *rotund* (R) or *strictly convex* if all points of $S(X)$ are extreme points of $B(X)$.

Now, we look at a slightly stronger version of an extreme point which has "some continuity" added in it.

Definition 1.0.6. A *slice* of a closed, bounded and convex set $C \subseteq X$ determined by $f \in S(X^*)$ is a set of the form

$$S(C, f, \delta) := \{x \in C : f(x) > \sup_{c \in C} f(c) - \delta\}$$

for some $0 < \delta < 1$. For $x \in S(X)$ and $C \subseteq X^*$, $S(C, x, \delta)$ is called a w*-slice of C.

In Chapter 3, we will use a slightly different notation to fit in the context.

Definition 1.0.7. We say that $x \in S(X)$ is a *denting point* of $B(X)$ if for every $\varepsilon > 0$, x is contained in a slice of $B(X)$ of diameter less than ε.

A w*-denting point of $B(X^*)$ is defined similarly using w*-slices.

Definition 1.0.8. We say that $x \in S(X)$ is a *semidenting point* of $B(X)$ if for every $\varepsilon > 0$, there exists a slice $S(B(X), f, \delta) \subseteq B(x, \varepsilon)$.

A w*-semidenting point of $B(X^*)$ can be defined similarly in terms of w*-slices.

Observe that the difference between a semidenting and a denting point is that in case of a semidenting point, the point itself need not belong to the slice.

If, in the above definition of a denting point, the same functional f defines slices of diameter less than ε for all $\varepsilon > 0$, then x is called a *strongly exposed point* and f is called the *strongly exposing functional*. Similarly, one can define *w*-strongly exposed points* and *w*-strongly exposing functionals*.

It's easy to observe that a denting point is an extreme point. Indeed,

Theorem 1.0.9. *[34] x is a denting point if and only if x is an extreme point and a point of continuity (PC) of the identity map $Id : (B(X), weak) \to (B(X), norm)$.*

Analogously, for the w*-topology on X^*, we have

Theorem 1.0.10. *$f \in S(X^*)$ is an w*-denting point if and only if f is an extreme point and a w*-PC.*

Now, let's have a look at the notions of local uniform rotundity and uniform rotundity.

Definition 1.0.11. $x \in S(X)$ is called a *locally uniformly rotund* (LUR) point if for every $\varepsilon > 0$, there is $\delta > 0$ such that whenever for any $y \in S(X)$, $\dfrac{\|x+y\|}{2} > 1 - \delta$, then $\|x - y\| < \varepsilon$.

X is said to be LUR if all points of $S(X)$ are LUR points.

Definition 1.0.12. A Banach space X is said to be *Uniformly Rotund* (UR) if given $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in S(X)$,

$$\frac{\|x+y\|}{2} > 1 - \delta \implies \|x - y\| < \varepsilon.$$

For $0 < t < 1$, the quantity defined as:

$$\delta_X(t) := \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x-y\| \geq t \right\}$$

is called the *modulus of convexity.* Clearly, X is uniformly convex if and only if $\delta_X(t) > 0$ for all $t > 0$. It's easy to observe that UR $\implies$ LUR $\implies$ R.

Dual to convexity, there are various notions of smoothness, which we now look at.

Definition 1.0.13. Let U be an open subset of X and $x \in U$. Let $f : U \to \mathbb{R}$ be a convex function. f is said to be *Gâteaux differentiable* (also called Gâteaux smooth or smooth) at x if given $\varepsilon > 0$ and $y \in X$, there is $\delta > 0$ satisfying

$$\sup_{0<\lambda<\delta} \frac{f(x+\lambda y) + f(x - \lambda y) - 2f(x)}{\lambda} < \varepsilon.$$

If in the above definition, there is a $\delta > 0$ which works uniformly for all $y \in B(X)$, then f is said to be *Fréchet smooth* at x.

If f is (Fréchet) smooth at all x in the domain, then f is said to be (Fréchet) smooth.

Definition 1.0.14. If, for a Banach space $(X, \|\cdot\|)$, $\|\cdot\|$ is (Fréchet) smooth on $X \setminus \{0\}$, then $(X, \|\cdot\|)$ is called a *(Fréchet) smooth space.*

Definition 1.0.15. X is said to be *uniformly smooth* if X is Fréchet smooth uniformly over $x \in S(X)$.

Following characterisations of smoothness and Fréchet smoothness are widely used as definitions.

Definition 1.0.16. For $x \in S(X)$, let

$$D(x) = \{f \in S(X^*) : f(x) = 1\}.$$

The multivalued map D is called the duality map. Any selection of D is called a support mapping.

Theorem 1.0.17. *$x \in S(X)$ is a Fréchet smooth point if and only if $f \in D(x)$ is a w^*-strongly exposed point of $B(X^*)$ exposed by x.*

Theorem 1.0.18. *A Banach space X is a smooth if and only if for each $x \in S(X)$, $D(x)$ is a singleton.*

Theorem 1.0.19. *The following are equivalent :*

(a) *X is Fréchet smooth*

(b) *Whenever a sequence $\{g_n\} \subseteq S(X^*)$ is such that for some $x \in S(X)$, $g_n(x) \to 1$, then $\{g_n\}$ converges in norm.*

(c) *The duality map is single valued and norm-norm continuous.*

Following object related to differentiability would be quite useful in our discussion of MIP and UMIP.

Definition 1.0.20. *[47]* For $\varepsilon, \delta > 0$, define

$$M_{\varepsilon,\delta}(X) = \left\{ x \in S(X) : \sup_{0<\|y\|<\delta} \frac{\|x+y\| + \|x-y\| - 2}{\|y\|} < \varepsilon \right\}.$$

From the definition of Fréchet smoothness, we observe that $\bigcap_{\varepsilon>0}\bigcup_{\delta>0} M_{\varepsilon,\delta}(X)$ is the set of all Fréchet smooth points in $S(X)$.

The famous notion of an Asplund space is defined in terms of differentiability of convex continuous functions.

Definition 1.0.21. A Banach space X is said to be an *Asplund space* if for every open subset U of X and real-valued continuous convex function f on U, the set of points of Fréchet differentiability of f is a dense $\mathcal{G}_\delta$ subset of U.

A popular characterisation of an Asplund space, which is easier to handle, is that X is Asplund if and only if for every separable subspace Y of X, Y^* is separable.

Definition 1.0.22. A Banach space X is said to have the *Radon-Nikodým property* (RNP) if for every bounded subset $C \subseteq X$ and $\varepsilon > 0$, C has slices of diameter less than ε.

The Asplund property and the RNP are dual to each other: X is Asplund if and only if X^* has the RNP.

Following theorem relates uniform convexity and uniform smoothness via duality.

Theorem 1.0.23.
(a) *X is uniformly convex if and only if X^* is uniformly smooth.*

(b) *X is uniformly smooth if and only if X^* is uniformly convex.*

Coming to asymptotic uniform convexity and asymptotic uniform smoothness, consider the following quantities:

$$\begin{aligned}
\overline{\Delta}_X(\varepsilon) &= \inf_{\|x\|=1} \sup_{dim(X/Y)<\infty} \inf_{\|y\|\geq\varepsilon} (\|x+y\|-1) \quad \text{and} \\
\overline{\rho}_X(\varepsilon) &= \sup_{\|x\|=1} \inf_{dim(X/Y)<\infty} \sup_{\|y\|\leq\varepsilon} (\|x+y\|-1).
\end{aligned}$$

Definition 1.0.24. A Banach space X is called

(a) *asymptotically uniformly convex* (AUC) if for every $\varepsilon > 0$, $\overline{\Delta}_X(\varepsilon) > 0$.

(b) *asymptotically uniformly smooth* (AUS) if $\lim\limits_{\varepsilon\to 0} \dfrac{\overline{\rho}_X(\varepsilon)}{\varepsilon} \to 0$.

These notions when considered in X^* and the subspaces are w*-closed, then we obtain the notion of w*-AUC and w*-AUS.

To discuss $\mathcal{C}$-MIP for a class $\mathcal{C}$ of sets, we need some compatibility assumptions on this class.

Definition 1.0.25. We call a class $\mathcal{C}$ of bounded subsets of X a *compatible class* if

(I) $C \in \mathcal{C}$, $\alpha \in \mathbb{R}$ and $x \in X \implies \alpha C + x \in \mathcal{C}$, $C \cup \{x\} \in \mathcal{C}$;

(II) $C \in \mathcal{C}$ implies the closed absolutely convex hull of $C \in \mathcal{C}$.

(III) $C \in \mathcal{C}$ and $A \subseteq C$ implies $A \in \mathcal{C}$.

(IV) $C_1, C_2 \in \mathcal{C}$ implies $C_1 \cup C_2 \in \mathcal{C}$.

Here is the formal definition of $\mathcal{C}$-MIP.

Definition 1.0.26. For a compatible class $\mathcal{C}$, X is said to have the *$\mathcal{C}$-MIP* if every *closed convex* $C \in \mathcal{C}$ is the intersection of closed balls containing it.

Notice that the families $\mathcal{K}$, $\mathcal{W}$ and $\mathcal{F}$ mentioned earlier are not exactly compatible classes with our definition. One has to consider such sets and subsets thereof. However, the property $\mathcal{C}$-MIP remains the same.

Definition 1.0.27. [11] Given a compatible class $\mathcal{C}$, $f \in S(X^*)$ is called a *$\mathcal{C}$-denting point* of $B(X^*)$ if for each $A \in \mathcal{C}$ and $\varepsilon > 0$, there exists a w*-slice S of $B(X^*)$ such that $f \in S$ and $diam_A(S) < \varepsilon$.

Cheng and Dong in [12] introduced the following notion of $\mathcal{C}$-semidenting points.

Definition 1.0.28. [12, Definition 1.1] We say that $f \in S(X^*)$ is a *$\mathcal{C}$-semidenting point* of $B(X^*)$ if for each $A \in \mathcal{C}$ and $\varepsilon > 0$, there exists a w*-slice S of $B(X^*)$ such that $S \subseteq cone(B_A(f, \varepsilon))$.

And we propose the following alternative definition.

Definition 1.0.29. We say that $f \in S(X^*)$ is a strong $\mathcal{C}$-semidenting point if for each $A \in \mathcal{C}$ and $\varepsilon > 0$, there exists a w*-slice S of $B(X^*)$ such that $S \subseteq B_A(f, \varepsilon)$.

In order to discuss residuality of equivalent norms, we need a metric on the set of equivalent norms.

Let $\mathcal{B}$ be the family of unit balls determined by the set of norms on X equivalent to $\|\cdot\|$. For $B \in \mathcal{B}$, let us denote the corresponding norm on both X and X^* by $\|\cdot\|_B$. In this thesis, we will consider $\mathcal{B}$ equipped with the Hausdorff metric. That is, we define h on $\mathcal{B}$ as follows: for $B_1, B_2 \in \mathcal{B}$,

$$h(B_1, B_2) := \inf\{\varepsilon > 0 : B_1 \subseteq B_2 + \varepsilon B(X) \text{ and } B_2 \subseteq B_1 + \varepsilon B(X)\}.$$

Or, equivalently,

$$h(B_1, B_2) := \inf\{\varepsilon > 0 : B_1 \subseteq B_2 + \varepsilon B_2 \text{ and } B_2 \subseteq B_1 + \varepsilon B_1\}.$$

This metric is discussed in [11]. It is well known that $(\mathcal{B}, h)$ is a complete metric space.

A slightly different metric is considered in [18] and [19]. This is defined as

$$d(\|\cdot\|_1, \|\cdot\|_2) := \sup\left\{\big|\|x\|_1 - \|x\|_2\big| : x \in B(X)\right\}.$$

From the discussion before [19, Theorem 1], it is clear that the two metrics are equivalent.

Chapter 2

Uniform Mazur's Intersection Property

2.1 Introduction

As we have already noticed in Chapter 1 that w*-denting type of characterisation for UMIP was missing in [47] which we feel hindered its study. In this chapter, we show that a Banach space X has UMIP if and only if every $f \in S(X^*)$ is uniformly w*-semidenting point of $B(X^*)$, thus filling a long felt gap. Thus, we obtain a crucial characterisation of UMIP which we think is important in understanding its structure. In the process, we also present simpler proofs of some characterisations in [47]. Analogous to w*-MIP, we introduce the notion of w*-UMIP and obtain similar characterisations. Later in the chapter, we talk about "Hyperplane-MIP (H-MIP)" (The property "$\mathcal{P} = \mathcal{H}$" in [20]). A very short and direct proof of the fact that H-MIP implies Fréchet smoothness is given. For the discussion of this chapter, reader is referred to [3].

2.2 Characterisation of UMIP

Definition 2.2.1. [47] We say that a Banach space X has the *Uniform Mazur Intersection Property (UMIP)* if for every $\varepsilon > 0$ and $M \geq 2$, there is $K > 0$ such that whenever a closed convex set $C \subseteq X$ and a point $p \in X$ are such that $diam(C) \leq M$

and $d(p,C) \geq \varepsilon$, there is a closed ball $B \subseteq X$ of radius $\leq K$ such that $C \subseteq B$ and $d(p,B) \geq \varepsilon/2$.

Remark 2.2.2. In the original definition in [47], $M = 1/\varepsilon$.

We begin with a variant of Phelps' Parallel Hyperplane Lemma [20, Lemma 2.1]. We include the well-known proof as we will use a step of the proof later.

Lemma 2.2.3. *For a normed linear space X, if for $f, g \in S(X^*)$ and $\varepsilon > 0$, $\{x \in B(X) : f(x) > \varepsilon\} \subseteq \{x \in X : g(x) > 0\}$, then $\|f - g\| < 2\varepsilon$.*

Proof. By the given condition, if $x \in B(X)$ and $g(x) = 0$, then $|f(x)| \leq \varepsilon$. That is $\|f|_{\ker(g)}\| \leq \varepsilon$. By the Hahn-Banach extension theorem, there exists $h \in X^*$ such that $\|h\| \leq \varepsilon$ and $h \equiv f$ on $\ker(g)$. It follows that $f - h = tg$ for some $t \in \mathbb{R}$. Then $\|f - tg\| = \|h\| \leq \varepsilon$. Now, if $y \in \{x \in B(X) : f(x) > \varepsilon\}$, then $g(y) > 0$ and

$$\varepsilon < f(y) \leq (f - tg)(y) + tg(y) \leq \|f - tg\| + tg(y) \leq \varepsilon + tg(y).$$

It follows that $t > 0$. And

$$|1 - t| \leq \big|\|f\| - \|tg\|\big| \leq \|f - tg\| \leq \varepsilon.$$

Thus,

$$\|f - g\| \leq \|f - tg\| + |1 - t| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

$\square$

The theorem below is equivalent to [11, Theorem 2.1] with a much simpler proof.

Theorem 2.2.4. *Let X be a Banach space. Let $A \subseteq X^{**}$ be a bounded set. Then there exists a closed ball $B^{**} \subseteq X^{**}$ with centre in X such that $A \subseteq B^{**}$ and $0 \notin B^{**}$ if and only if $d(0, A) > 0$ and there is a w^*-slice of $B(X^*)$ contained in*

$$\{f \in B(X^*) : x^{**}(f) > 0 \text{ for all } x^{**} \in A\}.$$

Proof. Let $A \subseteq B^{**}[x_0, r]$ and $0 \notin B^{**}[x_0, r]$. Then $\|x_0\| > r$. Clearly, $d(0, A) \geq \|x_0\| - r > 0$.

Let $S = \{f \in B(X^*) : f(x_0) > r\}$. Then S is a w*-slice of $B(X^*)$. And if $g \in S$, then for any $x^{**} \in A$,

$$g(x_0 - x^{**}) \leq \|x_0 - x^{**}\| \leq r$$

and hence,

$$x^{**}(g) \geq g(x_0) - r > 0.$$

Thus,

$$S \subseteq \{f \in B(X^*) : x^{**}(f) > 0 \text{ for all } x^{**} \in A\}.$$

Conversely, let $d = d(0, A) > 0$, $x_0 \in S_X$ and $0 < \varepsilon < 1$ be such that

$$\{f \in B(X^*) : f(x_0) > \varepsilon\} \subseteq \{f \in B(X^*) : x^{**}(f) > 0 \text{ for all } x^{**} \in A\}.$$

Let $M = \sup\{\|x^{**}\| : x^{**} \in A\}$. By the proof of Lemma 2.2.3, for all $x^{**} \in A$, there exists $t \in \mathbb{R}$ such that $1 - \varepsilon \leq t \leq 1 + \varepsilon$ and

$$\left\| \frac{tx^{**}}{\|x^{**}\|} - x_0 \right\| \leq \varepsilon.$$

Then for $\lambda \geq M/(1 - \varepsilon)$,

$$\begin{aligned} \|x^{**} - \lambda x_0\| &\leq \left\| x^{**} - \frac{\|x^{**}\|}{t} x_0 \right\| + \left| \frac{\|x^{**}\|}{t} - \lambda \right| \leq \frac{\varepsilon \|x^{**}\|}{t} + \lambda - \frac{\|x^{**}\|}{t} \\ &= \lambda - \frac{\|x^{**}\|}{t}(1 - \varepsilon) \leq \lambda - \frac{d(1 - \varepsilon)}{1 + \varepsilon}. \end{aligned}$$

Therefore,

$$A \subseteq B^{**}\left[\lambda x_0, \lambda - \frac{d(1 - \varepsilon)}{1 + \varepsilon}\right] \text{ and clearly, } 0 \notin B^{**}\left[\lambda x_0, \lambda - \frac{d(1 - \varepsilon)}{1 + \varepsilon}\right].$$

□

Remark 2.2.5. From Theorem 2.2.4, it can be shown that if every $f \in S(X^*)$ is a w*-semidenting point of $B(X^*)$, then X has the MIP. See Theorem 2.2.9 $(b) \Rightarrow (a)$ below.

Definition 2.2.6. For $\varepsilon, \delta > 0$ and $x \in S(X)$, denote by

$$\begin{aligned}
d_1(x,\delta) &= \sup_{0<\lambda<\delta,\ y\in B(X)} \frac{\|x+\lambda y\| + \|x-\lambda y\| - 2}{\lambda}, \\
d_2(x,\delta) &= diam(S(B(X^*),x,\delta)), \\
d_3(x,\delta) &= diam(D(S(X)\cap B(x,\delta))).
\end{aligned}$$

Definition 2.2.7. [47] For $\varepsilon, \delta > 0$, define

$$M_{\varepsilon,\delta}(X) = \{x \in S(X) : \sup_{0<\|y\|<\delta} \frac{\|x+y\| + \|x-y\| - 2}{\|y\|} < \varepsilon\}.$$

In other words, $M_{\varepsilon,\delta}(X) = \{x \in S(X) : d_1(x,\delta) < \varepsilon\}$.

The lemma below is quantitatively more precise than [46, Lemma 1]. It follows from [1, Lemma 2.1]. We include the details for completeness.

Lemma 2.2.8. *For any $\alpha, \delta > 0$, we have*

(*i*) $d_2(x,\alpha) \le d_1(x,\delta) + \dfrac{2\alpha}{\delta}$

(*ii*) $d_3(x,\delta) \le d_2(x,\delta)$

(*iii*) $d_1(x,\delta) \le d_3(x,2\delta)$.

Proof. (*i*). Fix α, $\delta > 0$. Let $d_2 = d_2(x,\alpha)$. For each $n \ge 1$, we can choose f_n, $g_n \in S(B(X^*),x,\alpha)$ such that $\|f_n - g_n\| > d_2 - \frac{1}{n}$. Choose $y_n \in B(X)$ such that $(f_n - g_n)(y_n) > d_2 - \frac{1}{n}$. Then

$$\frac{\|x+\delta y_n\| + \|x-\delta y_n\| - 2}{\delta} \ge \frac{f_n(x+\delta y_n) + g_n(x-\delta y_n) - 2}{\delta} \ge d_2 - \frac{1}{n} - \frac{2\alpha}{\delta}.$$

Thus, $d_1(x,\delta) \ge d_2 - \frac{2\alpha}{\delta}$.

(*ii*). Let $y \in S(X) \cap B(x,\delta)$ and $f_y \in D(y)$. Then,

$$0 \le 1 - f_y(x) = f_y(y-x) \le \|y-x\| < \delta.$$

Thus, $f_y \in S(B(X^*),x,\delta)$. Therefore, $D(S(X)\cap B(x,\delta)) \subseteq S(B(X^*),x,\delta)$.

(iii). Let $\lambda \leq \delta$. Observe that for any $y \in B(X)$,

$$\begin{aligned}\left\| \frac{x \pm \lambda y}{\|x \pm \lambda y\|} - x \right\| &\leq \left\| \frac{x \pm \lambda y}{\|x \pm \lambda y\|} - (x \pm \lambda y) \right\| + \lambda = \big|1 - \|x \pm \lambda y\|\big| + \lambda \\ &= \big|\|x\| - \|x \pm \lambda y\|\big| + \lambda \leq 2\lambda.\end{aligned}$$

Let $f \in D\left(\frac{x + \lambda y}{\|x + \lambda y\|}\right)$ and $g \in D\left(\frac{x - \lambda y}{\|x - \lambda y\|}\right)$. Then

$$\begin{aligned}\frac{\|x + \lambda y\| + \|x - \lambda y\| - 2}{\lambda} &= \frac{f(x + \lambda y) + g(x - \lambda y) - 2}{\lambda} \\ &\leq (f - g)(y) \leq \|f - g\|.\end{aligned}$$

□

We now come to our main theorem. The conditions (c)–(e) are reformulations of (ii)–(iv) of [47, Theorem 1] in the language of [21, Theorem 2.1]. We give a self-contained proof of $(a) \Leftrightarrow (b)$, our main result. The proofs of $(c) \Rightarrow (e)$ and $(c) \Leftrightarrow (d)$ are essentially from [47, Theorem 1], which we include for the sake of completeness and we will need them in the next section. For the rest, our arguments are either new or simpler.

Theorem 2.2.9. *For a Banach space X, the following are equivalent :*

- (a) *X has the UMIP.*
- (b) *Every $f \in S(X^*)$ is uniformly w^*-semidenting, i.e., given $\varepsilon > 0$, there exists $0 < \delta < 1$ such that for any $f \in S(X^*)$, there exists $x \in S(X)$ such that*

$$S(B(X^*), x, \delta) \subseteq B(f, \varepsilon).$$

- (c) *The duality map is uniformly quasicontinuous, i.e., given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $f \in S(X^*)$, there exists $x \in S(X)$ such that $D(S(X) \cap B(x, \delta)) \subseteq B(f, \varepsilon)$.*
- (d) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that every support mapping maps a δ-net in $S(X)$ to an ε-net in $S(X^*)$.*
- (e) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $f \in S(X^*)$, there exists $x \in M_{\varepsilon,\delta}(X)$ such that $D(x) \subseteq B(f, \varepsilon)$.*

Proof. $(a) \Rightarrow (b)$. We use an idea from [20, Lemma 2.2(ii)].

Let $0 < \varepsilon < 1$ and $M \geq 2$ be given. Choose K as given by (a) for $\varepsilon/3$. We may assume w.l.o.g. that $K \geq 1$.

Let $f \in S(X^*)$. Consider $C := \{x \in B(X) : f(x) \geq \varepsilon/3\}$. Then, $d(0, C) \geq \varepsilon/3$. Also, $diam(C) \leq 2 \leq M$.

So, we have $B = B[x_0, r]$ containing C, $r \leq K$ and $d(0, B) \geq \varepsilon/6$.

Since $d(0, B[x_0, r]) \geq \varepsilon/6$, $\|x_0\| \geq r + \varepsilon/6 > r + \varepsilon/9$.

Claim : $S := S(B(X^*), x_0/\|x_0\|, 1 - K/(K + \varepsilon/9)) \subseteq B(f, \varepsilon)$, that is, (b) holds for $x = x_0/\|x_0\|$ and $\delta = 1 - K/(K + \varepsilon/9)$.

Let $g \in S$. Since, $r/(r + \varepsilon/9) \leq K/(K + \varepsilon/9)$, $g(x_0/\|x_0\|) > r/(r + \varepsilon/9)$, which implies $g(x_0) > r\|x_0\|/(r + \varepsilon/9) > r$. That is, $\inf g(B) > 0$. So,

$$\{x \in B(X) : f(x) > \varepsilon/3\} \subseteq C \subseteq B \subseteq \{x : g(x) > 0\}.$$

By Lemma 2.2.3, $\|f - g/\|g\|\| < 2\varepsilon/3$. Also, $g \in S$ implies $\|g\| \geq g(x_0/\|x_0\|) > K/(K + \varepsilon/9)$. Hence,

$$\|f - g\| \leq \|f - g/\|g\|\| + (1 - \|g\|) \leq 2\varepsilon/3 + \varepsilon/(9K + \varepsilon) < \varepsilon.$$

This proves the claim.

$(b) \Rightarrow (a)$. Given $\varepsilon > 0$ and $M \geq 2$, it suffices to show that there is $K > 0$ such that whenever a closed convex set $C \subseteq X$ is such that $diam(C) \leq M$ and $d(0, C) \geq \varepsilon$, there is a closed ball $B \subseteq X$ of radius $\leq K$ such that $C \subseteq B$ and $d(0, B) \geq \varepsilon/2$.

Let $\varepsilon > 0$ be given. Let $L = M + \varepsilon$. Choose $0 < \delta < 1$ for $\varepsilon/4L$ obtained from (b). Let $K = L/\delta + 1$. We will show that this K works.

Case I : $C \setminus B[0, M + \varepsilon/2] \neq \emptyset$.

Choose $z \in C \setminus B[0, M + \varepsilon/2]$. Then $C \subseteq B[z, M]$, $d(0, B[z, M]) \geq \varepsilon/2$ and $M \leq K$.

Case II : $C \subseteq B[0, M + \varepsilon/2]$.

Define $D = \overline{C + \frac{\varepsilon}{2}B(X)}$. Then $D \subseteq B[0, L]$ and $d(0, D) \geq \varepsilon/2$, and hence, D is disjoint from $B(0, \varepsilon/2)$. By the separation theorem, there exists $f \in S(X^*)$ such that

$\inf f(D) \geq \varepsilon/2$. By choice of δ, there exists $x_0 \in S(X)$ such that

$$S(B(X^*), x_0, \delta) \subseteq B(f, \varepsilon/4L).$$

It follows that if $g \in B(X^*) \cap B(f, \varepsilon/4L)$ and $z \in D$, then $g(z) \geq f(z) - \|f-g\|\|z\| \geq \varepsilon/2 - \varepsilon/4 = \varepsilon/4 > 0$. Therefore,

$$\begin{aligned} S(B(X^*), x_0, \delta) &\subseteq B(X^*) \cap B(f, \varepsilon/4L) \\ &\subseteq \{g \in B(X^*) : g(x) > 0 \text{ for all } x \in D\}. \end{aligned}$$

By the proof of Theorem 2.2.4, for $K \geq \lambda \geq L/\delta$,

$$D \subseteq B\left[\lambda x_0, \lambda - \frac{d(0,D)\delta}{2-\delta}\right].$$

It follows that

$$\begin{gathered} C \subseteq B = B\left[\lambda x_0, \lambda - \frac{\varepsilon}{2} - \frac{d(0,D)\delta}{2-\delta}\right], \quad \text{and} \\ d(0,B) = \frac{\varepsilon}{2} + \frac{d(0,D)\delta}{2-\delta} \geq \frac{\varepsilon}{2}. \end{gathered}$$

This completes the proof.

$(b) \Rightarrow (c)$. As observed in the proof of Lemma 2.2.8(ii), $D(S(X) \cap B(x,\delta)) \subseteq S(B(X^*), x, \delta)$.

$(c) \Rightarrow (e)$. Let $\varepsilon > 0$ be given. Then from (c) for $\varepsilon/3$, there exists $\delta > 0$ such that for any $f \in S(X^*)$, there exists $x \in S(X)$ such that $D(S(X) \cap B(x, 2\delta)) \subseteq B(f, \varepsilon/3)$. This implies $d_3(x, 2\delta) \leq 2\varepsilon/3 < \varepsilon$. By Lemma 2.2.8($iii$), $d_1(x,\delta) < \varepsilon$, *i.e.*, $x \in M_{\varepsilon,\delta}(X)$. And $D(x) \subseteq B(f, \varepsilon/3) \subseteq B(f, \varepsilon)$.

$(e) \Rightarrow (b)$. Let $\varepsilon > 0$ be given. By (e), there exists $\delta > 0$ such that for every $f \in S(X^*)$, there is an $x \in M_{\varepsilon/4,\delta}(X)$ such that $D(x) \subseteq B(f, \varepsilon/4)$. So, $d_1(x,\delta) < \varepsilon/4$. By Lemma 2.2.8($i$), $d_2(x,\alpha) < \varepsilon/2$, for $\alpha < \delta\varepsilon/8$. That is, $diam(S(B(X^*), x, \alpha)) < \varepsilon/2$. Now, $D(x) \subseteq S(B(X^*), x, \alpha)$. Hence for any $g \in S(B(X^*), x, \alpha)$ and $f_x \in D(x)$

$$\|f - g\| \leq \|f - f_x\| + \|f_x - g\| \leq \varepsilon/4 + \varepsilon/2 < \varepsilon.$$

Therefore,

$$S(B(X^*), x, \alpha) \subseteq B(f, \varepsilon).$$

$(c) \Rightarrow (d)$ is obvious.

$(d) \Rightarrow (c)$. If (c) does not hold, there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $f_\delta \in S(X^*)$ such that for all $x \in S(X)$, there exists $z_x \in S(X)$ and $f_{z_x} \in D(z_x)$ such that $\|z_x - x\| < \delta/2$ and $\|f_\delta - f_{z_x}\| \geq \varepsilon$. Then $\{z_x : x \in S(X)\}$ is a δ-net, but $\{f_{z_x} : x \in S(X)\}$ is not an ε-net in $S(X^*)$. □

2.3 Characterisation of w*-UMIP

In this section, we study the w*-version of UMIP defined above.

Definition 2.3.1. We say that a dual Banach space X^* has the *w*-Uniform Mazur Intersection Property (w*-UMIP)* if for every $\varepsilon > 0$ and $M \geq 2$, there is $K > 0$ such that whenever a w*-compact convex set $C \subseteq X^*$ and a point $f \in X^*$ are such that $diam(C) \leq M$ and $d(f, C) \geq \varepsilon$, there is a closed ball $B \subseteq X^*$ of radius $\leq K$ such that $C \subseteq B$ and $d(f, B) \geq \varepsilon/2$.

For $f \in S(X^*)$, the inverse duality mapping is defined as

$$D^{-1}(f) = \begin{cases} \{x \in S(X) : f(x) = 1\} & \text{if} \quad f \in D(S(X)) \\ \emptyset & \text{if} \quad f \notin D(S(X)) \end{cases}$$

The fact that $D^{-1}(f) = \emptyset$ for some $f \in S(X^*)$ introduces some technicalities in the discussion of w*-UMIP.

Definition 2.3.2. For $\varepsilon, \delta > 0$ and $f \in S(X^*)$, denote by

$$\begin{aligned} d_1^*(f, \delta) &= \sup_{0<\lambda<\delta,\ g \in B(X^*)} \frac{\|f + \lambda g\| + \|f - \lambda g\| - 2}{\lambda}, \\ d_2^*(f, \delta) &= diam(S(B(X), f, \delta)), \\ d_3^*(f, \delta) &= diam(D^{-1}(D(S(X)) \cap B(f, \delta))). \end{aligned}$$

Lemma 2.3.3. *For any $\alpha, \delta > 0$, we have,*

(i) $d_2^*(f, \alpha) \leq d_1^*(f, \delta) + \dfrac{2\alpha}{\delta}$

(*ii*) $d_3^*(f,\delta) \le d_2^*(f,\delta)$

(*iii*) $d_1^*(f,\alpha) \le \dfrac{d_3^*(f,\delta)+2\alpha}{1-\alpha^2}$ *if* $\alpha < \sqrt{\delta+1}-1$.

Proof. (i) follows from Lemma 2.2.8(i).

(ii) holds since $D^{-1}(D(S(X)) \cap B(f,\delta)) \subseteq S(B(X), f, \delta)$.

(iii) Let $d_3^* = d_3^*(f,\delta)$. Choose $0 < \alpha < \sqrt{\delta+1}-1$. Then $\alpha^2 + 2\alpha < \delta$. Let $g \in S(X^*)$, $0 < \lambda < \alpha$. As before,

$$\left\| \frac{f \pm \lambda g}{\|f \pm \lambda g\|} - f \right\| \le 2\lambda.$$

Find $h_1, h_2 \in D(S(X))$ such that

$$\left\| \frac{f + \lambda g}{\|f + \lambda g\|} - h_1 \right\| \le \lambda^2 \quad \text{and} \quad \left\| \frac{f - \lambda g}{\|f - \lambda g\|} - h_2 \right\| \le \lambda^2$$

and $x_1, x_2 \in S(X)$ such that $h_i \in D(x_i)$, $i = 1, 2$.

Observe that $\|h_i - f\| \le \lambda^2 + 2\lambda \le \alpha^2 + 2\alpha < \delta$, i.e., $h_1, h_2 \in D(S(X)) \cap B(f,\delta)$. Thus $\|x_1 - x_2\| \le d_3^*$. Now,

$$0 \le 1 - x_1\left(\frac{f+\lambda g}{\|f+\lambda g\|}\right) = x_1\left(h_1 - \frac{f+\lambda g}{\|f+\lambda g\|}\right) \le \left\| \frac{f+\lambda g}{\|f+\lambda g\|} - h_1 \right\| \le \lambda^2.$$

So, $x_1(f+\lambda g) \ge (1-\lambda^2)\|f+\lambda g\|$. Similarly, $x_2(f-\lambda g) \ge (1-\lambda^2)\|f-\lambda g\|$. So, we have

$$\begin{aligned} \frac{\|f+\lambda g\| + \|f-\lambda g\| - 2}{\lambda} &\le \frac{x_1(f+\lambda g) + x_2(f-\lambda g) - 2(1-\lambda^2)}{\lambda(1-\lambda^2)} \\ &= \frac{(x_1+x_2)(f) - 2 + \lambda(x_1-x_2)(g) + 2\lambda^2}{\lambda(1-\lambda^2)} \\ &\le \frac{\|x_1-x_2\| + 2\lambda}{1-\lambda^2} \le \frac{d_3^*+2\lambda}{1-\lambda^2} \le \frac{d_3^*+2\alpha}{1-\alpha^2} \end{aligned}$$

(since $\dfrac{d_3^*+2\lambda}{1-\lambda^2}$ is increasing in λ). Thus,

$$d_1^*(f,\alpha) \le \frac{d_3^*+2\alpha}{1-\alpha^2}.$$

□

Remark 2.3.4. The proof of (iii) is adapted from the proof of [2, Lemma 2]. Notice that unlike the proof of [21, Lemma 3.1], our proof uses just the Bishop-Phelps Theorem and not Bollobás' extensions.

We now come to our characterisation theorem. Though the proof is similar to that of Theorem 2.2.9, there are added technicalities as pointed out earlier.

Theorem 2.3.5. *For a Banach space X, the following statements are equivalent :*

(a) *X^* has w*-UMIP.*

(b) *Every $x \in S(X)$ is uniformly semidenting, i.e., given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in S(X)$, there exists $f \in S(X^*)$ such that*

$$S(B(X), f, \delta) \subseteq B(x, \varepsilon).$$

(c) *The inverse duality map is uniformly quasicontinuous, i.e., given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in S(X)$, there exists $f \in D(S(X))$ such that $D^{-1}(D(S(X)) \cap B(f, \delta)) \subseteq B(x, \varepsilon)$.*

(d) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that every support mapping on X^* that maps $D(S(X))$ into $S(X)$ maps any $\delta \subseteq D(S(X))$ that is a δ-net in $S(X^*)$ to an ε-net in $S(X)$.*

(e) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in S(X)$, there exists $f \in M_{\varepsilon,\delta}(X^*) \cap D(S(X))$ such that $D^{-1}(f) \subseteq B(x, \varepsilon)$.*

Proof. $(a) \Rightarrow (b)$ is similar to Theorem 2.2.9 $(a) \Rightarrow (b)$.

$(b) \Rightarrow (a)$. Define $D = C + \frac{\varepsilon}{2} B(X^*)$. Then D is w*-compact, and $d(0, D) \geq \varepsilon/2$. Therefore, D is disjoint from the w*-compact convex set $3\varepsilon/8B(X^*)$. Hence there exists $x \in S(X)$ such that $\inf \hat{x}(D) \geq 3\varepsilon/8$.

Rest of the proof is similar to Theorem 2.2.9 $(b) \Rightarrow (a)$.

$(b) \Rightarrow (c)$. Let $\varepsilon > 0$ be given. By (b), there exists $\delta > 0$ such that for any $x \in S(X)$, there exists $g \in S(X^*)$ such that

$$S(B(X), g, 2\delta) \subseteq B(x, \varepsilon).$$

As in [9, Lemma 3.2.6], if we choose $0 < \eta < \delta/4$ then for any $h \in B(g,\eta)$,

$$S(B(X), h, \delta) \subseteq S(B(X), g, 2\delta).$$

By Bishop-Phelps Theorem, there exists $f \in D(S(X)) \cap B(g,\eta)$. Then

$$S(B(X), f, \delta) \subseteq S(B(X), g, 2\delta) \subseteq B(x,\varepsilon).$$

And $D^{-1}(D(S(X)) \cap B(f,\delta)) \subseteq S(B(X), f, \delta)$. Therefore, (c) holds.

$(c) \Rightarrow (e)$. Let $\varepsilon > 0$ be given. Then from (c) for $\varepsilon/4$, there exists $\delta > 0$ such that for any $x \in S(X)$, there exists $f \in D(S(X))$ such that $D^{-1}(D(S(X)) \cap B(f,\delta)) \subseteq B(x,\varepsilon/4)$. This implies $d_3^*(f,\delta) \leq \varepsilon/2$. By Lemma 2.3.3$(iii)$,

$$\begin{aligned} d_1^*(f,\alpha) \quad &\leq \quad \frac{d_3^*(f,\delta)+2\alpha}{1-\alpha^2} \leq \frac{\varepsilon/2+2\alpha}{1-\alpha^2} < \varepsilon \\ &\qquad \text{if } \alpha < \sqrt{\delta+1}-1 \text{ and } \frac{\varepsilon/2+2\alpha}{1-\alpha^2} < \varepsilon. \end{aligned}$$

That is, $f \in M_{\varepsilon,\alpha}(X^*)$. Clearly, the choice of α depends only on ε. And $D^{-1}(f) \subseteq B(x,\varepsilon/4) \subseteq B(x,\varepsilon)$.

$(e) \Rightarrow (b)$. Let $\varepsilon > 0$ be given. Choose $0 < \delta < 1$ obtained from (e) for $\varepsilon/4$. So, given $x \in S(X)$, there is $f \in M_{\varepsilon/4,\delta}(X^*) \cap D(S(X))$ such that $D^{-1}(f) \subseteq B(x,\varepsilon/4)$. As $f \in M_{\varepsilon/4,\delta}(X^*)$, $d_1^*(f,\delta) < \varepsilon/4$. By Lemma 2.3.3, $d_2^*(f,\alpha) < \varepsilon/2$ for $0 < \alpha < \varepsilon\delta/8$. That is, $diam(S(B(X), f, \alpha)) < \varepsilon/2$.

Also, $D^{-1}(f) \subseteq B(x,\varepsilon/4)$ and $f \in D(S(X))$. Hence for any $y \in S(B(X), f, \alpha)$ and $z \in D^{-1}(f)$,

$$\|x-y\| \leq \|x-z\| + \|y-z\| \leq \varepsilon/4 + \varepsilon/2 < \varepsilon.$$

Therefore,

$$S(B(X), f, \alpha) \subseteq B(x,\varepsilon).$$

$(c) \Rightarrow (d)$ is obvious.

$(d) \Rightarrow (c)$. If (c) does not hold, there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x_\delta \in S(X)$ such that for all $f \in D(S(X))$, there exists $g_f \in D(S(X))$ and $y_f \in D^{-1}(g_f)$ such that $\|f - g_f\| < \delta/2$ and $\|x_\delta - y_f\| \geq \varepsilon$.

Define $\delta = \{g_f : f \in D(S(X))\}$. Clearly, δ forms a $\delta/2$-net in $D(S(X))$. Since $D(S(X))$ is dense in $S(X^*)$, δ forms a δ-net in $S(X^*)$ as well. Define a support mapping $\Phi : D(S(X)) \to S(X)$ as follows :

(i) If $g \in \delta$, we choose any $f \in D(S(X))$ such that $g = g_f$ and define $\Phi(g) = y_f$. Observe that, by definition of δ, there is at least one such $f \in D(S(X))$.

(ii) If $g \in D(S(X)) \setminus \delta$, define $\Phi(g) = y$ for some $y \in D^{-1}(g)$.

Now, as $\|x_\delta - \Phi(g)\| \geq \varepsilon$ for all $g \in \delta$, $\Phi(\delta)$ does not form an ε-net in $S(X)$. So, (d) does not hold. $\square$

2.4 Fréchet smoothness via ball separation

Fréchet smoothness implying MIP was observed by Mazur himself. So, it makes quite a sense to ask whether the converse is true. But, the converse does not hold is apparent even from finite dimensional examples. So, we consider a stronger hypothesis. That is, the case when all points of $S(X^*)$ are w*-denting points of $B(X^*)$. The later condition implying Fréchet differentiability can be observed by the work of Hu and Lin in [26, Theorem 3.1] and [27, Theorem 4]. But, this approach is quite indirect and long. In this section, we characterise Fréchet smoothness in terms of w*-denting points. As a corollary, we obtain the result discussed in [27, Theorem 4]. Our approach in this case is quite direct and short. Also, using the ball separation characterisation of a w*-denting point by Chen and Lin in [11, Corollary 1.6], we obtain a ball separation characterisation of Fréchet smoothness.

First, let us recall the definition of the condition "$\mathcal{P} = \mathcal{H}$" as defined in [20]. We will call this property the hyperplane Mazur intersection property (H-MIP).

Definition 2.4.1. We say that a Banach space X has the Hyperplane MIP (H-MIP) if for every closed convex bounded set $C \subseteq X$ and $f \in S(X^*)$ such that $\inf f(C) > 0$, there exists a closed ball $B[x_0, r_0]$ in X such that $C \subseteq B[x_0, r_0]$ and $\inf f(B[x_0, r_0]) > 0$.

The following characterisation of the above property is already known:

Theorem 2.4.2. *[20, Theorem 4.6] X has the H-MIP if and only if every $f \in S(X^*)$ is a w^*-denting point of $B(X^*)$.*

Now, we observe that, restricted to norm attaining functionals, the notion of w*-denting and w*-strongly exposed points coincide.

Let NA denote the set of all $f \in S(X^*)$ which attain their norm on $B(X)$, i.e., $NA = \cup\{D(x) : x \in S(X)\}$.

Following theorem characterises Fréchet smoothness in terms of w*-denting points of $B(X^*)$.

Theorem 2.4.3. *(a) X is smooth if and only if every $f \in NA$ is an extreme point of $B(X^*)$.*

(b) X is Fréchet smooth if and only if every $f \in NA$ is a w^*-denting point of $B(X^*)$.

Proof. (a): Let X be smooth and $f \in NA$. Let $x \in S(X)$ be such that $f \in D(x)$. Since $D(x)$ is a singleton face of $B(X^*)$, f is an extreme point.

Conversely, let every element of NA be extreme. Let $x \in S(X)$ and $g, h \in D(x)$. Then, $\frac{(h+g)(x)}{2} = 1$. Thus, $\frac{g+h}{2} \in D(x) \subseteq NA$. Therefore, $\frac{g+h}{2}$ is an extreme point. Hence, $g = h$. So, X is smooth.

(b): Suppose X is Fréchet smooth. Let $f \in NA$. Let $x \in S(X)$ be such that $f \in D(x)$. Since x is a point of Fréchet differentiability, f is w*-strongly exposed by x (Refer to Theorem 1.0.17). Hence, f is a w*-denting point of $B(X^*)$.

Conversely, let all $f \in NA$ be w*-denting. So, every $f \in NA$ is extreme. Thus, by part (a), X is smooth. Hence, the duality map is singleton and norm-w* continuous. But, by Theorem 1.0.10 norm and w*-topology coincide on NA. Hence, the duality map is norm-norm continuous on NA. So X is Fréchet smooth by Theorem 1.0.19. $\square$

Now, we can characterise Fréchet smoothness as a ball separation property using the characterisation of w*-denting points in [11, Corollary 1.6] and Theorem 2.4.3 .

Theorem 2.4.4. *X is Fréchet smooth if and only if for every closed bounded set C and $f \in NA$ satisfying* $\inf f(C) > 0$*, we have $z_0 \in X$ and $r_0 > 0$ such that $C \subseteq z_0 + r_0 B(X)$ and* $\inf f(z_0 + r_0 B(X)) > 0$.

In the next chapter, we will take this discussion to the uniform versions of the above notions and establish parallel analogies and perhaps more!

Chapter 3

Ball Separation Properties via Some Moduli

3.1 Introduction

In the previous chapter, we characterised UMIP in terms of uniform w*-semidenting points. In this chapter, we continue our investigation of Banach spaces with UMIP. We introduce two moduli of w*-semidenting points and characterise MIP and UMIP in terms of these moduli. One of these moduli finds its origin in the work of Dutta and Lin in [17]. The other moduli, in terms of w*-slices, is motivated by the characterisation of MIP and UMIP in terms of w*-semidenting points.

Also, as seen in the previous chapter that H-MIP is characterised by all points of $S(X^*)$ being w*-denting points of $B(X^*)$, we look at the uniform version of this result. Based on the modulus of uniform denting by Dutta and Lin, we introduce a notion of uniform w*-denting and prove that the uniform version of H-MIP (H-UMIP) is characterised by uniform w*-denting. Also, we develop upon the fact that H-MIP implies Fréchet smoothness to prove that H-UMIP is equivalent to uniform smoothness. Hence, we obtain a ball separation characterisation of uniform smoothness. This will be used to prove the residuality of uniformly smooth norms in Chapter 5. We also look at the asymptotic version of uniform smoothness (AUS) and characterise it in terms of asymptotic notions of uniform w*-denting points and a ball separation property similar to H-UMIP.

Throughout this chapter, we use the following notation for a slice which is different from the notation used in the previous chapter. For $f \in S(X^*)$ and $0 < \alpha < 1$, we write

$$S(B(X), f, \alpha) := \{y \in B(X) : f(y) > \alpha\}.$$

Materials of this chapter has appeared in [5, 24].

3.1.1 Modulus of Denting Point

We look at the modulus of denting and uniformly denting points discussed by Dutta and Lin [17]. This serves as a background for the moduli we define later.

Definition 3.1.1. For $0 < t < 2$, $f \in S(X^*)$ and $x \in S(X)$, define

$$s(x, f, t) := \inf\{\|x + y\| - 1 : y \in \ker f, \|y\| \geq t/4\}.$$

For $0 < t < 2$ and $x \in S(X)$, define

$$\begin{aligned} d(x,t) &:= \sup_{f \in S(X^*)} s(x, f, t) \quad \text{and} \\ d(t) &:= \inf_{x \in S(X)} d(x,t). \end{aligned}$$

Note that for $0 < t < 2$, $x \in S(X)$, $f \in D(x)$ and $y \in \ker f$,

$$1 = f(x) = f(x + y) \leq \|x + y\|.$$

It follows that $s(x, f, t) \geq 0$ and hence, $d(x, t) \geq 0$.

The following lemma ([17, Lemma 3.1]) sets our base for the discussion of modulus of denting points. It relates the size of a slice with the quantity $s(x, f, t)$ defined above. For the sake of completeness, we include a proof filling up some of the details missing in [17].

Lemma 3.1.2. *[17, Lemma 3.1] Let $x \in S(X)$ and $f \in S(X^*)$ be such that $f(x) > 0$ and for some $0 < t < 2$, $s(x, f, t) > 0$, then*

$$diam(S(B(X), f, f(x)(1 - s(x, f, t)))) < 2t.$$

Proof. Let $Z = \ker(f)$. Suppose $s = s(x, f, t) > 0$. Let $v \in Z$ with $\|v\| = t/4$. Clearly, $s \leq \|x\| + \|v\| - 1 \leq t/4 < 1$.

Claim : $f(x) \geq 1 - t/4$.

Let, if possible, $f(x) < 1 - t/4$. Let $0 < \varepsilon < \min\{s/(1+s), 1 - t/4 - f(x)\}$.

Let $v \in S(X)$ be such that $f(v) > 1 - \varepsilon$. Put $z = x - \frac{f(x)}{f(v)}v$. Then, $z \in Z$ and

$$\|z\| \geq 1 - \frac{f(x)}{f(v)} \geq 1 - \frac{1 - t/4 - \varepsilon}{1 - \varepsilon} = \frac{t/4}{1 - \varepsilon} > t/4.$$

It follows that

$$1 + s \leq \|x - z\| \leq \frac{f(x)}{f(v)} \leq \frac{1}{1 - \varepsilon} < 1 + s.$$

This is a contradiction. Hence, the claim.

Let $y \in S = S(B(X), f, f(x)(1 - s))$. Then, $f(y) > f(x)(1 - s)$. Consider $z = y - \frac{f(y)}{f(x)}x$. Then $z \in Z$. Also,

$$\|x + z\| - 1 = \left\|x + y - \frac{f(y)}{f(x)}x\right\| - 1 \leq \left|1 - \frac{f(y)}{f(x)}\right|.$$

Case 1 : Suppose $f(y) \leq f(x)$. Then, $|1 - f(y)/f(x)| = 1 - f(y)/f(x) < s$. It follows that $\|z\| < t/4$, and hence,

$$\|y - x\| = \left\|z + \frac{f(y)}{f(x)}x - x\right\| \leq \|z\| + \left|1 - \frac{f(y)}{f(x)}\right| < t/4 + s \leq t/4 + t/4 = t/2.$$

Case 2 : If $1 \geq f(y) > f(x)$, put $z_1 = f(x)z = f(x)y - f(y)x$. Then $z_1 \in Z$. Also,

$$\|x + z_1\| - 1 = \|x + f(x)y - f(y)x\| - 1 \leq f(x) + 1 - f(y) - 1 \leq 0 < s.$$

Thus, $\|z_1\| = \|f(x)z\| < t/4$, i.e., $\|z\| < (t/4)/f(x)$. Therefore, by the claim above,

$$\|y - x\| \leq \|z\| + \left|1 - \frac{f(y)}{f(x)}\right| < \frac{t/4}{f(x)} + \frac{1}{f(x)} - 1 \leq \frac{t/4 + 1}{1 - t/4} - 1 < t.$$

Since $x \in S$, $diam(S) < 2t$. $\square$

In the following lemma, we observe the converse of the above lemma.

Lemma 3.1.3. *Given $x \in S(X)$ and $0 < t < 2$, suppose there exists $f \in S(X^*)$ and $\alpha \in (0,1)$ such that $x \in S := S(B(X), f, \alpha)$ and diam$(S) < t/5$. Then*

$$s(x, f, t) \geq \beta := \min\left\{\frac{f(x) - \alpha}{\alpha}, \frac{t}{20}\right\} > 0.$$

Proof. Let $y \in \ker f$ such that $z = x + y \in B(X)$. Then $z \in S$. Thus, $\|y\| = \|z - x\| < t/5$. This implies $s(x, f, t) \geq 0$.

Suppose $0 \leq s(x, f, t) < \beta$. Choose $\varepsilon > 0$ such that $s(x, f, t) < \varepsilon < \beta$.

Then $1 \geq f(x) > \alpha(1 + \varepsilon) > \alpha$.

Now, since $0 \leq s(x, f, t) < \varepsilon$, there exists $y_0 \in \ker f$ such that $\|y_0\| \geq t/4$ and $1 \leq \|x + y_0\| < 1 + \varepsilon$.

Put $z_0 = x + y_0$. Then, $f(z_0) > \alpha(1 + \varepsilon)$. Therefore, $f(z_0/\|z_0\|) > \alpha$.

This implies, $z_0/\|z_0\| \in S$, and hence, $\|x - z_0/\|z_0\|\| < t/5$. Therefore,

$$\|y_0\| = \|x - z_0\| \leq \left\|x - \frac{z_0}{\|z_0\|}\right\| + \left\|z_0 - \frac{z_0}{\|z_0\|}\right\| < \frac{t}{5} + (\|z_0\| - 1) < \frac{t}{5} + \varepsilon < \frac{t}{4}.$$

This is a contradiction, proving $s(x, f, t) \geq \beta$. □

Theorem 3.1.4. *[17, Proposition 3.3 (a)] $x \in S(X)$ is a denting point of $B(X)$ if and only if $d(x, t) > 0$ for all $0 < t < 2$.*

Proof. Let $0 < t < 2$ be given. Suppose $d(x, t) > 0$. Then there exists $f \in S(X^*)$ such that $s = s(x, f, t) > 0$.

If $f(x) = 0$, then $(t/4)(-x) \in \ker f$. So, $s \leq \|x - (t/4)x\| - 1 \leq -t/4 < 0$, a contradiction. Hence, $f(x) \neq 0$. Note that $s(x, f, t) = s(x, -f, t)$. Therefore, without loss of generality, we may assume $f(x) > 0$. Hence, by Lemma 3.1.2, $x \in S = S(B(X), f, f(x)(1 - s))$ and *diam*$(S) < 2t$. Since $0 < t < 2$ is arbitrary, x is a denting point of $B(X)$.

Conversely, let $0 < t < 2$ be given. Since, x is denting point, there exists $f \in S(X^*)$ and $\alpha \in (0,1)$ such that $x \in S := S(B(X), f, \alpha)$ and *diam*$(S) < t/5$. By Lemma 3.1.3, $s(x, f, t) \geq \min\{(f(x) - \alpha)/\alpha, t/20\} > 0$ which implies that $d(x, t) > 0$. □

Following definition of uniform denting was given in [17].

Definition 3.1.5. X is uniformly denting if $d(t) > 0$ for all $0 < t < 2$.

Dutta and Lin [17] sketch an argument to show that if X is uniformly denting then X has an equivalent uniformly convex norm. We will elaborate and improve on this in Section 3.6.

We end this section with another observation regarding the above modulus.

Proposition 3.1.6. *If D is a dense set in $S(X)$, then*

$$d(t) = \inf_{x \in D} d(x,t) \text{ for all } 0 < t < 2.$$

Proof. Let $d_D(t) := \inf_{x \in D} d(x,t)$. Clearly, $d_D(t) \geq d(t)$ for all $0 < t < 2$.

Let $\varepsilon > 0$ and $x \in S(X)$. Since D is dense in $S(X)$, there exists $z \in D$ such that $\|x - z\| < \varepsilon$. Then for any $f \in S(X^*)$ and $y \in \ker f$ with $\|y\| \geq t/4$, we have $\|z + y\| \leq \|x+y\|+\varepsilon$. It follows that $s(z,f,t) \leq s(x,f,t)+\varepsilon$ and hence, $d_D(t) \leq d(z,t) \leq d(x,t)+\varepsilon$. Therefore, $d_D(t) \leq d(t) + \varepsilon$. $\square$

3.2 Modulus of w*-semidenting point

In this section, we are going to study MIP and UMIP with the help of some moduli motivated by the quantities discussed in the last section.

We begin with the analogues of the moduli defined previously.

Definition 3.2.1. For $0 < t < 2$, $f \in S(X^*)$ and $x \in S(X)$, define

$$s^*(f,x,t) := \inf\{\|f + h\| - 1 : \|h\| \geq t/4, h(x) = 0\}.$$

For $0 < t < 2$ and $f \in S(X^*)$, define

$$\begin{aligned} d^*(f,t) &:= \sup_{x \in S(X)} s^*(f,x,t) \quad \text{and} \\ d^*(t) &:= \inf_{f \in S(X^*)} d^*(f,t). \end{aligned}$$

It follows as in the previous section that

Lemma 3.2.2. *(a) Let $0 < t < 2$, $x \in S(X)$ and $f \in S(X^*)$ be such that $f(x) > 0$ and $s^*(f,x,t) > 0$, then*

$$diam(S(B(X^*), x, f(x)(1 - s^*(f,x,t)))) < 2t.$$

(b) $f \in S(X^*)$ is a w^*-denting point of $B(X^*)$ if and only if $d^*(f,t) > 0$ for all $0 < t < 2$.

The w*-notion of uniform denting is defined as follows:

Definition 3.2.3. X^* is called w*-denting if $d^*(t) > 0$ for all $0 < t < 2$.

Based on the modulus of uniform w*-denting, we obtain the following notion of uniform w*-denting.

Definition 3.2.4. X is said to be uniformly w*-denting (I) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $f \in S(X^*)$, there exists $x \in S(X)$ such that

$$diam(S(B(X^*), x, f(x)(1 - \delta))) < \varepsilon.$$

The following theorem relates the modulus of uniform w*-denting and uniform w*-denting (I).

Theorem 3.2.5. *For a Banach space X, the following are equivalent :*

(a) For every $0 < t < 2$, $d^*(t) > 0$.

(b) X^* is uniformly w^*-denting (I).

Proof. $(a) \implies (b)$. Let $d^*(t) > 0$ for all $0 < t < 2$.

Let $0 < t < 2$. So, $\delta := d^*(t/2)/2 > 0$. Then for any $f \in S(X^*)$, $d^*(f, t/2) > \delta$. By definition, there exists $x \in S(X)$ such that $s^*(f, x, t/2) > \delta$. So, by Lemma 3.2.2, we have that

$$diam(S(B(X^*), x, f(x)(1 - \delta))) < t.$$

So, X^* is uniformly w*-denting (I).

$(b) \implies (a)$. Let $0 < t < 2$ be given. So, there exists $\delta > 0$ such that for every $f \in S(X^*)$, there exists $x \in S(X)$ satisfying

$$diam(S(B(X^*), x, f(x)(1-\delta))) < t/5.$$

Now, by w*-analogue of Lemma 3.1.3, $s^*(f, x, t) \geq \beta := \min\left\{\frac{f(x)-f(x)(1-\delta)}{f(x)(1-\delta)}, \frac{t}{20}\right\}$. That is, $s(x, f, t) \geq \min\{\frac{\delta}{1-\delta}, t/20\}$.

Clearly, $d^*(t) \geq \beta > 0$. □

Another uniform version of w*-denting points can be defined by uniformising the quantities in the definition of w*-denting point as follows:

Definition 3.2.6. X^* is said to be uniformly w*-denting (II) if for every $\varepsilon > 0$, there exists $0 < \alpha < 1$ such that for every $f \in S(X^*)$, there is $x \in S(X)$ such that

$$f(x) > \alpha \text{ and } diam(S(B(X^*), x, \alpha)) < \varepsilon.$$

We will later observe the equivalence of the two notions.

Now, using the above modulus of w*-denting, we define new moduli in order to characterise w*-semidenting and uniform w*-semidenting points.

Definition 3.2.7. For $0 < t < 2$ and $f \in S(X^*)$, define

$$\begin{aligned} d_0^*(f,t) &:= \sup\{s^*(g,x,t) : x \in S(X),\ g \in S(X^*),\ \|f-g\| < t\} \\ &= \sup\{d^*(g,t) : g \in S(X^*),\ \|f-g\| < t\} \quad \text{and} \\ d_0^*(t) &:= \inf_{f \in S(X^*)} d_0^*(f,t). \end{aligned}$$

Theorem 3.2.8. *(a) $f \in S(X^*)$ is a w*-semidenting point of $B(X^*)$ if and only if for any $0 < t < 2$, $d_0^*(f,t) > 0$.*

(b) X has MIP if and only if $d_0^*(f,t) > 0$ for every $f \in S(X^*)$ and $t \in (0,2)$.

(c) X has UMIP if and only if for any $t \in (0,2)$, $d_0^*(t) > 0$.

Proof. (a). Let $0 < t < 2$ be given. Since $d_0^*(f, t/4) > 0$, there exist $x \in S(X)$ and $g \in S(X^*)$ such that $\|f-g\| < t/4$ and $s^*(g, x, t/4) > 0$.

As before, $g(x) \neq 0$ and we may assume $g(x) > 0$.

By Lemma 3.1.2, we have a w*-slice S of diameter smaller than $t/2$ containing g.

Now, $\|f - g\| < t/2$. Thus, we have $S \subseteq B(f, t)$.

Hence, f is a w*-semidenting point.

Conversely, let $0 < t < 2$ and f be a w*-semidenting point. So, there exist $0 < \alpha < 1$ and $x \in S(X)$ such that $S = S(B(X^*), x, \alpha) \subseteq B(f, t/10)$.

Choose $g_0 \in S$ such that $g_0 \in S(X)$. Then $g_0 \in S$ and $diam(S) < t/5$. By w*-version of Lemma 3.1.3, $s^*(g_0, x, t) > 0$ and $\|g_0 - f\| < t/10 < t$. Hence, $d_0^*(f, t) > 0$.

(b) follows from (a) and [20, Theorem 4.1].

(c). Let X have the UMIP. By Theorem 2.2.9 (b), for any $t \in (0, 2)$, there exists $0 < \alpha(t/10) < 1$ such that for every $f \in S(X^*)$, there is an $x \in S(X)$ such that

$$S := S(B(X^*), x, \alpha(t/10)) \subseteq B(f, t/10).$$

Clearly, $diam(S) < t/5$. Choose $g_0 \in D(x)$. Then $g_0 \in S$. By w*-version of Lemma 3.1.3, $s^*(g_0, x, t) \geq \beta(t) := \min\left\{\dfrac{1}{\alpha(t)} - 1, \dfrac{t}{20}\right\} > 0$. Now, since $g_0 \in S$, $\|f - g_0\| < t$, and hence, $d_0^*(f, t) \geq \beta(t)$. It follows that $d_0^*(t) \geq \beta(t) > 0$.

Conversely, let $0 < t < 2$. Let $\gamma(t) = \frac{1}{2} d_0^*(t/4) > 0$. Let $f \in S(X^*)$. Then $d_0^*(f, t/4) \geq d_0^*(t/4) > \gamma(t)$. So, there exists $x \in S(X)$ and $g \in S(X^*)$ such that $\|f - g\| < t/4$ and $s^* = s^*(g, x, t/4) > \gamma(t)$. As before, changing x to $-x$ if necessary, we may assume $g(x) > 0$. Let $S := S(B(X^*), x, g(x)(1 - s^*))$. Then $g \in S$ and by Lemma 2.2.8, we have $diam(S) < t/2$. Since $\|f - g\| < t/4$, we have $S \subseteq B(f, t)$. Also, $0 < g(x) \leq 1$ and $s^* > \gamma(t)$ imply that

$$S(B(X^*), x, 1 - \gamma(t)) \subseteq S(B(X^*), x, 1 - s^*) \subseteq S \subseteq B(f, t).$$

It follows that $\alpha(t) = 1 - \gamma(t)$ works. $\square$

Remark 3.2.9. Note that if X has UMIP, it is uniformly w*-semidenting, also the set D of w*-denting points of $B(X^*)$ is norm dense in $S(X^*)$. However, it does not follow that these w*-denting points are uniform over D, that is, $\inf_{f\in D} d^*(f,t) > 0$, as by w*-version of Proposition 3.1.6, this would imply that X^* is uniformly w*-denting, which, in general, is a much stronger condition, as finite dimensional examples show.

3.3 Modulus of UMIP by slices

In this section, we define another modulus of w*-semidenting points and characterise MIP and UMIP in its terms.

Definition 3.3.1. For $f \in S(X^*)$, $x \in S(X)$ and $0 < t < 1$, define

$$\begin{aligned}
\beta(f,x,t) &:= \inf\{1 - g(x) : g \in B(X^*), \|f - g\| \geq t\}, \\
\beta(f,t) &:= \sup_{x\in S(X)} \beta(f,x,t) \text{ and,} \\
\beta(t) &:= \inf_{f\in S(X^*)} \beta(f,t).
\end{aligned}$$

Lemma 3.3.2. *Let $x \in S(X)$, $f \in S(X^*)$ and $0 < t < 1$ be such that $\beta(f,x,t) > 0$, then*

$$S(B(X^*), x, 1 - \beta(f,x,t)) \subseteq B(f,t)$$

Proof. Let $g \in S(B(X^*), x, 1 - \beta(f,x,t))$. Then $1 - g(x) < \beta(f,x,t)$. By definition of $\beta(f,x,t)$, it follows that $\|f - g\| < t$. □

Theorem 3.3.3. *X has MIP if and only if for every $f \in S(X^*)$ and $0 < t < 1$, $\beta(f,t) > 0$.*

Proof. Let $f \in S(X^*)$. If for every $0 < t < 1$, $\beta(f,t) > 0$, then there exists $x \in S(X)$ such that $\beta(f,x,t) > 0$. By the above lemma, $f \in S(X^*)$ is a w*-semidenting point of $B(X^*)$. Thus, X has MIP.

Conversely, let X have the MIP. Then, every $f \in S(X^*)$ is a w*-semidenting point of $B(X^*)$. Hence, for every $f \in S(X^*)$ and $t > 0$, there exist $0 < \alpha < 1$ and $x \in S(X)$ such that $S(B(X), x, \alpha) \subseteq B(f,t)$. So, if $g \in B(X^*)$ and $\|g - f\| \geq t$, then $g(x) \leq \alpha$, i.e., $0 < 1 - \alpha \leq 1 - g(x)$.

So, $\beta(f,x,t) = \inf\{1 - g(x) : g \in B(X^*), \|f - g\| \geq t\} \geq 1 - \alpha > 0$. Thus, $\beta(f,t) > 0$. □

Theorem 3.3.4. *X has UMIP if and only if for every $0 < t < 1$, $\beta(t) > 0$.*

Proof. Let X have UMIP. Let $t > 0$. There exists $0 < \alpha < 1$ such that for any $f \in S(X^*)$, there exists $x \in S(X)$ such that if $g \in B(X^*)$ and $g(x) > \alpha$, then $\|f-g\| < t$. Hence, for every $f \in S(X^*)$, there exists $x \in S(X)$ such that $\beta(f, x, t) \geq 1 - \alpha$. Then, $\beta(t) \geq 1 - \alpha > 0$.

Conversely, let $\beta(t) > 0$ for every $0 < t < 1$. Then, for every $f \in S(X^*)$, there exists $x \in S(X)$ such that $\beta(f, x, t) \geq \beta(t)/2 > 0$. Hence, $g \in S(B(X^*), x, 1 - \beta(t)/2)$ implies $1 - g(x) < \beta(f, x, t)$. Therefore,

$$S(B(X^*), x, 1 - \beta(t)/2) \subseteq B(f, t).$$

Thus X has UMIP. $\square$

Remark 3.3.5. It is to be noted that for $0 < \varepsilon_1 < \varepsilon_2$, $\beta(\varepsilon_1) \leq \beta(\varepsilon_2)$.

It is proved in [47] that if X^* is uniformly convex, then X has UMIP. Since H-UMIP is stronger than UMIP, this also follows from Theorem 3.6.1 below. We now obtain a quantitative relationship between $\delta_{X^*}(t)$ (the modulus of convexity of X^*) and $\beta(t)$ when X^* is uniformly convex, which gives a much simpler proof of the above result.

Theorem 3.3.6. *If X^* is uniformly convex, then $\beta(t) \geq 2\delta_{X^*}(t) > 0$.*

Proof. Let X^* be uniformly convex. Then, X is reflexive. Let $0 < t < 1$. We have $\delta_{X^*}(t) > 0$. Let $f \in S(X^*)$ and $x \in S(X)$ be such that $f(x) = 1$.

Let $g \in B(X^*)$ be such that $g(x) > 1 - 2\delta_{X^*}(t)$. Then

$$\frac{\|f+g\|}{2} \geq \frac{f(x)+g(x)}{2} > 1 - \delta_{X^*}(t).$$

It follows that $\|f-g\| < t$. And hence, $\beta(f, x, t) \geq 2\delta_{X^*}(t)$. Therefore, $\beta(t) \geq 2\delta_{X^*}(t) > 0$. $\square$

Remark 3.3.7. It follows from the above proof that, if $f \in S(X^*)$ and $x \in S(X)$ are such that $f(x) = 1$, then $f \in S := S(B(X^*), x, 1 - \delta_{X^*}(t/2))$ and $diam(S) < t$. This fact will be used later in Chapter 6 while proving stability of UMIP under ℓ_p-sum.

3.4 A few results on slices

Since we are going to use this notion frequently, let us denote by

$$\mathcal{Z} =: \{Z \subseteq X^* : Z \text{ is a w*-closed finite co-dimensional subspace of } X^*\}.$$

Following elementary lemma is an analogue of the Hahn-Banach theorem.

Lemma 3.4.1. *Let $Z \subseteq \mathcal{Z}$. If $u \in X$ and $\|u\|_Z = 1$, then there exists $x \in S(X)$ such that $x|_Z = u|_Z$.*

Proof. Since $Z \in \mathcal{Z}$, there exists a finite dimensional subspace $Y \subseteq X$ such that $Z = Y^\perp = (X/Y)^*$ and

$$1 = \|u\|_Z = \|u + Y\| = d(u, Y).$$

Since Y is finite dimensional, it is proximinal. So, there exists $y \in Y$ such that $d(u, Y) = \|u + y\|$. Therefore, $x = u + y$ works. $\square$

The proof of (b) below is essentially contained in that of [13, Fact 2.2].

Lemma 3.4.2. *Let $\varepsilon > 0$, $0 < \beta < \alpha < 1$, $x \in S(X)$, $Z \in \mathcal{Z}$ and $f \in S(B(Z), x, \alpha)$.*

(a) If $diam(S(B(Z), x, \alpha)) < \varepsilon$ then $\alpha \geq f(x) - \varepsilon$.

(b) $diam(S(B(Z), x, \beta)) \leq \left(\dfrac{f(x) - \beta}{f(x) - \alpha}\right) diam(S(B(Z), x, \alpha))$.

Proof. (a). If possible, let $\alpha < f(x) - \varepsilon$. Choose $g \in B(Z)$ such that $\alpha < g(x) < f(x) - \varepsilon$. Then $f, g \in S(B(Z), x, \alpha)$ but $\|f - g\| \geq f(x) - g(x) > \varepsilon$.

(b). Let $g, h \in S(B(Z), x, \beta)$ and $\lambda = \frac{\alpha - \beta}{f(x) - \beta}$. Let $g_\lambda := \lambda f + (1 - \lambda)g$ and $h_\lambda := \lambda f + (1 - \lambda)h$. We have,

$$g_\lambda(x) > \lambda + (1 - \lambda)\beta \geq \alpha.$$

So, $g_\lambda \in S(B(Z), x, \alpha)$. Similarly, $h_\lambda \in S(B(Z), x, \alpha)$. Therefore,

$$\begin{aligned} \|h - g\| &= \frac{\|g_\lambda - h_\lambda\|}{1 - \lambda} \leq \frac{diam(S(B(Z), x, \alpha))}{1 - \lambda} \\ &= \left(\frac{f(x) - \beta}{f(x) - \alpha}\right) diam(S(B(Z), x, \alpha)). \end{aligned}$$

Hence,

$$diam(S(B(Z),x,\beta)) \leq \left(\frac{f(x)-\beta}{f(x)-\alpha}\right) diam(S(B(Z),x,\alpha)).$$

$\square$

Corollary 3.4.3. *Let $\varepsilon > 0$, $0 < \beta < \alpha < 1$, $x \in S(X)$ and $Z \in \mathcal{Z}$ be such $\|x\|_Z = 1$.*

(a) *If $diam(S(B(Z),x,\alpha)) < \varepsilon$, then $\alpha \geq 1-\varepsilon$.*

(b) *$diam(S(B(Z),x,\beta)) \leq \left(\frac{1-\beta}{1-\alpha}\right) diam(S(B(Z),x,\alpha))$.*

Proof. Since, Z is w*-closed and $\|x\|_Z = 1$, we can choose $f \in S(Z)$ such that $f(x) = 1$. Then, it follows directly from Lemma 3.4.2 $\square$

Corollary 3.4.4. *Let $\gamma > 0$, $k \in \mathbb{N}$ be such that $1-2k(1-\gamma) > 0$, $Z \in \mathcal{Z}$ and $x \in X$ be such that $\|x\|_Z = 1$, then*

$$diam(S(B(Z),x,1-2k(1-\gamma))) \leq 2k(diam(S(B(Z),x,\gamma))).$$

When $Z = X^*$, we obtain the following results as corollary to the above results.

Corollary 3.4.5. *Let $\varepsilon > 0$, $0 < \beta < \alpha < 1$ and $x \in S(X)$.*

(a) *If $diam(S(B(X^*),x,\alpha)) < \varepsilon$, then $\alpha \geq 1-\varepsilon$.*

(b) *$diam(S(B(X^*),x,\beta)) \leq \left(\frac{1-\beta}{1-\alpha}\right) diam(S(B(X^*),x,\alpha))$.*

Corollary 3.4.6. *Let $\gamma > 0$ and $k \in \mathbb{N}$ be such that $1-2k(1-\gamma) > 0$, then*

$$diam(S(B(X^*),x,1-2k(1-\gamma))) \leq 2k(diam(S(B(X^*),x,\gamma))).$$

3.5 Characterising asymptotic uniform smoothness

In Section 3.2, we looked at the notion of uniform w*-denting (I). In this section, we introduce the related asymptotic notion.

We begin this section with the duality between AUS and w*-AUC which follows from [16, Proposition 2.1]. We will use this relationship while proving the ball separation characterisation of AUS.

Theorem 3.5.1. *X is AUS if and only if X^* is w*-AUC.*

Let us define an asymptotic analogue of uniform denting discussed in Section 3.2.

Definition 3.5.2. We say that X is asymptotically uniformly denting (I) (AUD (I)) if for every $0 < \varepsilon \leq 2$, there exists $\delta(\varepsilon) > 0$ such that for every $x \in S(X)$, there exist a w*-closed finite co-dimensional subspace Z of X containing x and $f \in S(X^*)$ such that

$$diam(S(B(Z), f, f(x)(1-\delta))) < \varepsilon.$$

We say that X^* is w*-AUD (I) if the finite co-dimensional subspace of X^* is w*-closed and the functional defining the slice comes from X.

The following theorem relates w*-AUC and w*-AUD (I):

Theorem 3.5.3. *X^* is w*-AUD (I) if and only if X^* is w*-AUC.*

Proof. Let X^* be w*-AUC and $0 < \varepsilon \leq 2$. We have, $\overline{\delta}^*(\varepsilon/8) > 0$. Let $0 < \delta < \overline{\delta}^*(\varepsilon/8)$. So, for $x^* \in S(X^*)$, there exists $Y \in \mathcal{Z}$ such that

$$\inf_{y^* \in Y, \|y^*\| = \varepsilon/8} \|x^* + y^*\| - 1 > \delta.$$

Claim : $\inf_{y^* \in Y, \|y^*\| \geq \varepsilon/8} \|x^* + y^*\| = \inf_{y^* \in Y, \|y^*\| = \varepsilon/8} \|x^* + y^*\|.$

If not, then there exists $y_0^* \in Y$ with $\|y_0^*\| > \varepsilon/8$ such that

$$\|x^* + y_0^*\| < \inf_{y^* \in Y, \|y^*\| = \varepsilon/8} \|x^* + y^*\| = A \text{ (say)}.$$

Let $0 < \beta, \gamma < 1$ be such that $\gamma < A - \|x^* + y_0^*\|$ and $\|\beta y_0^*\| = \varepsilon/8$.

Choose $x \in S(X)$ such that $(x^* + \beta y_0^*)(x) > \|x^* + \beta y_0^*\| - \max\{\gamma, \delta/2\}$. So, we have $(x^* + \beta y_0^*)(x) > 1 + \delta/2$ which implies $y_0^*(x) > 0$. Also, $\|x^* + y_0^*\| < \|x + \beta y_0\|$.

We have $(x^* + y_0^*)(x) \leq \|x^* + y_0^*\| < \|x^* + \beta y_0^*\| - \gamma < (x^* + \beta y_0^*)(x)$. That is, $y_0^*(x) < \beta y_0^*(x)$. But, $0 < \beta < 1$. So, $y_0^*(x) < 0$. This is a contradiction.

Therefore, $\inf_{y^* \in Y, \|y^*\| \geq \varepsilon/8} \|x^* + y^*\| - 1 > \delta$.

Clearly, $x^* \notin Y$. Let $Z = span(Y \cup \{x^*\})$. We have that, Z is w*-closed. So, there exists $x \in S(X)$ such that $x^*(x) > 0$ and $Y = \ker(x|_Z)$.

Now, by Lemma 3.1.2,

$$diam(S(B(Z), x, x^*(x)(1-\delta))) < \varepsilon.$$

So, X^* is w*-AUD (I).

Conversely, let X^* be w*-AUD (I) and $0 < \varepsilon \leq 2$. Let $\delta = \delta(\varepsilon/5)$ obtained from the hypothesis.

Let $\beta := \min\left\{\frac{\delta}{1-\delta}, \frac{\varepsilon}{20}\right\}$.

For any $x^* \in S(X^*)$, there is $Z \in \mathcal{Z}$ containing x^* and $x \in S(X)$ such that

$$diam(S(B(Z), x, x^*(x)(1-\delta))) < \varepsilon/5.$$

Let $Y = Z \cap \ker(x)$. Clearly, $Y \in \mathcal{Z}$. And $Y = \ker(x|_Z)$.

Thus, by slight modification in the arguments of Lemma 3.1.3,

$$\inf_{y^* \in Y, \|y^*\| = \varepsilon} \|x^* + y^*\| - 1 \geq \inf_{y^* \in Y, \|y^*\| \geq \varepsilon} \|x^* + y^*\| - 1 > \beta.$$

So, $\overline{\delta}^*(\varepsilon) \geq \beta > 0$. Hence, X^* is w*-AUC. $\square$

Analogous to the above theorem, we have the following results relating AUC and AUD (I).

Theorem 3.5.4. *X is AUC if and only if X is AUD (I).*

In order to obtain a ball separation characterisation of AUS, we need another notion of asymptotic uniform w*-denting (w*-AUD(II)), along the lines of uniform w*-denting (II), defined below:

Definition 3.5.5. X^* is said to be w*-AUD (II) if for every $\varepsilon > 0$, there is $\alpha < 1$ such that for every $f \in S(X^*)$, there is $Z \in \mathcal{Z}$ containing f and $x \in S(X)$ with $\|x\|_Z = 1$ such that

$$f \in S(B(Z), x, \alpha) \text{ and } diam(S(B(Z), x, \alpha)) < \varepsilon.$$

Lemma 3.5.6. *If X^* is w*-AUD (II), then for every $\varepsilon > 0$, there exist $0 < \beta < \alpha < 1$ such that for every $f \in S(X^*)$, there exists $Z \in \mathcal{Z}$ containing f and $x \in S(X)$ with $\|x\|_Z = 1$ satisfying*

$$diam(S(B(Z), x, \beta)) < \varepsilon \text{ and } f(x) > \alpha.$$

Proof. Let X^* be w*-AUD (II). Let $\varepsilon > 0$ be given. There exists $\alpha < 1$ such that given $f \in S(X^*)$, there exists $Z \in \mathcal{Z}$ containing f and $x \in S(X)$ with $\|x\|_Z = 1$ such that

$$diam(S(B(Z), x, \alpha)) < \varepsilon/2 \text{ and } f(x) > \alpha.$$

Let $\beta > 0$ be such that $\alpha > \beta > 2\alpha - 1$. By Corollary 3.4.3 (b), we have that

$$diam(S(B(Z), x, \beta)) \leq \left(\frac{1-\beta}{1-\alpha}\right) diam(S(B(Z), x, \alpha)) < \varepsilon.$$

□

With similar arguments, we can show that

Lemma 3.5.7. *If X^* is uniformly w*-denting (II), then for every $\varepsilon > 0$, there exist $0 < \beta < \alpha < 1$ such that for every $f \in S(X^*)$, there is $x \in S(X)$ satisfying*

$$diam(S(B(X^*), x, \beta)) < \varepsilon \text{ and } f(x) > \alpha.$$

Now, we consider the following ball separation properties. The first property is a uniform version of H-MIP (H-UMIP) and the second is an asymptotic version of H-UMIP (AHUMIP). In this section, we characterise AUS in terms AHUMIP and in the next section similar results are obtained for uniform smoothness and H-UMIP.

Definition 3.5.8. We say that X has the hyperplane uniform Mazur intersection property (H-UMIP) if for every $\varepsilon > 0$ and $M \geq 1$, there is $K > 0$ such that whenever a closed convex set $C \subseteq X$ and $f \in S(X^*)$ are such that $\sup\{\|x\| : x \in C\} \leq M$ and $\inf f(C) \geq \varepsilon$, there is a closed ball $B[x_0, r_0]$ in X such that $C \subseteq B[x_0, r_0]$, $\inf f(B[x_0, r_0]) \geq \varepsilon/2$ and $r_0 \leq K$.

Remark 3.5.9. Note that unlike the UMIP, in this case, the uniform bound is not on the diameter of C, but its farthest distance from the point being separated, which is an apparently stronger assumption.

Definition 3.5.10. X is said have the asymptotic hyperplane uniform Mazur intersection property (AHUMIP) if for every $\varepsilon > 0$, $M \geq 1$, there exists $\gamma, K > 0$ such that for every $f \in S(X^*)$, there exists $Z \in \mathcal{Z}$ such that $f \in Z$ and for any closed convex set $C \subseteq X$ with $C \subseteq B_Z[0, M]$ and $\inf f(C) \geq \varepsilon$, there exists $z_0 \in X$ and $r_0 > 0$ such that $r_0 \leq K$, $C \subseteq B_Z[z_0, r_0] \leq r_0$ and $\inf f(B[z_0, r_0]) \geq \gamma$.

The proof of the following lemma uses the techniques presented in [28, Lemma 2.1].

Lemma 3.5.11. *Let $\varepsilon > 0$, $0 < \beta < 1$, $f \in S(X^*)$. Let $x \in S(X)$ and $Z \in \mathcal{Z}$ be such that $f \in S(B(Z), x/\|x\|_Z, \beta)$ and diam$(S(B(Z), x/\|x\|_Z, \beta)) < \varepsilon$. Then for any $1 > \gamma \geq \beta$, there exists $u \in S(X)$ with $\|u\|_Z = 1$ such that diam$(S(B(Z), u, \gamma)) < 2\varepsilon$ and $f \in S(B(Z), u, \gamma)$.*

Proof. Let $\delta = 1 - \gamma$. Put $x_1 = \frac{x}{\|x\|_Z}$. Let $z \in S(X)$ be such that $f(z) > 1 - \delta/2$.

Define $G(\alpha) := \dfrac{1 + \alpha\beta}{\|z + \alpha x_1\|_Z}$. We have, $G(0) \geq 1$ and $\lim\limits_{\alpha \to \infty} G(\alpha) = \beta$.

Since $1 > 1 - \delta/2 > \beta$, there exists $\alpha_0 > 0$ such that $G(\alpha_0) = 1 - \delta/2$, i.e.,

$$\frac{1 + \alpha_0\beta}{\|z + \alpha_0 x_1\|_Z} = 1 - \delta/2.$$

Let

$$y := \frac{z + \alpha_0 x_1}{\|z + \alpha_0 x_1\|_Z}.$$

By Lemma 3.4.1, there exists $u \in S(X)$ such that $u|_Z = y|_Z$.

Consider $h \in S(B(Z), u, 1 - \delta/2)$. Then,

$$\begin{aligned} 1 + \alpha_0 h(x_1) &\geq h(z) + \alpha_0 h(x_1) = h(z + \alpha_0 x_1) = h(y)\|z + \alpha_0 x_1\|_Z \\ &> (1 - \delta/2)\|z + \alpha_0 x_1\|_Z = 1 + \alpha_0\beta. \end{aligned}$$

So, $h \in S(B(Z), x_1, \beta)$. That is, $S(B(Z), u, 1-\delta/2) \subseteq S(B(Z), x_1, \beta)$ and hence, $diam(S(B(Z), u, 1-\delta/2)) < \varepsilon$. By Corollary 3.4.4, $diam(S(B(Z), u, \gamma)) < 2\varepsilon$. Also,

$$\begin{aligned} f(u) &= f(y) = f\left(\frac{z+\alpha_0 x_1}{\|z+\alpha_0 x_1\|_Z}\right) \geq \frac{f(z)+f(z)\alpha_0 f(x_1)}{\|z+\alpha_0 x_1\|_Z} \\ &> f(z)\frac{1+\alpha_0\beta}{\|z+\alpha_0 x_1\|_Z} = f(z)(1-\delta/2) > (1-\delta/2)^2 > 1-\delta = \gamma. \end{aligned}$$

So, $f \in S(B(Z), u, \gamma)$. □

We will need the following analogues of Lemma 2.2.3 and Theorem 2.2.4.

Lemma 3.5.12. *Let X be a Banach space and $Z \in \mathcal{Z}$. If for $x, y \in X$ with $\|x\|_Z = \|y\|_Z = 1$ and $\varepsilon > 0$,*

$$\{f \in B(Z) : f(x) > \varepsilon\} \subseteq \{g \in X^* : g(y) > 0\},$$

then $\|x-y\|_Z < 2\varepsilon$.

Proof. Let $u = x|_Z$, $v = y|_Z \in S(Z^*)$. Following the arguments in the proof of Lemma 2.2.3, there exists $t > 0$ such that

$$|1-t| \leq \|u-tv\| \leq \varepsilon.$$

And

$$\|x-y\|_Z = \|u-v\| \leq 2\varepsilon.$$

□

Corollary 3.5.13. *Let $Z \in \mathcal{Z}$. If for $x, y \in X$ and $\varepsilon > 0$, if*

$$\emptyset \neq \{f \in B(Z) : f(x) > \varepsilon\} \subseteq \{g \in X^* : g(y) > 0\},$$

then

$$\left\| \frac{x}{\|x\|_Z} - \frac{y}{\|y\|_Z} \right\|_Z < \frac{2\varepsilon}{\|x\|_Z}.$$

Theorem 3.5.14. *Let X be a Banach space and $Z \in \mathcal{Z}$. Let $A \subseteq X$ be bounded in $\|\cdot\|_Z$ seminorm and $d_Z(0, A) > 0$. If there is a non-empty w*-slice of $B(Z)$ contained in*

$$\{f \in B(X^*) : f(x) > 0 \textit{ for all } x \in A\},$$

then there exists a closed ball $B[x_0, r_0] \subseteq X$ such that $A \subseteq B_Z[x_0, r_0]$ and $0 \notin B[x_0, r_0]$.

Proof. The proof is an adaptation of the proof of Theorem 2.2.4.

Let $x_0 \in S(X)$ and $0 < \varepsilon < 1$ be such that

$$\emptyset \neq \{f \in B(Z) : f(x_0) > \varepsilon\} \subseteq \{f \in X^* : f(x) > 0 \text{ for all } x \in A\}.$$

Let $M = \sup\{\|x\|_Z : x \in A\}$. As in Lemma 3.5.12, for all $x \in A$, there exists $t > 0$ such that $1 - \varepsilon/\|x_0\|_Z \leq t \leq 1 + \varepsilon/\|x_0\|_Z$ and

$$\left\| \frac{tx}{\|x\|_Z} - \frac{x_0}{\|x_0\|_Z} \right\|_Z \leq \frac{\varepsilon}{\|x_0\|_Z}.$$

That is,

$$\left\| \frac{t\|x_0\|_Z}{\|x\|_Z} x - x_0 \right\|_Z \leq \varepsilon.$$

Also, since $\varepsilon < \|x_0\|_Z \leq 1$, we have $0 < 1 - \varepsilon/\|x_0\|_Z \leq t < 2$.

Then for $\lambda \geq M/\varepsilon(1 - \varepsilon/\|x_0\|_Z)$ and $0 < d < d_Z(0, A)$,

$$\begin{aligned}
\|x - \lambda x_0\|_Z &\leq \left\| x - \frac{\|x\|_Z}{t\|x_0\|_Z} x_0 \right\|_Z + \left| \frac{\|x\|_Z}{t\|x_0\|_Z} - \lambda \right| \|x_0\|_Z \\
&\leq \frac{\varepsilon\|x\|_Z}{t\|x_0\|_Z} + \lambda\|x_0\|_Z - \frac{\|x\|_Z}{t} \\
&\leq \lambda - \frac{d(1 - \varepsilon/\|x_0\|_Z)}{2}.
\end{aligned}$$

So, $B\left[\lambda x_0, \lambda - \frac{d(1-\varepsilon/\|x_0\|_Z)}{2}\right]$ satisfies the desired conditions. ∎

Now, we come to the main characterisation theorem of this section

Theorem 3.5.15. *The following are equivalent:*

(*a*) *X is AUS.*

(*b*) *X^* is w*-AUC.*

(c) *X^* is w*-AUD (I).*

(d) *X^* is w*-AUD (II).*

(e) *X^* is AHUMIP.*

Proof. $(a) \iff (b)$: This is Theorem 3.5.1.

$(b) \iff (c)$: This is Theorem 3.5.3.

$(c) \implies (d)$: Let X^* has w*-AUD (I).

Let $\varepsilon > 0$ be given. Then, there exists $\delta > 0$ such that for every $f \in S(X^*)$ there is $Z \in \mathcal{Z}$ containing f and $x \in S(X)$ such that

$$S(B(Z), x, f(x)(1-\delta)) < \varepsilon/2.$$

So,

$$S(B(Z), x/\|x\|_Z, f(x/\|x\|_Z)(1-\delta)) < \varepsilon/2.$$

Now, $1 - \delta \geq f(x/\|x\|_Z)(1-\delta)$.

By Lemma 3.5.11, there exists $y \in S(X)$ with $\|y\|_Z = 1$ such that

$$diam(S(B(Z), y, 1-\delta)) < \varepsilon \text{ and } f(y) > 1 - \delta.$$

So, X^* is w*-AUD (II).

$(d) \implies (e)$: Let $\varepsilon > 0$ and $M \geq 2$ be given. Let $M_1 := M + 3\varepsilon/4$. Since X^* is w*-AUD (II), by Lemma 3.5.6, there exist $0 < \alpha < 1$ and $\delta > 0$ such that for every $f \in S(X^*)$, there exists $Z_1 \in \mathcal{Z}$ containing f and $x \in S(X)$ with $\|x\|_{Z_1} = 1$ satisfying

$$diam(S(B(Z_1), x, \alpha - \delta)) < \varepsilon/4M_1 \text{ and } f(x) > \alpha.$$

Choose $k \in \mathbb{N}$ such that $1/2(k-1) < \varepsilon/4M_1$. By Corollary 3.4.4, there exists $0 < \gamma < 1$ such that for every $f \in S(X^*)$, there is $Z_2 \in \mathcal{Z}$ containing f and $x \in S(X)$ with $\|x\|_{Z_2} = 1$ such that

$$diam(S(B(Z_1)), x, \gamma) < \delta/10k \text{ and } f(x) > 1 - \frac{1-\gamma}{2}.$$

Then, by Corollary 3.4.3(a), $(1-\gamma) \leq \delta/10k < 1/2k < \varepsilon/4M_1$. Hence, $1-2k(1-\gamma) > 0$.

Let $f \in S(X^*)$. So, there exist $Z_1, Z_2 \in \mathcal{Z}$ containing f and $x_1, x_2 \in S(X)$ with $\|x_1\|_{Z_1} = \|x_2\|_{Z_2} = 1$ such that

$$\begin{aligned} f(x_1) > \alpha, &\qquad diam(S(B(Z_1), x_1, \alpha - \delta)) < \varepsilon/4M_1, \text{ and} \\ f(x_2) > 1 - \frac{1-\gamma}{2}, &\qquad diam(S(Z_2), x_2, \gamma) < \delta/10k. \end{aligned}$$

Using Lemma 3.5.11, we can assume that $\gamma \geq \sqrt{1 - 1/4k^2} > 1 - 1/2k$.

Let $Z = Z_1 \cap Z_2$. Then $Z \in \mathcal{Z}$ containing f and

$$diam(S(B(Z), x_1, \alpha - \delta)) < \varepsilon/4M_1 \text{ and } diam(S(B(Z), x_2, \gamma)) < \delta/10k.$$

If $g \in B(Z) \setminus S(B(Z), x_1, \alpha - \delta/2)$, then $\|f - g\| \geq f(x_1) - g(x_1) > \delta/2$.

Let $\eta = \gamma(1 - 2k(1-\gamma)) > 0$. We have,

$$\eta = \gamma - 2k\gamma + 2k\gamma^2 \geq \gamma - 2k\gamma + 2k(1 - 1/4k^2) \geq \gamma - 1/2k \geq 1 - 1/k.$$

Also, $\dfrac{f(x_2) - \eta}{f(x_2) - \gamma} \leq \dfrac{1-\eta}{(1-\gamma)/2} = 2\dfrac{1-\eta}{1-\gamma} \leq 2(2k+1)$.

So, by Lemma 3.4.2,

$$diam(S(B(Z), x_2, \eta)) \leq \frac{f(x_2) - \eta}{f(x_2) - \gamma} diam S(B(Z), x_2, \gamma) < \delta/2.$$

Since $f \in S(B(Z), x_2, \eta)$, it follows that $g \notin S(B(Z), x_2, \eta)$ (otherwise, $\|f - g\| < \delta/2$, which is a contradiction). Thus, we obtain that

$$\begin{aligned} f \in S(B(Z), x_2, \gamma) &\subseteq S_1 = S(B(Z), x_2, \eta) \\ &\subseteq S(B(Z), x_1, \alpha - \delta/2) \subseteq S(B(Z), x_1, \alpha - \delta). \end{aligned}$$

Therefore, $diam(S_1) < \varepsilon/4M_1$.

Let C be a closed convex set such that $C \subseteq B_Z[0, M]$ and $\inf f(C) \geq \varepsilon$.

Define $D = \overline{C + 3\varepsilon/4B(X)}$. Then $D \subseteq B_Z[0, M_1]$ and $\inf f(D) \geq \varepsilon/4$. Hence, for $z \in D$ and $g \in B(Z) \cap B(f, \varepsilon/4M_1)$,

$$g(z) \geq f(z) - \|f - g\|\|z\|_Z > \varepsilon/4 - \frac{\varepsilon M_1}{4M_1} = 0.$$

Therefore,

$$S_1 \subseteq B(Z) \cap B(f, \varepsilon/4M_1) \subseteq \{g \in B(Z) : g(z) > 0 \text{ for all } z \in D\}.$$

Since, $f \in S(B(Z), x_2, \gamma)$, $\|x_2\|_Z > \gamma > \eta$. Thus, $(1 - \eta/\gamma) \leq (1 - \eta/\|x_2\|_Z)$. Let

$$d = \frac{d_Z(0, D)(1 - \eta/\gamma)}{2} > 0.$$

Then by Theorem 3.5.14, for $\lambda = M_1/2k(1-\gamma)\eta = M_1/\eta(1-\eta/\gamma) \geq M_1/\eta(1-\eta/\|x_2\|_Z)$, let $B_1 = B\left[\lambda x_2, \lambda - d - \frac{3\varepsilon}{4}\right]$.

So, we have,

$$\sup_{y \in D} \|y - \lambda x_2\|_Z \leq \lambda - d$$

i.e.

$$\sup_{c \in C} \|c - \lambda x_2\|_Z \leq \lambda - d - 3\varepsilon/4.$$

Since $1/2k\eta < 1/2(k-1) < \varepsilon/4M_1$, i.e., $M_1/2k\eta < \varepsilon/4$, we have

$$\begin{aligned} \inf f(B_1) &= f(\lambda x_2) - \lambda + d + \frac{3\varepsilon}{4} > \lambda(\gamma - 1) + d + \frac{3\varepsilon}{4} \\ &= -\frac{M_1}{2k\eta} + d + \frac{3\varepsilon}{4} \geq d + \frac{3\varepsilon}{4} - \varepsilon/4 > \varepsilon/2. \end{aligned}$$

Also,

$$\frac{M_1}{2k\eta(1-\gamma)} - d - \frac{3\varepsilon}{4} \leq \frac{M_1}{2k\eta(1-\gamma)} = K.$$

Hence, X has the AHUMIP.

$(e) \implies (c)$: Let $\varepsilon > 0$ be given. Choose $\gamma, K > 0$ obtained from the hypothesis for $\varepsilon/4$ and $M = 1$. Let $\delta = 1 - \frac{K}{K+\gamma}$.

Let $f \in S(X^*)$. So, there exists $Z \in \mathcal{Z}$ containing f such that for any closed convex set $C \subseteq B_Z[0, M]$ with $f(C) \geq \varepsilon/4$, there exists a closed ball $B[z_0, r_0]$ containing C with

$r_0 \leq K$ and $\inf f(B[z_0, r_0]) \geq \gamma$.

Let $A_\varepsilon := \{x \in B(X) : f(x) \geq \varepsilon/4\}$. Clearly, $A_\varepsilon \subseteq B_Z[0,1]$ and $f(A_\varepsilon) \geq \varepsilon/4$.

So, there exists a closed ball $B[x_0, r_0]$ with $r_0 \leq K$, $A_\varepsilon \subseteq B_Z[x_0, r_0]$ and $\inf f(B[x_0, r_0]) \geq \gamma$.

Let $z_0 = x_0/\|x_0\|$.

Consider $S := S(B(Z), z_0, f(z_0)(1-\delta))$.

Choose $g \in S$. So, $g \in Z$ and $g(z_0) > f(z_0)(K/(K+\gamma))$. So, $g(x_0) > f(z_0)(r_0/(r_0 + \gamma)) > r_0$. So, $g(B[x_0, r_0]) > 0$. Thus, $g(x) > 0$ for all $x \in A_\varepsilon$. So, $\|f - g\| < \varepsilon$.

Thus X^* is w*-AUD (I). □

3.6 Uniform smoothness and H-UMIP

In the last section, we obtained a ball separation characterisation of AUS via asymptotic w*-denting notions. In this section, we follow the same path to prove the equivalence of H-UMIP, uniform w*-denting notions and uniform smoothness.

For $\varepsilon > 0$ and a bounded subset C of X, let

$$F_{C,\varepsilon} := C \setminus \bigcup \{S : S \text{ is a slice of } C \text{ with } diam(S) < \varepsilon\}.$$

If C is a bounded subset of X^* and slices are restricted to w*-slices, then we denote the above quantity as $F^*_{C,\varepsilon}$. Moreover, we use the notation F^*_ε for $F^*_{B(X^*),\varepsilon}$.

Theorem 3.6.1. *The following statements are equivalent :*

(*a*) *X^* is uniformly w*-denting (I).*

(*b*) *For every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$F^*_\varepsilon \subseteq (1 - \delta(\varepsilon))B(X^*).$$

(*c*) *X^* is uniformly convex.*

(*d*) *X^* is uniformly w*-denting (II).*

(e) *X has the H-UMIP.*

Proof. $(a) \implies (b)$. Let X^* be uniformly w*-denting (I) and $\varepsilon > 0$. So, there exists $\delta > 0$ such that for every $f \in S(X^*)$, there is $x \in S(X)$ such that

$$diam(S(B(X^*), x, f(x)(1-\delta))) < \varepsilon.$$

Let $f \in F^*_\varepsilon$. Now, for $h = f/\|f\|$, there exists $x \in S(X)$ such that

$$diam(S(B(X^*), x, h(x)(1-\delta))) < \varepsilon.$$

Since, $f \notin S(B(X^*), x, h(x)(1-\delta))$, $f(x) \leq h(x)(1-\delta)$. Then, $\|f\| \leq 1-\delta$. Therefore,

$$F^*_\varepsilon \subseteq (1-\delta)B(X^*).$$

$(b) \implies (c)$. Let $\varepsilon > 0$ be given. Let $\delta > 0$ be such that

$$F^*_{\varepsilon/2} \subseteq (1-\delta)B(X^*).$$

Claim : If $f, g \in S(X^*)$ and $\|f-g\| \geq \varepsilon$, then $\left\|\dfrac{f+g}{2}\right\| \leq 1-\delta$.

Suppose not. Then, $\left\|\dfrac{f+g}{2}\right\| > 1-\delta$ and hence, $\dfrac{f+g}{2} \notin F_{\varepsilon/2}$.

Thus, there exists $S = S(B(X^*), x, \alpha)$ such that $\dfrac{f+g}{2} \in S$ and $diam(S) < \varepsilon/2$.

Now, $\dfrac{f(x)+g(x)}{2} > \alpha$ implies either $f(x) > \alpha$ or $g(x) > \alpha$, that is, at least one of f or $g \in S$.

Since $\dfrac{f+g}{2} \in S$ and $diam(S) < \varepsilon/2$, $\left\|\dfrac{f-g}{2}\right\| < \varepsilon/2$, or $\|f-g\| < \varepsilon$. Contradiction. Hence, X^* is uniformly convex.

$(c) \implies (d)$. Let $\varepsilon > 0$ be given. Then, there exists $0 < \delta \leq \varepsilon/4$ such that for any $f, g \in S(X^*)$ with

$$\frac{\|f+g\|}{2} > 1-\delta,$$

we have $\|f-g\| \leq \varepsilon/4$.

Let $f \in S(X^*)$. Since X^* is uniformly convex, X is reflexive. Therefore, there exists $x \in S(X)$ such that $f(x) = 1$. Let $g \in B(X^*)$ be such that $g(x) > 1 - \delta$. We have $\|g\| > 1 - \delta \geq 1 - \varepsilon/4$ and therefore, $1 - \|g\| < \varepsilon/4$ and $\frac{g}{\|g\|}(x) > \frac{1}{\|g\|}(1-\delta) > 1 - \delta$. Now,

$$\begin{aligned} \frac{\|f + (g/\|g\|)\|}{2} &\geq \frac{(f + (g/\|g\|))(x)}{2} = \frac{1}{2}(1 + (g/\|g\|)(x)) \\ &> \frac{1}{2}(1 + 1 - \delta) = 1 - \delta/2 > 1 - \delta. \end{aligned}$$

Then, $\|f - \frac{g}{\|g\|}\| < \varepsilon/4$, and hence,

$$\|f - g\| = \|f - \frac{g}{\|g\|}\| + 1 - \|g\| < \varepsilon/2.$$

Hence, $diam(S(B(X^*), x, 1-\delta)) < \varepsilon$ and $f \in S(B(X^*), x, 1-\delta)$. So, X^* is uniformly w*-denting (II).

$(d) \implies (e)$. Let $\varepsilon > 0$ and $M \geq 2$ be given. Let $M_1 := M + 3\varepsilon/4$. Since X^* is uniformly w*-denting (II), by Lemma 3.5.7, there exist $0 < \alpha < 1$ and $\delta > 0$ such that for every $f \in S(X^*)$, there exists $x \in S(X)$ satisfying

$$diam(S(B(X^*), x, \alpha - \delta)) < \varepsilon/4M_1 \text{ and } f(x) > \alpha.$$

Choose $k \in \mathbb{N}$ such that $1/2k < \varepsilon/4M_1$. Again, there exists $0 < \gamma < 1$ such that for every $f \in S(X^*)$, there is $x \in S(X)$ such that

$$f(x) > \gamma \text{ and } diam(S(B(X^*), x, \gamma)) < \delta/4k.$$

Then, by Lemma 3.4.5, part (a), $(1-\gamma) \leq \delta/4k < 1/2k < \varepsilon/4M_1$. Hence, $1 - 2k(1 - \gamma) > 0$.

It suffices to show that for $K = \frac{M_1}{2k(1-\gamma)}$, whenever a closed convex set C and $f \in S(X^*)$ are such that $\sup\{\|x\| : x \in C\} \leq M$ and $\inf f(C) \geq \varepsilon$, there is a closed ball $B[x_0, r_0]$ in X such that $C \subseteq B[x_0, r_0]$, $\inf f(B[x_0, r_0]) \geq \varepsilon/2$ and $r_0 \leq K$.

Let C and $f \in S(X^*)$ be as above. For f, there exists $x_1 \in S(X)$ such that

$$diam(S(B(X^*), x_1, \alpha - \delta)) < \varepsilon/4M_1 \text{ and } f(x_1) > \alpha.$$

Again, there exists $x_2 \in S(X)$ such that

$$diam(S(B(X^*), x_2, \gamma)) < \delta/4k \text{ and } f(x_2) > \gamma.$$

If $g \notin S(B(X^*), x_1, \alpha - \delta/2)$, then $\|f - g\| \geq f(x_1) - g(x_1) > \delta/2$.

Let $\eta = 1 - 2k(1-\gamma) > 0$. By Corollary 3.4.6,

$$diam(S(B(X^*), x_2, \eta)) \leq 2k(diam(S(B(X^*), x_2, \gamma))) < \delta/2.$$

Since $f \in S(B(X^*), x_2, \eta)$, it follows that $g \notin S(B(X^*), x_2, \eta)$. Thus, we obtain that

$$\begin{aligned} f \in S(B(X^*), x_2, \gamma) &\subseteq S_1 = S(B(X^*), x_2, \eta) \\ &\subseteq S(B(X^*), x_1, \alpha - \delta/2) \subseteq S(B(X^*), x_1, \alpha - \delta). \end{aligned}$$

Therefore, $diam(S_1) < \varepsilon/4M_1$.

Define $D = \overline{C + 3\varepsilon/4B(X)}$. Then $D \subseteq B[0, M_1]$ and $\inf f(D) \geq \varepsilon/4$. Hence, for $z \in D$ and $g \in B(X^*) \cap B(f, \varepsilon/4M_1)$,

$$g(z) \geq f(z) - \|f - g\|\|z\| > \varepsilon/4 - \frac{\varepsilon M_1}{4M_1} = 0.$$

Therefore,

$$S_1 \subseteq B(X^*) \cap B(f, \varepsilon/4M_1) \subseteq \{g \in B(X^*) : g(z) > 0 \text{ for all } z \in D\}.$$

Let

$$d = \frac{d(0, D)(1-\eta)}{1+\eta} > 0.$$

Then as in Theorem 2.2.4, for $\lambda = M_1/(1-\eta) = M_1/2k(1-\gamma)$,

$$D \subseteq B\left[\lambda x_2, \lambda - d\right].$$

i.e.

$$C \subseteq B\left[\lambda x_2, \lambda - d - \frac{3\varepsilon}{4}\right] = B_1.$$

Since, $1/2k < \varepsilon/4M_1$, i.e., $M_1/2k < \varepsilon/4$, we have

$$\begin{aligned} \inf f(B_1) &= f(\lambda x_2) - \lambda + d + \frac{3\varepsilon}{4} > \lambda(\gamma - 1) + d + \frac{3\varepsilon}{4} \\ &= -\frac{M_1}{2k} + d + \frac{3\varepsilon}{4} \geq d + \frac{3\varepsilon}{4} - \varepsilon/4 > \varepsilon/2. \end{aligned}$$

Also,

$$\frac{M_1}{2k(1-\gamma)} - d - \frac{3\varepsilon}{4} \leq \frac{M_1}{2k(1-\gamma)} = K.$$

Hence, X has the H-UMIP.

$(e) \implies (a)$. Let $0 < \varepsilon < 1$ and $M \geq 1$ be given. Then, there is $K = K > 0$ such that whenever a closed convex set $C \subseteq X$ and $f \in S(X^*)$ are such that $\sup\{\|x\| : x \in C\} \leq M$ and $\inf f(C) \geq \varepsilon/4$, there is a closed ball $B[x_0, r_0]$ in X such that $C \subseteq B[x_0, r_0]$, $\inf f(B[x_0, r_0]) \geq \varepsilon/8$ and $r_0 \leq K$. We may assume w.l.o.g. that $K \geq 1$.

Let $f \in S(X^*)$. Consider $A := \{x \in B(X) : f(x) \geq \varepsilon/4\}$. Then, $\inf f(A) = \varepsilon/4$. Also, $\sup\{\|x\| : x \in C\} \leq 1 \leq M$. So, there exist $B = B[x_0, r]$ containing A such that $r \leq K$ and $\inf f(B) \geq \varepsilon/8$. Put $\alpha = K/(K + \varepsilon/9)$. We claim that $\delta = 1 - \alpha$ works. Note that $r/(r + \varepsilon/9) \leq K/(K + \varepsilon/9) = \alpha$.

Since $\inf f(B) \geq \varepsilon/8$, we have $f(x_0) \geq r + \varepsilon/8 > r + \varepsilon/9$.

Put $z_0 = x_0/\|x_0\|$. Let $S = S(B(X^*), z_0, f(z_0)(1 - \delta))$.

Let $g \in S$. Then, $g(z_0) > f(z_0)\alpha$, which implies, $g(x_0) > f(x_0)\alpha > f(x_0)(r/(r + \varepsilon/9)) \geq r$. That is, $\inf g(B) > 0$. Thus,

$$\{x \in B(X) : f(x) > \varepsilon/3\} \subseteq A \subseteq B \subseteq \{x : g(x) > 0\}.$$

It follows from Lemma 2.2.3 that $\|f - g\| \leq 2\varepsilon/3$. But $f \in S$. Therefore,

$$diam(S) < 4\varepsilon/3.$$

Therefore, X^* is uniformly w*-denting (I). $\square$

Remark 3.6.2. $(a) \implies (b)$ above is hinted in [17, p 11], but the details are from a personal communication from Late Prof. Sudipta Dutta.

$(b) \implies (c)$ was pointed out to us by Prof. Gilles Lancien recently.

Since, X is uniformly smooth if and only if X^* is uniformly convex, we have the following corollary of the above theorem.

Corollary 3.6.3. *X has H-UMIP if and only if X is uniformly smooth.*

Chapter 4

Generalised MIP

4.1 Introduction

We have already explored the uniform version of MIP in Chapter 2. In this chapter, we look at some generalisations of MIP. We show that strong $\mathcal{C}$-MIP is actually a more satisfactory generalisation of MIP for general families. It can be characterised in terms of a more natural notion of the (strong) $\mathcal{C}$-semidenting points. Moreover, one can also obtain complete analogue of [21, Theorem 2.1].

We also introduce the uniform version of $\mathcal{C}$-MIP and its stronger form and characterise them analogously. Even in this case, strong $\mathcal{C}$-UMIP appears have richer characterisations than $\mathcal{C}$-UMIP. In the last section, we summarise the interrelations between various notions discussed in this chapter with examples and counterexamples.

Contents of this chapter has appeared in [4].

4.2 Preliminaries

Throughout this chapter, $\mathcal{C}$ will denote a compatible class of sets (as defined in Definition 1.0.25). We will denote by $\tau_{\mathcal{C}}$, or, simply τ when there is no ambiguity, the topology on X^* of uniform convergence on the sets of $\mathcal{C}$, or, in other words, the topology on X^* generated by the family of seminorms $\{\|\cdot\|_C : C \in \mathcal{C}\}$. Since $\mathcal{C}$ is a compatible class, the family $\{B_C(0,\varepsilon) : C \in \mathcal{C}, \varepsilon > 0\}$ forms a local base for $\tau_{\mathcal{C}}$ at 0. For the classes $\mathcal{C} = \mathcal{F}$, $\mathcal{K}$, $\mathcal{W}$ and $\mathcal{B}$, the topology $\tau_{\mathcal{C}}$ coincides with the w*-, bw*-, Mackey and norm topologies, respectively. We note that the topologies $\tau_{\mathcal{F}}$ (w*-) and $\tau_{\mathcal{K}}$ (bw*-) coincide on bounded sets.

Definition 4.2.1. (*a*) For $\varepsilon, \delta > 0$, $x \in S(X)$ and $C \in \mathcal{C}$ denote by

$$\begin{aligned} d_1(C,x,\delta) &= \sup_{0<\lambda<\delta,\ y\in C} \frac{\|x+\lambda y\| + \|x-\lambda y\| - 2}{\lambda} \\ d_2(C,x,\delta) &= diam_C(S(B(X^*),x,\delta)) \\ d_3(C,x,\delta) &= diam_C(D(S(X)\cap B(x,\delta))) \end{aligned}$$

(*b*) [2] For $\varepsilon, \delta > 0$ and $C \in \mathcal{C}$

$$M_{\varepsilon,\delta,C}(X) = \{x \in S(X) : \sup_{0<\|y\|<\delta, y\in C} \frac{\|x+y\| + \|x-y\| - 2}{\|y\|} < \varepsilon\}.$$

In other words, $M_{\varepsilon,\delta,C}(X) = \{x \in S(X) : d_1(C,x,\delta) < \varepsilon\}$. Define

$$M_{\varepsilon,C}(X) = \bigcup_{\delta>0} M_{\varepsilon,\delta,C}(X).$$

(*c*) Define $H = \bigcap\{\overline{D(M_{\varepsilon,C}(X))}^{\tau} : C \in \mathcal{C}, \varepsilon > 0\}$.

Following lemma is a generalisation of Lemma 2.2.8. Details can be found in [1, Lemma 2.1]:

Lemma 4.2.2. *For any $\alpha, \delta > 0$, we have,*

1. $d_2(C,x,\alpha) \leq d_1(C,x,\delta) + \dfrac{2\alpha}{\delta}$.
2. $d_3(C,x,\delta) \leq d_2(C,x,\delta)$.
3. $d_1(C,x,\delta) \leq d_3(C,x,2\delta)$.

Known results on $\mathcal{C}$-MIP [2, 11, 12] are summarised in our notation and terminology below. $\mathcal{C}$-denting and $\mathcal{C}$-semidenting points were already defined earlier (Definition 1.0.27).

Theorem 4.2.3. *[2, Theorem 1, Corollary 1], [11, Theorem 1.10], [12, Theorem 2.3] Suppose $\mathcal{C}$ is a compatible class of bounded sets. Consider the following statements :*

(*a*) *The cone generated by $\mathcal{C}$-denting points of $B(X^*)$ is τ-dense in X^*.*

(b) *The cone generated by* H *is* τ*-dense in* X^**.*

(c) *If* $C_1, C_2 \in \mathcal{C}$ *are closed convex such that there exists* $f \in X^*$ *with* $\sup f(C_1) < \inf f(C_2)$*, then there exist disjoint closed balls* B_1, B_2 *such that* $C_i \subseteq B_i$*,* $i = 1, 2$*.*

(d) X *has* $\mathcal{C}$*-MIP.*

(e) *Every* $f \in S(X^*)$ *is a* $\mathcal{C}$*-semidenting point of* $B(X^*)$*.*

(f) *For every norm dense subset* $A \subseteq S(X)$ *and every support mapping* ϕ*, the cone generated by* $\phi(A)$ *is* τ*-dense in* X^**.*

Then $(a) \implies (b) \implies (c) \implies (d) \iff (e) \implies (f)$*. Moreover, if*

$$A = \{x \in S(X) : D(x) \text{ contains a } \mathcal{C}\text{-denting point of } B(X^*)\}$$

is norm dense in $S(X)$*, then all the statements are equivalent.*

Remark 4.2.4. In [11, Theorem 1.10], it is shown that (a) and (d) are equivalent under a weaker assumption that every w*-slice of $B(X^*)$ contains a $\mathcal{C}$-denting point.

4.3 Strong $\mathcal{C}$-MIP

Definition 4.3.1. A Banach space X is said to have strong $\mathcal{C}$-MIP if for every closed convex $C \in \mathcal{C}$ and $\beta \geq 0$, $\overline{C + \beta B(X)}$ is the intersection of closed balls containing it.

Vanderwerff [45] called this $\mathcal{K}$-IP and $\mathcal{W}$-IP for $\mathcal{C} = \mathcal{K}$ and $\mathcal{W}$, respectively. Clearly, for $\mathcal{C} = \mathcal{B}$, $\mathcal{C}$-MIP and strong $\mathcal{C}$-MIP coincide. See Section 4.6 for some examples and counterexamples.

Let us recall the definition of a strong $\mathcal{C}$-semidenting point from the Introduction (Definition 1.0.29).

Definition 4.3.2. We say that $f \in S(X^*)$ is a strong $\mathcal{C}$-semidenting point of $B(X^*)$ if for each $A \in \mathcal{C}$ and $\varepsilon > 0$, there exists a w*-slice S of $B(X^*)$ such that $S \subseteq B_A(f, \varepsilon)$.

We begin by observing that the strong $\mathcal{C}$-semidenting points can be characterised completely analogous to [21, Lemma 2.2].

Lemma 4.3.3. *For* $f \in S(X^*)$*, the following are equivalent:*

(a) $f \in H$*, that is,* $f \in \bigcap_{\varepsilon>0} \overline{D(M_{\varepsilon,C}(X))}^{\tau}$ *for every* $C \in \mathcal{C}$.

(b) f *is a strong* $\mathcal{C}$*-semidenting point.*

(c) For every $\beta \geq 0$ *and closed convex* $C \in \mathcal{C}$ *such that* $\inf f(\overline{C + \beta B(X)}) > 0$*, there exists a closed ball* $B[x_0, r_0]$ *in* X *containing* $C + \beta B(X)$ *with* $0 \notin B[x_0, r_0]$.

(d) For every $\varepsilon > 0$ *and* $C \in \mathcal{C}$*, there exist* $\delta > 0$ *and* $x \in S(X)$ *such that* $D(S(X) \cap B(x, \delta)) \subseteq B_C(f, \varepsilon)$.

Proof. $(a) \implies (b)$: For $C \in \mathcal{C}$, let $f \in \bigcap_{\varepsilon>0} \overline{D(M_{\varepsilon,C}(X))}^{\tau}$. Let $\varepsilon > 0$ be given. We have $f \in \overline{D(M_{\varepsilon/4,C}(X))}^{\tau}$. So, there exist $\delta > 0$ and $x \in M_{\varepsilon/4,\delta,C}(X)$ and $g \in D(x)$ such that $\|f - g\|_C < \varepsilon/4$.

Since $x \in M_{\varepsilon/4,\delta,C}(X)$, we have $d_1(C, x, \delta) < \varepsilon/4$. By Lemma 4.2.2$(i)$, for $\alpha = \varepsilon\delta/8$,

$$d_2(C, x, \alpha) \leq d_1(C, x, \delta) + \frac{2\alpha}{\delta} < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

Hence, $diam_C(S(B(X^*), x, \alpha)) < \varepsilon/2$.

Since $g \in S(B(X^*), x, \delta)$, $S(B(X^*), x, \delta) \subseteq B_C(f, \varepsilon)$.

$(b) \implies (c)$: Let $\beta \geq 0$ and $C \in \mathcal{C}$ such that $\inf f(\overline{C + \beta B(X)}) > 0$.

Let $\inf f(\overline{C + \beta B(X)}) = \alpha > 0$. Now, there exist $\delta > 0$ and $x \in S(X)$ such that

$$S := S(B(X^*), x, \delta) \subseteq B_C(f, \alpha).$$

Choose $g \in S$ and $c \in C$. We have,

$$g(c) = f(c) - \|f - g\|_C > \alpha + \beta - \alpha = \beta.$$

Hence, for $y \in \overline{C + \beta B(X)}$, $g(y) > 0$.

So, by Theorem 2.2.4, there is a closed ball $B[x_0, r_0]$ containing $C + \beta B(X)$ such that $0 \notin B[x_0, r_0]$.

$(c) \implies (b)$: Let $C \in \mathcal{C}$ and $\varepsilon > 0$ be given.

Let $z_0 \in X$ be such that $\|z_0\| = 1+\varepsilon/2$ and $f(z_0) > 1+\varepsilon/4$. Let $D = \overline{aco}(C \cup \{z_0\})$. We have $D \in \mathcal{C}$.

Consider $A := \{x \in D : f(x) \geq \varepsilon/4\}$. We have $A \in \mathcal{C}$.

For $\beta = \frac{\varepsilon/4}{1+\varepsilon/4}$, $f(\overline{A + \beta B(X)}) \geq \varepsilon/4 - \beta > 0$. So, there exists a closed ball $B[x_0, r_0]$ containing $\overline{A + \beta B(X)}$ such that $0 \notin B[x_0, r_0]$.

So, for $r_1 = r_0 - \beta$ and $x_1 = x_0$, $A \subseteq B[x_1, r_1]$ and $d(0, B[x_1, r_1]) > \beta$.

Consider $S = \{g \in B(X^*) : g(x_1/\|x_1\|) > (r_1 + \beta)/\|x_1\|\}$.

For $g \in S$, $g(x_1/\|x_1\|) > (r_1 + \beta)/\|x_1\|$. we have, $g(x_1) > r_1 + \beta$ which implies $\inf g(B[x_1, r_1]) \geq \beta$. Thus, $g(x) > 0$ for all $x \in A$. Therefore, applying Lemma 2.2.3 to the semi-norm $\|\cdot\|_D$,

$$\left\| f - \frac{\|f\|_D}{\|g\|_D} g \right\|_D < \varepsilon/2.$$

Consider $y_0 = \dfrac{\varepsilon}{4} \dfrac{z_0}{f(z_0)}$. Since $f(z_0) > \varepsilon/4$, D is absolutely convex, and $f(y_0) = \varepsilon/4$, $y_0 \in A$, and hence, $g(y_0) > \beta$.

Therefore, $g(z_0) = \dfrac{f(z_0) g(y_0)}{\varepsilon/4} > \dfrac{(1+\varepsilon/4)\beta}{\varepsilon/4} = 1$ and so, $\|g\|_D \geq 1$.

Since $f, g \in B(X^*)$ and $D \subseteq (1+\varepsilon/2)B(X)$,

$$1 \leq \|f\|_D, \|g\|_D \leq 1 + \varepsilon/2.$$

Hence,

$$\big| \|f\|_D - \|g\|_D \big| < \varepsilon/2.$$

Finally,

$$\|f - g\|_C \leq \|f - g\|_D \leq \left\| f - \frac{\|f\|_D}{\|g\|_D} g \right\|_D + \big| \|f\|_D - \|g\|_D \big| < \varepsilon.$$

$(b) \implies (d)$: Let $f \in S(X^*)$ be a strong $\mathcal{C}$-semidenting point. Let $\varepsilon > 0$ and $C \in \mathcal{C}$ be given. There exist $\delta > 0$ and $x \in S(X)$ such that

$$S(B(X^*), x, \delta) \subseteq B_C(f, \varepsilon/4).$$

So, $d_2(C,x,\delta) = diam_C(S(B(X^*),x,\delta)) < \varepsilon/2$ and $D(x) \subseteq B_C(f,\varepsilon/4)$. Now, by Lemma 4.2.2, $d_3(C,x,\delta) = diam_C(D(S(X)\cap B(x,\delta))) < \varepsilon/2$. Hence,

$$D(S(X)\cap B(x,\delta)) \subseteq B_C(f,\varepsilon).$$

$(d) \implies (a)$: To show $f \in \overline{D(M_{\varepsilon,C}(X))}^{\tau}$ for every $C \in \mathcal{C}$ and $\varepsilon > 0$, let $0 < \eta < \varepsilon$ and $C_1 \in \mathcal{C}$. Enough to show that there exist $x \in M_{\varepsilon,C}$ and $g \in D(x)$ such that $\|f-g\|_{C_1} < \eta$.

Let $C_0 = C \cup C_1 \in \mathcal{C}$. By (d), there exist $\delta > 0$ and $x \in S(X)$ such that $D(S(X)\cap B(x,\delta)) \subseteq B_{C_0}(f,\eta/2)$.

So, $d_3(C_0,x,\delta) \leq \eta$ and $D(x) \subseteq B_{C_0}(f,\eta/2)$. By Lemma 4.2.2, $d_1(C_0,x,\delta/2) \leq d_3(C_0,x,\delta) \leq \eta$. Hence, $x \in M_{\eta,\delta/2,C_0} \subseteq M_{\varepsilon,\delta/2,C} \subseteq M_{\varepsilon,C}$ and $\|f-g\|_{C_1} < \eta$ for any $g \in D(x)$. ∎

Definition 4.3.4. The duality map on X is said to be norm-τ quasicontinuous if for every $f \in S(X^*)$, $\varepsilon > 0$ and $C \in \mathcal{C}$, there exist $\delta > 0$ and $x \in S(X)$ such that $D(S(X)\cap B(x,\delta)) \subseteq B_C(f,\varepsilon)$.

We are now ready for our main theorem characterising strong $\mathcal{C}$-MIP. Again, we obtain a complete analogue of [21, Theorem 2.1].

Theorem 4.3.5. *For a Banach space X, the following are equivalent:*

(a) X has strong $\mathcal{C}$-MIP.

(b) Every $f \in S(X^*)$ is a strong $\mathcal{C}$-semidenting point.

(c) For every $\varepsilon > 0$ and $C \in \mathcal{C}$, $D(M_{\varepsilon,C}(X))$ is τ-dense in $S(X^*)$.

(d) The duality map on X is norm-τ quasicontinuous.

(e) Every support mapping in X maps norm dense sets in $S(X)$ to τ-dense sets in $S(X^*)$.

Proof. Equivalence of (a), (b), (c) and (d) follows from Lemma 4.3.3.

$(d) \implies (e)$: Let $A \subseteq S(X)$ be norm dense in $S(X)$. Let $\phi : S(X) \to S(X^*)$ be a support mapping. Let $f \in S(X^*)$, $C \in \mathcal{C}$ and $\varepsilon > 0$. It is enough to find $x_0 \in A$ such that $\|f - \phi(x_0)\|_C < \varepsilon$.

By (d), there exist $\delta > 0$ and $x \in S(X)$ such that $D(S_X \cap B(x, \delta)) \subseteq B_C(f, \varepsilon)$. Since, A is dense in $S(X)$, there exists $x_0 \in A$ such that $\|x_0 - x\| < \delta$. If follows that $D(x_0) \subseteq B_C(f, \varepsilon)$. Therefore, for any support mapping ϕ, $\|f - \phi(x_0)\|_C < \varepsilon$.

$(e) \implies (d)$: Suppose the duality map is not norm-τ quasicontinuous. Then there exist $f \in S(X^*)$, $\varepsilon > 0$ and $C \in \mathcal{C}$, such that for any $x \in S(X)$ and $n \in \mathbb{N}$, there exists $y_{n,x} \in S(X)$ and $g_{n,x} \in D(y_{n,x})$ such that $\|y_{n,x} - x\| < 1/n$ and $\|g_{n,x} - f\|_C \geq \varepsilon$.

Let $\phi : S(X) \to S(X^*)$ be a support mapping that maps $y_{n,x}$ to $g_{n,x}$. Let $A = \{y_{n,x} : x \in S(X), n \in \mathbb{N}\}$. Then A is norm dense in $S(X)$, but $\phi(A)$ is not τ-dense in $S(X^*)$. $\square$

Remark 4.3.6. For $\mathcal{C} = \mathcal{K}$ or $\mathcal{W}$, Vanderwerff [45] observed that strong $\mathcal{C}$-MIP is equivalent to the following condition:

(b') For every $\varepsilon > 0$ and $C \in \mathcal{C}$, the points in $S(X^*)$ that lie in w*-slices of $B(X^*)$ having C-diameter $\leq \varepsilon$ is τ-dense in $S(X^*)$.

It is easy to see that this condition is equivalent to (b) above and holds for any compatible family $\mathcal{C}$. Now consider the following condition:

(b'') The $\mathcal{C}$-denting points of $B(X^*)$ are τ-dense in $S(X^*)$.

Clearly, $(b'') \implies (b)$. Moreover, if every w*-slice of $B(X^*)$ contains a $\mathcal{C}$-denting point, then both are equivalent. Notice that this condition holds in any Asplund space.

Since the $\tau_{\mathcal{F}}$ and $\tau_{\mathcal{K}}$ topologies coincide on bounded sets, we obtain

Corollary 4.3.7. *For a Banach space X, the following are equivalent:*

(a) *X has the strong $\mathcal{K}$-MIP.*

(b) *X has the strong $\mathcal{F}$-MIP.*

(c) *The extreme points of $B(X^*)$ are w*-dense in $S(X^*)$.*

Though we have given enough evidence that the strong $\mathcal{C}$-semidenting points are more natural than the $\mathcal{C}$-semidenting points, one question remains unanswered: Can we characterise $\mathcal{C}$-MIP in terms of strong $\mathcal{C}$-semidenting points?

So let us consider the condition:

(a') The cone generated by strong $\mathcal{C}$-semidenting points of $B(X^*)$ is τ-dense in X^*.

It is clear from Theorem 4.2.3 that this condition is generally stronger than $\mathcal{C}$-MIP and is equivalent to it if every w*-slice of $B(X^*)$ contains a strong $\mathcal{C}$-semidenting point, a condition slightly weaker than the one that gives equivalence in Theorem 4.2.3.

4.4 $\mathcal{C}$-UMIP and strong $\mathcal{C}$-UMIP

In this section, we discuss uniform versions of $\mathcal{C}$-MIP and strong $\mathcal{C}$-MIP.

The following quantitative result is contained in the proof of Theorem 2.2.4.

Lemma 4.4.1. *Let $A \subseteq X$ be such that $M = \sup\{\|x\| : x \in A\}$ and $d = d(0, A) > 0$. Suppose there exist $x_0 \in S(X)$ and $0 < \gamma < 1$ such that*

$$S = \{f \in B(X^*) : f(x_0) > \gamma\} \subseteq \{f \in B(X^*) : f(x) > 0 \textit{ for all } x \in A\}.$$

Then for $\lambda \geq M/(1-\gamma)$,

$$A \subseteq B\left[\lambda x_0, \lambda - \frac{d(1-\gamma)}{1+\gamma}\right].$$

We observe that there are three ways of defining uniform versions of strong $\mathcal{C}$-MIP ((a), (b) and (c) below) and they are equivalent.

Theorem 4.4.2. *In a Banach space X, the following are equivalent:*

- (a) *For every $\varepsilon > 0$, $\beta \geq 0$, $M \geq 2$ and $0 < \eta < \varepsilon$, there exists $K > 0$ such that for every closed convex $C \in \mathcal{C}$ with $diam(C) \leq M$ and $d(0, C + \beta B(X)) \geq \varepsilon$, there exist $z_0 \in X$, $0 < r \leq K$ such that $C + \beta B(X) \subseteq B[z_0, r]$ and $d(0, B[z_0, r]) > \eta$.*
- (b) *For every $\varepsilon > 0$, $M \geq 2$ and $0 < \eta < \varepsilon$, there exists $K > 0$ such that for every closed convex $C \in \mathcal{C}$ with $diam(C) \leq M$ and $d(0, C) \geq \varepsilon$, there exist $z_0 \in X$, $0 < r \leq K$ such that $C \subseteq B[z_0, r]$ and $d(0, B[z_0, r]) > \eta$.*
- (c) *For every $\varepsilon > 0$, $\beta \geq 0$ and $M \geq 2$, there exist $K > 0$ and $0 < \eta < \varepsilon$, such that for every closed convex $C \in \mathcal{C}$ with $diam(C) \leq M$ and $d(0, C + \beta B(X)) \geq \varepsilon$, there exist $z_0 \in X$, $0 < r \leq K$ such that $C + \beta B(X) \subseteq B[z_0, r]$ and $d(0, B[z_0, r]) > \eta$.*

(*d*) *For every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for every* $C \in \mathcal{C}$*,* $C \subseteq B(X)$ *and* $f \in S(X^*)$*, there is* $x \in S(X)$ *such that*

$$S(B(X^*), x, \delta) \subseteq B_C(f, \varepsilon).$$

(*e*) *For every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for every* $C \in \mathcal{C}$*,* $C \subseteq B(X)$ *and* $f \in S(X^*)$*, there is* $x \in S(X)$ *such that* $D(S_X \cap B(x, \delta)) \subseteq B_C(f, \varepsilon)$*.*

(*f*) *For every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for every* $C \in \mathcal{C}$*,* $C \subseteq B(X)$ *and* $f \in S(X^*)$*, there exists* $x \in M_{\varepsilon,\delta,C}$ *such that* $D(x) \subseteq B_C(f, \varepsilon)$*.*

Proof. Notice that for any bounded $C \subseteq X$ and $r, \beta > 0$,

$$C \subseteq B[z_0, r] \iff C + \beta B(X) \subseteq B[z_0, r + \beta]$$

and

$$d(0, B[z_0, r + \beta]) = d(0, B[z_0, r]) - \beta$$

provided $d(0, B[z_0, r]) > \beta$.

Clearly, $(a) \implies (b)$ and $(a) \implies (c)$.

$(b) \implies (d)$: This is a quantitatively more precise version of the proof of Lemma 4.3.3 ($(c) \implies (b)$).

Let $\varepsilon > 0$ be given. Let $\alpha = \dfrac{\varepsilon}{4+\varepsilon} > 0$ and $\eta \in (\alpha, \varepsilon/4)$. Choose K for $\varepsilon/4$, η and $M = 4$. That is, for every $C \in \mathcal{C}$ with $d(0, C) \geq \varepsilon/4$ and $diam(C) \leq 4$, there exists a closed ball $B[z_0, r]$ containing C with $r \leq K$ and $d(0, B[z_0, r]) > \eta$. We will show that $\delta = \dfrac{\eta - \alpha}{K + \eta}$ works.

Let $C \in \mathcal{C}$ such that $C \subseteq B(X)$ and $f \in S(X^*)$. As in the proof of Lemma 4.3.3 ($(c) \implies (b)$), let $z_0 \in X$ be such that $\|z_0\| = 1 + \varepsilon/2$ and $f(z_0) > 1 + \varepsilon/4$ and $D = \overline{aco}(C \cup \{z_0\})$. Again, $D \in \mathcal{C}$ and $\|f\|_D \geq 1$. For $A = \{y \in D : f(y) \geq \varepsilon/4\}$, $A \in \mathcal{C}$, diam $(A) \leq 4$ and $d(0, A) \geq \varepsilon/4$.

By assumption, there exist $x_0 \in X$ and $r > 0$ such that $A \subseteq B[x_0, r]$, $d(0, B[x_0, r]) > \eta$ and $r \leq K$. Thus, $\|x_0\| > r + \eta$. Now let

$$S := S\left(B(X^*), \frac{x_0}{\|x_0\|}, \delta\right) = \left\{g \in B(X^*) : g\left(\frac{x_0}{\|x_0\|}\right) > \frac{K+\alpha}{K+\eta}\right\}.$$

Then, for $g \in S$, we have $g(x_0/\|x_0\|) > (K+\alpha)/(K+\eta) > (r+\alpha)/(r+\eta)$. So, $g(x_0) > (r+\alpha)\|x_0\|/(r+\eta)$. But, $\|x_0\| > r+\eta$. Thus, $g(x_0) > r+\alpha$. So, $\inf g(B[x_0, r]) > g(x_0) - r > \alpha > 0$. And hence, again as in Lemma 4.3.3 ($(c) \implies (b)$), applying Lemma 2.2.3 to $\|.\|_D$,

$$\left\| f - \frac{\|f\|_D}{\|g\|_D} g \right\|_D < \varepsilon/2.$$

Now, arguing as in Lemma 4.3.3 ($(c) \implies (b)$), we have that for $g \in S$,

$$\|f - g\|_C \leq \|f - g\|_D < \varepsilon.$$

$(c) \implies (d)$: This is a minor variant of the above proof.

Let $\varepsilon > 0$ be given. Let $\alpha = \frac{\varepsilon}{4+\varepsilon} > 0$ and $\beta \in (\alpha, \varepsilon/4)$. Choose K and η for $\varepsilon/4 - \beta$, β and $M = 4$. That is, for every $C \in \mathcal{C}$ with $d(0, C + \beta B(X)) \geq \varepsilon/4 - \beta$ and $diam(C) \leq 4$, there exists a closed ball $B[z_0, r]$ containing $C + \beta B(X)$, $d(0, B[z_0, r]) > \eta > 0$ and $r \leq K$.

We will show that $\delta = \frac{\beta - \alpha}{K + \beta}$ works.

Let $C \in \mathcal{C}$ such that $C \subseteq B(X)$ and $f \in S(X^*)$.

As before, let $z_0 \in X$ be such that $\|z_0\| = 1 + \varepsilon/2$ and $f(z_0) > 1 + \varepsilon/4$. Let $D = \overline{aco}(C \cup \{z_0\})$ and $A = \{y \in D : f(y) \geq \varepsilon/4\}$. Then $D, A \in \mathcal{C}$, $\|f\|_D \geq 1$, $\mathrm{diam}(A) \leq 4$ and $d(0, A + \beta B(X)) \geq \varepsilon/4 - \beta$.

By assumption, there exist $x_0 \in X$ and $0 < r \leq K$ such that $A + \beta B(X) \subseteq B[x_0, r]$, and $d(0, B[x_0, r]) > \eta$. Thus, $\|x_0\| > r + \eta$ and $A \subseteq B[x_0, r - \beta]$, $d(0, B[x_0, r - \beta]) \geq \eta + \beta$ and $r - \beta \leq K$.

Rest of the proof is identical as above and shows that

$$S\left(B(X^*), \frac{x_0}{\|x_0\|}, \delta\right) \subseteq B_C(f, \varepsilon).$$

$(d) \implies (a)$: Let $\varepsilon > 0$, $\beta \geq 0$, $M \geq 2$ and $0 < \eta < \varepsilon$ be given.

Let $L = M + \varepsilon + 2\beta$. Clearly, $L > 1$.

Let $\varepsilon_1 = \varepsilon/L$ and $\eta_1 = \eta/L$. Choose $0 < \delta < 1$ such that for any $C_1 \in \mathcal{C}$ with $C_1 \subseteq B(X)$ and $f \in S(X^*)$, there exists $x \in S(X)$ such that

$$S(B(X^*), x, \delta) \subseteq B_{C_1}(f, \varepsilon_1 - \eta_1).$$

Let $K = (L + \varepsilon)/\delta$.

Let $C \in \mathcal{C}$ with $diam(C) \leq M$ and $d(0, C + \beta B(X)) \geq \varepsilon$. We will show that there is a closed ball B of radius $\leq K$ such that $C + \beta B(X) \subseteq B$ and $d(0, B) > \eta$.

Case I: $C + \beta B(X) \setminus B[0, L] \neq \emptyset$.

If $z \in C + \beta B(X) \setminus B[0, M + \varepsilon + 2\beta]$, then $d(0, B[z, M + 2\beta]) \geq \varepsilon > \eta$, $C + \beta B(X) \subseteq B[z, M + 2\beta]$ and $M + 2\beta \leq K$.

Case II: $C + \beta B(X) \subseteq B[0, L]$.

Let $\widetilde{C} = \frac{1}{L}C$ and $\alpha = \beta/L$. Then, $\widetilde{C} \subseteq D := \frac{1}{L}(C + \beta B(X)) = \widetilde{C} + \alpha B(X) \subseteq B(X)$ and $d(0, D) \geq \varepsilon_1$. Thus, $d(0, \widetilde{C}) \geq \varepsilon_1 + \alpha$. It suffices to show that there is a closed ball B of radius $\leq K/L$ such that $D \subseteq B$ and $d(0, B) > \eta_1$. Let $D' = D + \eta_1 B(X) = \widetilde{C} + (\eta_1 + \alpha)B(X)$.

Choose $f \in S(X^*)$ such that $\inf f(\widetilde{C}) \geq \varepsilon_1 + \alpha$. For this f, there exists $x \in S(X)$ such that

$$S := S(B(X^*), x, \delta) \subseteq B_{\widetilde{C}}(f, \varepsilon_1 - \eta_1).$$

Let $h \in S$. For $y \in \widetilde{C}$, we have,

$$h(y) \geq f(y) - \|f - h\|_{\widetilde{C}} > (\varepsilon_1 + \alpha) - (\varepsilon_1 - \eta_1) = \eta_1 + \alpha.$$

That is, $h(y) > 0$ for all $y \in D'$. So,

$$S \subseteq \{g \in X^* : g(x) > 0 \text{ for all } x \in D'\}.$$

By Lemma 4.4.1, we have that for $\lambda = \frac{(1+\varepsilon_1)}{\delta}$

$$D' \subseteq B\left[\lambda x, \lambda - \frac{d(0,D')\delta}{2-\delta}\right].$$

Hence,

$$D \subseteq B\left[\lambda x, \lambda - \frac{d(0,D')\delta}{2-\delta} - \eta_1\right].$$

Also,

$$d(0, B\left[\lambda x, \lambda - \frac{d(0,D')\delta}{2-\delta} - \eta_1\right]) \geq \frac{d(0,D')\delta}{2-\delta} + \eta_1 > \eta_1$$

and

$$\lambda - \frac{d(0,D')\delta}{2-\delta} - \eta_1 \leq \lambda = \frac{1+\varepsilon_1}{\delta} = K/L.$$

This completes the proof.

Equivalence of (d), (e) and (f) follows from Lemma 4.2.2. $\square$

Definition 4.4.3. For a compatible class $\mathcal{C}$, X is said to have strong $\mathcal{C}$-UMIP if any of the equivalent conditions of Theorem 4.4.2 is satisfied.

We again observe that

Corollary 4.4.4. *Strong $\mathcal{K}$-UMIP and strong $\mathcal{F}$-UMIP are equivalent.*

Now we come to a uniform version of $\mathcal{C}$-MIP.

Definition 4.4.5. For a compatible class $\mathcal{C}$, X is said to have $\mathcal{C}$-UMIP if for every $\varepsilon > 0$ and $M \geq 2$, there exist $K > 0$ and $0 < \eta \leq \varepsilon$ such that for every closed convex $C \in \mathcal{C}$ with $diam(C) \leq M$ and $d(0, C) \geq \varepsilon$, there exist $z_0 \in X$, $0 < r \leq K$ such that $C \subseteq B[z_0, r]$ and $d(0, B[z_0, r]) \geq \eta$.

It can be similarly characterised in terms of the uniform version of the semidenting point discussed in [12]. The proof is a slight modification of the arguments of Theorem 4.4.2. We omit the details.

Theorem 4.4.6. *For a compatible class $\mathcal{C}$, the following are equivalent:*

(a) *X has the $\mathcal{C}$-UMIP*

(b) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $C \in \mathcal{C}$, $C \subseteq B(X)$ and $f \in S(X^*)$, there is $x \in S(X)$ such that*

$$S(B(X^*), x, \delta) \subseteq cone\ (B_C(f, \varepsilon)).$$

4.5 $\mathcal{C}$-MIP and $\mathcal{C}$-smoothness

We know that a Banach space with a Fréchet smooth norm has MIP. Also, a separable MIP space is Asplund. In this section, we look at their analogues for (strong) $\mathcal{C}$-MIP.

Definition 4.5.1. Let D be a bounded subset of X.

(a) $F : X \to \mathbb{R}$ is D-differentiable at $x \in X$ if there exists $f \in X^*$ such that

$$\lim_{\lambda \to 0^+} \sup_{d \in D} \left| \frac{F(x + \lambda d) - F(x)}{\lambda} - f(d) \right| = 0.$$

(b) D is said to have the Asplund Property (AP) if every continuous convex function $F : X \to \mathbb{R}$ is D-differentiable on a residual subset of X.

(c) D is said to have the Stegall Property (SP) if for every countable $A \subseteq D$, the seminormed space $(X^*, \|\cdot\|_A)$ is separable.

It is known that the AP and SP are equivalent [9, Theorem 5.2.1].

Let $\mathcal{C}$ be a compatible class.

(d) The norm on X is $\mathcal{C}$-smooth if for every $D \in \mathcal{C}$, it is D-differentiable at every $x \in S(X)$.

(e) X is a $\mathcal{C}$-Asplund space if every $C \in \mathcal{C}$ has the AP/SP.

Proposition 4.5.2. *If the norm on X is $\mathcal{C}$-smooth, then X has strong $\mathcal{C}$-MIP.*

Proof. Let $\varepsilon > 0$, $C \in \mathcal{C}$. Since the norm is C-differentiable at every $x \in S(X)$, it follows that $S(X) \subseteq M_{\varepsilon, C}$. The result now follows from the Bishop-Phelps Theorem and Theorem 4.3.5(c). ☐

Remark 4.5.3. For $\mathcal{C} = \mathcal{K}$ and $\mathcal{W}$, this was observed by Vanderwerff in [45].

Theorem 4.5.4. *If X is a separable $\mathcal{C}$-MIP space, then X is $\mathcal{C}$-Asplund.*

Proof. Let $D \subseteq S(X)$ be a countable dense set. Let $\phi : S(X) \to S(X^*)$ be a support mapping. By Theorem 4.2.3(f), $\mathrm{cone}(\phi(D))$ is τ-dense in X^*. It follows that X^* is τ-separable. And hence, for every $C \in \mathcal{C}$, $(X^*, \|\cdot\|_C)$ is separable.

Let $C \in \mathcal{C}$. Let $A \subseteq C$ be countable. Then $A \in \mathcal{C}$ and hence, $(X^*, \|\cdot\|_A)$ is separable. Thus, C has the SP. $\square$

We know that a separable Asplund space has an equivalent MIP norm. Is an analogue of this also true? That is, if X is a separable $\mathcal{C}$-Asplund space, does it have an equivalent (strong) $\mathcal{C}$-MIP renorming?

4.6 Summary

In this section, we summarise the interrelations between various notions discussed above. We give counterexamples to some of the reverse implications. As far as we know, the other implications are open.

$$\begin{array}{ccccccc}
\text{MIP} & \overset{\Rightarrow}{\not\Leftarrow} & \text{strong } \mathcal{W}\text{-MIP} & \Rightarrow & \text{strong } \mathcal{K}\text{-MIP} & \Leftrightarrow & \text{strong } \mathcal{F}\text{-MIP} \\
\Updownarrow & & \Downarrow & & \Downarrow\not\Uparrow & & \Downarrow \\
\text{MIP} & \overset{\Rightarrow}{\not\Leftarrow} & \mathcal{W}\text{-MIP} & \Rightarrow & \mathcal{K}\text{-MIP} & \Rightarrow & \mathcal{F}\text{-MIP}
\end{array}$$

Examples

- Vanderwerff in [45, Theorem 2.6] shows that there is a Banach space which has $\mathcal{W}$-MIP but it does not have the strong $\mathcal{K}$-MIP. Thus, the $\mathcal{K}$-MIP does not imply the strong $\mathcal{K}$-MIP. This also shows that there exist Banach spaces with the $\mathcal{W}$-MIP but not MIP.

- Vanderwerff in [45, Theorem 1.1] also shows that every Banach space can be renormed to have the $\mathcal{W}$-MIP, and hence, the $\mathcal{K}$-MIP. This again shows that the $\mathcal{W}$-MIP does not imply MIP.

- In ℓ_1, Gâteaux differentiability and weak Hadamard (that is, $\mathcal{W}$-) differentiability coincide (see [22]). Now, ℓ_1, being a separable space, has an equivalent norm that is Gâteaux smooth, and hence, $\mathcal{W}$-smooth, and therefore, has the strong $\mathcal{W}$-MIP. But, being a separable non-Asplund space, it has no MIP renorming.

We also have similar interrelations between the uniform versions. In this case, most of the reverse implications are open.

$$\begin{array}{ccccccc}
\text{UMIP} & \Rightarrow & \text{strong } \mathcal{W}\text{-UMIP} & \Rightarrow & \text{strong } \mathcal{K}\text{-UMIP} & \Leftrightarrow & \text{strong } \mathcal{F}\text{-UMIP} \\
\Updownarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{UMIP} & \Rightarrow & \mathcal{W}\text{-UMIP} & \Rightarrow & \mathcal{K}\text{-UMIP} & \Rightarrow & \mathcal{F}\text{-UMIP}
\end{array}$$

Chapter 5

Some Residuality Results

5.1 Introduction

This chapter is dedicated to the study of some residuality properties. As mentioned in the Introduction, even though properties like rotundity, LUR or uniform rotundity are isometric and not isomorphic properties, still 'almost' every equivalent norm has the property if there is one norm with it. In this chapter, we prove similar residuality results for AUS, uniformly smooth and UMIP norms. We make fine use of ball separation property in each of the cases. Ball separation characterisations for AUS norms and uniform smoothness obtained in Chapter 3 play a crucial role in the residuality of uniformly smooth and AUS norms. Most of the techniques are inspired by the residuality results in [11]. The content of this chapter can be found in [5].

For a subspace Z of X^*, $B \in \mathcal{B}$, $C \subseteq X$, and $x \in X$, let

(a) $\|x\|_{Z,B} := \sup\{g(x) : g \in Z, \|g\|_B^* \leq 1\}$;

(b) $d_{Z,B}(x, C) = \inf\{\|x - z\|_{Z,B} : z \in C\}$ and

(c) $diam_{Z,B}(C) = \sup\{\|x - y\|_{Z,B} : x, y \in C\}$.

Let us consider following easy observation on some quantities.

Lemma 5.1.1. *Let $B_1, B_2 \in \mathcal{B}$. For every $\varepsilon_1 > 0$ and $M_1 \geq 1$, there exists $\varepsilon_2 > 0$ and $M_2 \geq 1$ such that for any $A \subseteq X$ closed convex and a subspace Z of X^*, we have the following relations:*

(a) $d_{Z,B_1}(0, A) \geq \varepsilon_1 \implies d_{Z,B_2}(0, A) \geq \varepsilon_2$.

(b) $diam_{Z,B_1}(A) \leq M_1 \implies diam_{Z,B_2}(A) \leq M_2$.

In particular, for $Z = X^*$, we have the following:

Lemma 5.1.2. *Let $B_1, B_2 \in \mathcal{B}$. For every $\varepsilon_1 > 0$, $M_1 \geq 1$ and $K_1 \geq 1$, there exists $\varepsilon_2 > 0$, $M_2 \geq 1$ and $K_2 \geq 1$ such that for any $A \subseteq X$ closed, bounded and convex,*

(a) $d_{B_1}(0, A) \geq \varepsilon_1 \implies d_{B_2}(0, A) \geq \varepsilon_2$

(b) $diam_{B_1}(A) \leq M_1 \implies diam_{B_2}(A) \leq M_2$

(c) $A \subseteq K_1 B_1 \implies A \subseteq K_2 B_2$.

We may drop the subscript in cases when the subspace is X^* or the norm under consideration is clear from the context.

5.2 Residuality of norms with UMIP

First, we start with the residuality of norms with UMIP.

Following lemma is an easy equivalence of UMIP and hence we skip its proof.

Lemma 5.2.1. *X has the UMIP if and only if for every $\varepsilon > 0$ and $M \geq 2$, there exist $K > 0$ and $0 < \delta \leq \varepsilon$ such that for every closed, bounded, convex $A \subseteq X$ with $d(0, A) \geq \varepsilon$ and $diam(A) \leq M$, we have a closed ball $B[z_0, r]$ with $r \leq K$ such that $A \subseteq B[z_0, r]$ and $d(0, B[z_0, r]) \geq \delta$.*

Theorem 5.2.2. *If $(X, \|\cdot\|)$ has the UMIP, then there is a dense G_δ subset $\mathcal{B}_0$ of $\mathcal{B}$ such that $(X, \|\cdot\|_B)$ has the UMIP for each $B \in \mathcal{B}_0$.*

Proof. For $n, k, \in \mathbb{N}$, $k \geq 2$ let

$$\begin{aligned} \mathcal{A}_{n,k} &:= \{A \subseteq X : A \text{ is closed, convex and bounded, } d(0, A) \geq 1/n, \\ &\quad\;\; diam(A) \leq k\} \text{ and} \\ \mathcal{B}_{n,k} &:= \{B \in \mathcal{B} : \text{ there exist } 0 < \gamma < 1/n,\ K > 0 \text{ such that for every} \\ &\quad\;\; A \in \mathcal{A}_{n,k},\ \text{we have } z_0 \in X,\ r \leq K,\ \text{so that } A \subseteq z_0 + rB, \\ &\quad\;\; d_B(0, z_0 + rB) > \gamma\}. \end{aligned}$$

Then,

$$\mathcal{B}_0 = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \mathcal{B}_{n,k}$$

is a dense G_δ set in $\mathcal{B}$ and each element of $\mathcal{B}_0$ has the UMIP.

We first prove that $\mathcal{B}_{n,k}$ is open in $\mathcal{B}$.

Let $B_0 \in \mathcal{B}_{n,k}$. So, there exists $0 < \gamma < 1/n$ and $K > 0$ such that for any $A \in \mathcal{A}_{n,k}$, $A \subseteq z_0 + rB_0$ for some $z_0 \in X$ and $r \leq K$ and $d_{B_0}(0, z_0 + rB_0) > \gamma$.

Choose $1 > \delta > 0$ such that

$$\frac{\gamma - K\delta(\delta+2)}{1+\delta} > \frac{\gamma}{2}$$

Let $B \in \mathcal{B}$ such that $h(B_0, B) < \delta$. Then,

$$\frac{1}{(1+\delta)} B_0 \subseteq B \subseteq (1+\delta)B_0.$$

It follows that

$$\frac{1}{(1+\delta)} \|\cdot\|_B \leq \|\cdot\|_{B_0} \leq (1+\delta)\|\cdot\|_B$$

and

$$A \subseteq z_0 + rB_0 \subseteq z_0 + r(1+\delta)B.$$

Now, $r(1+\delta) \leq 2K$ and

$$\begin{aligned} d_B(0, z_0 + r(1+\delta)B) &= \|z_0\|_B - r(1+\delta) \geq \frac{\|z_0\|_{B_0}}{1+\delta} - r(1+\delta) \\ &> \frac{\gamma + r}{1+\delta} - r(1+\delta) = \frac{\gamma - r\delta(2+\delta)}{1+\delta} \\ &> \frac{\gamma - K\delta(2+\delta)}{1+\delta} > \frac{\gamma}{2}. \end{aligned}$$

So, $B \in \mathcal{B}_{n,k}$. Hence, $\mathcal{B}_{n,k}$ is open.

Now, we will show that $\mathcal{B}_{n,k}$ is dense in $\mathcal{B}$.

Let $B_0 \in \mathcal{B}$ and $\varepsilon > 0$. We need to show that there exists $B \in \mathcal{B}_{n,k}$ such that $h(B, B_0) < \varepsilon$.

Let $\lambda > h(B_0, B(X))$ and let $\delta = \varepsilon/2(1+\lambda)$. Consider

$$B = \overline{B_0 + \delta B(X)}.$$

It follows that

$$B_0 \subseteq B \subseteq (1+\varepsilon/2)B_0$$

and hence,

$$h(B_0, B) \leq \varepsilon/2 < \varepsilon.$$

Also, $h(B, B(X)) < \varepsilon + \lambda = \beta$ (say).

We will prove that $B \in \mathcal{B}_{n,k}$.

Since X has the UMIP, $B(X) \in \mathcal{B}_{n,k}$. And hence, there exist $K > 0$ and $\gamma > 0$ such that for any $A \in \mathcal{A}_{n,k}$, there exists $z_0 \in X$ and $r > 0$ such that $A \subseteq z_0 + rB(X)$, $r \leq K$ and $d(0, z_0 + rB(X)) > \gamma$.

Let $A \in \mathcal{A}_{n,k}$. So, there exists $z_0 \in X$ and $r > 0$ such that $A \subseteq z_0 + rB(X)$, $r \leq K$ and $d(0, z_0 + rB(X)) > \gamma$.

Choose $f \in X^*$ such that $\|f\| = 1$ and $f(z_0) = \|z_0\|$. It follows that

$$\inf f(z_0 + rB(X)) = \|z_0\| - r = d(0, z_0 + rB(X)) > \gamma.$$

Let $x \in B_0$ be such that

$$\frac{K}{\delta}(f(x) + \|f\|_{B_0}) < \gamma/2.$$

Then,

$$z_0 - \frac{r}{\delta}x + \frac{r}{\delta}B \supseteq z_0 - \frac{r}{\delta}x + \frac{r}{\delta}(x + \delta B(X)) \supseteq z_0 + rB(X) \supseteq A.$$

Hence,

$$A \subseteq z_0 - \frac{r}{\delta}x + \frac{r}{\delta}B \quad \text{and} \quad \frac{r}{\delta} \leq \frac{K}{\delta}.$$

Also,

$$\begin{aligned}\inf f(z_0+\frac{r}{\delta}B-\frac{r}{\delta}x) &= \inf f(z_0+rB(X))+\frac{r}{\delta}\inf f(B_0)-\frac{r}{\delta}f(x)\\ &> \gamma-\frac{r}{\delta}\|f\|_{B_0}-\frac{r}{\delta}f(x)\geq\gamma-\frac{K}{\delta}(\|f\|_{B_0}+f(x))\\ &\geq \gamma-\frac{\gamma}{2}=\frac{\gamma}{2}\end{aligned}$$

Hence,

$$d_B(0,z_0-\frac{r}{\delta}x+\frac{r}{\delta}B)\geq\inf\frac{f}{\|f\|_B}(z_0+\frac{r}{\delta}B-\frac{r}{\delta}x)\geq\frac{\gamma}{2\|f\|_B}\geq\frac{\gamma}{2(1+\beta)}.$$

So, $B\in\mathcal{B}_{n,k}$.

By the Baire Category Theorem,

$$\mathcal{B}_0=\bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty}\mathcal{B}_{n,k}$$

is dense in $\mathcal{B}$.

Now, let $B\in\mathcal{B}_0$. We will show that $(X,\|\cdot\|_B)$ has the UMIP.

Let $\varepsilon>0$ and $M\geq 2$. By Lemma 5.1.2, there exist $n,k\in\mathbb{N}$ such that if $A\subseteq X$ is closed, bounded, convex with $d_B(0,A)\geq\varepsilon$ and $diam_B(A)\leq M$, then $A\in\mathcal{A}_{n,k}$.

Since $B\in\mathcal{B}_0$, there exist $K>0$ and $\gamma>0$ such that for each $A\in\mathcal{A}_{n,k}$, there is a z_0 and $r>0$ such that z_0+rB contains A, $r\leq K$ and $d_B(0,z_0+rB)>\gamma$.

Thus, $(X,\|\cdot\|_B)$ has the UMIP. $\square$

5.3 Residuality of uniformly smooth norms

In Theorem 3.6.1, we obtained a ball separation characterisation of uniform smoothness. We use that to prove residuality of uniformly smooth norms.

Following lemma can be obtained by slight perturbation of the quantities involved in H-UMIP, hence we would skip its proof.

Lemma 5.3.1. *X has the H-UMIP if and only if for every $\varepsilon > 0$ and $M \geq 1$, there exists $0 < \gamma_0, \gamma_1 \leq \varepsilon/2$, $K > 0$ such that for every closed convex set $\sup\{\|c\| : c \in C\} \leq M$ and $f \in S(X^*)$ satisfying $\inf f(C) \geq \varepsilon - \gamma_0$, there exists $z_0 \in X$ and $r_0 \leq K$ with $C \subseteq B[z_0, r_0]$ and $f(z_0) - r_0 \geq \gamma_1$.*

Following theorem establishes the residuality of uniformly smooth norms.

Theorem 5.3.2. *If $(X, \|\cdot\|)$ has the H-UMIP, then there is a dense G_δ subset $\mathcal{B}'_0$ of $\mathcal{B}$ such that $(X, \|\cdot\|_B)$ has the H-UMIP for each $B \in \mathcal{B}'_0$.*

Proof. For $n, k \in \mathbb{N}$, $k \geq 2$, let

$$\begin{aligned} \mathcal{B}'_{n,k} \quad := \quad & \{B \in \mathcal{B} : \text{ there exist } 0 < \gamma_0, \gamma_1 < 1/2n \text{ and } K > 0 \text{ such that for} \\ & \text{every } A \text{ closed, convex with } A \subseteq B[0,k], \text{ and } f \in S(X^*, \|\cdot\|_B^*) \\ & \text{with } \inf f(A) \geq 1/n - \gamma_0, \text{we have } z_0 \in X,\ r \leq K, \text{ so that} \\ & A \subseteq z_0 + rB, \text{ and } \inf f(z_0 + rB) > \gamma_1\}. \end{aligned}$$

We will show that

$$\mathcal{B}'_0 = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \mathcal{B}'_{n,k}$$

is a G_δ dense subset of $\mathcal{B}$ and each element of $\mathcal{B}'_0$ has the H-UMIP.

Claim 1: Each $B \in \mathcal{B}'_0$ has the H-UMIP.

Let $\varepsilon > 0$ and $M \geq 1$. By Lemma 5.1.2, there exists $M_1 \geq 1$ such that for any $A \subseteq X$ closed, convex with $A \subseteq B_B[0, M]$ implies $A \subseteq B[0, M_1]$. Choose $n \in \mathbb{N}$ such that $1/n < \varepsilon$ and $k \geq M_1$. Now, $B \in B'_{n,k}$. Choose $0 < \gamma_0, \gamma_1 < 1/2n$ and $K > 0$ as in the definition of $B'_{n,k}$.

Let $A \subseteq X$ closed, convex with $A \subseteq B_B[0, M]$ and $f \in S(X^*, \|\cdot\|_B)$ be such that $\inf f(A) \geq \varepsilon$. Then $A \subseteq B[0, k]$ and $\inf f(A) \geq 1/n - \gamma_0$. Thus, by the definition of $B'_{n,k}$, there exists $z_0 \in X$ and $r \leq K$ such that $A \subseteq z_0 + rB$ with $\inf f(z_0 + rB) \geq \gamma_1$. Thus, $(X, \|\cdot\|_B)$ has the H-UMIP.

Claim 2: For each $n, k \in \mathbb{N}$, $\mathcal{B}'_{n,k}$ is open in $\mathcal{B}$.

Fix $n, k \in \mathbb{N}$ and $B_0 \in \mathcal{B}'_{n,k}$. By definition, there exist $0 < \gamma_0, \gamma_1 < 1/2n$ and $K > 0$ such that for every closed, convex set A with $A \subseteq B[0, k]$, and $f \in S(X^*, \|\cdot\|_{B_0})$ with $\inf f(A) \geq 1/n - \gamma_0$, we have $z \in X$, $r \leq K$, so that $A \subseteq z + rB_0$, and $\inf f(z + rB_0) > \gamma_1$.

Choose $0 < \alpha_0 < \gamma_0$. Let $0 < \delta < \alpha_0$ be such that

$$\frac{\gamma_1}{K} > \delta(\delta + 2) \text{ and } \gamma_0 > \frac{\delta/n + \alpha_0}{1 + \delta}$$

Let $B \in \mathcal{B}$ be such that $h(B, B_0) < \delta$.

We show that $B \in \mathcal{B}'_{n,k}$ with α_0, $\alpha_1 = \frac{1}{1+\delta}(\gamma_1 - K\delta(\delta + 2))$ and $K_1 = 2K$ as the desired quantities for B. Note that $0 < \alpha_1 < \gamma_1$.

Let A be a closed convex set such that $A \subseteq B[0, k]$ and $\|f\|_B = 1$ such that $\inf f(A) \geq 1/n - \alpha_0$. It follows that $\frac{1}{1+\delta} \leq \|f\|_{B_0} \leq 1 + \delta$.

Let $g = f/\|f\|_{B_0}$. Then $g \in S(X^*, \|\cdot\|_{B_0})$ and

$$\inf g(A) \geq \frac{(1/n - \alpha_0)}{\|f\|_{B_0}} \geq \frac{1/n - \alpha_0}{1 + \delta} > 1/n - \gamma_0.$$

The last inequality follows from $\gamma_0 > \frac{\delta/n+\alpha_0}{1+\delta}$.

Since $B_0 \in \mathcal{B}'_{n,k}$, there exists $z_0 \in X$, $r_0 \leq K$, so that $A \subseteq z_0 + r_0B_0$, and $\inf g(z_0 + r_0B_0) = g(z_0) - r_0 > \gamma_1$. Thus, $f(z_0) - r_0\|f\|_{B_0} \geq \gamma_1\|f\|_{B_0}$.

Now, $A \subseteq z_0 + r_0B_0 \subseteq z_0 + r_0(1 + \delta)B$. Also, $r_0(1 + \delta) \leq K_1$. And,

$$\begin{aligned}
\inf f(z_0 + r_0(1 + \delta)B) &= f(z_0) - r_0(1 + \delta) \\
&= f(z_0) - r_0\|f\|_{B_0} + r_0\|f\|_{B_0} - r_0(1 + \delta) \\
&\geq \gamma_1\|f\|_{B_0} - r_0(1 + \delta - \|f\|_{B_0}) \\
&\geq \frac{\gamma_1}{1 + \delta} - K\left(1 + \delta - \frac{1}{1 + \delta}\right) \\
&= \frac{\gamma_1}{1 + \delta} - K\left(\frac{\delta(\delta + 2)}{1 + \delta}\right) = \alpha_1
\end{aligned}$$

Thus, $B \in \mathcal{B}'_{n,k}$, and hence, $\mathcal{B}'_{n,k}$ is open in $\mathcal{B}$.

Claim 3: For each $n, k \in \mathbb{N}$, $\mathcal{B}'_{n,k}$ is dense in $\mathcal{B}$.

Fix $n, k \in \mathbb{N}$ and let $B_1 \in \mathcal{B}$ and $\varepsilon > 0$. We will show that there exists $B \in \mathcal{B}'_{n,k}$ with $h(B, B_1) < \varepsilon$.

Let $\lambda > h(B_1, B(X))$ and let $\delta = \varepsilon/2(1+\lambda)$. Consider

$$B = \overline{B_1 + \delta B(X)}.$$

It follows that

$$B_1 \subseteq B = \overline{B_1 + \delta B(X)} \subseteq B_1 + \delta(1+\lambda)B_1 = (1+\varepsilon/2)B_1$$

and hence,

$$h(B_1, B) \leq \varepsilon/2 < \varepsilon.$$

We will prove that $B \in \mathcal{B}'_{n,k}$.

Also, $h(B, B(X)) < \varepsilon + \lambda = \beta$ (say). So, $\frac{1}{1+\beta}\|\cdot\|_B \leq \|\cdot\| \leq (1+\beta)\|\cdot\|_B$.

Since X has the H-UMIP, $B(X) \in \mathcal{B}'_{n',k}$ where n' is such that $1/n' < 1/n(1+\beta)$. And hence, there exist $0 < \gamma_0, \gamma_1 \leq 1/2n'$ such that for any closed convex $A \subseteq B[0,k]$ and for $\|f\| = 1$ satisfying $\inf f(A) \geq 1/n' - \gamma_0$, there exists $z_0 \in X$ and $r > 0$ such that $A \subseteq z_0 + rB(X)$, $r \leq K$ and $\inf f(z_0 + rB(X)) > \gamma_1$.

We show that $\alpha_0 = \gamma_0$, $\alpha_1 = \frac{\gamma_1}{2(1+\beta)}$ and $K_1 = K/\delta$ as the desired quantities for B.

Let $A \subseteq B[0,k]$ and $\|f\|_B = 1$ be such that $\inf f(A) \geq 1/n - \alpha_0 = 1/n - \gamma_0$. Put $g = \frac{f}{\|f\|}$. Then, $\|g\| = 1$ and $\inf g(A) \geq \frac{1}{n(1+\beta)} - \frac{\gamma_0}{1+\beta} > 1/n' - \gamma_0$. So, there exists $z_0 \in X$ and $r > 0$ such that $A \subseteq z_0 + rB(X)$, $r \leq K$ and $\inf g(z_0 + rB(X)) > \gamma_1$. That is, $\inf f(z_0 + rB(X)) > \gamma_1/(1+\beta)$.

Let $x \in B_1$ be such that

$$\frac{K}{\delta}(f(x) + \|f\|_{B_1}) < \gamma_1/2(1+\beta).$$

Then,

$$A \subseteq z_0 + rB(X) = z_0 - \frac{r}{\delta}x + \frac{r}{\delta}(x + \delta B(X)) \subseteq z_0 - \frac{r}{\delta}x + \frac{r}{\delta}B$$

and $\frac{r}{\delta} \leq \frac{K}{\delta} = K_1$. Also,

$$\begin{aligned}
\inf f(z_0 - \frac{r}{\delta}x + \frac{r}{\delta}B) &= \inf f(z_0 + rB(X)) + \frac{r}{\delta}\inf f(B_1) - \frac{r}{\delta}f(x) \\
&> \frac{\gamma_1}{1+\beta} - \frac{r}{\delta}\|f\|_{B_1} - \frac{r}{\delta}f(x) \\
&\geq \frac{\gamma_1}{1+\beta} - \frac{K}{\delta}(\|f\|_{B_1} + f(x)) \\
&\geq \frac{\gamma_1}{1+\beta} - \frac{\gamma_1}{2(1+\beta)} = \frac{\gamma_1}{2(1+\beta)}
\end{aligned}$$

So, $B \in \mathcal{B}'_{n,k}$. □

Corollary 5.3.3. *If $(X, \|\cdot\|)$ is uniformly smooth, then the set of equivalent uniformly smooth norms is residual in the set of all equivalent norms on X.*

Proof. This follows from the characterisation of uniform smoothness in Theorem 3.6.1 and Theorem 5.3.2. □

Notice that if $B \in \mathcal{B}$ then the corresponding dual ball $B^* = B^\circ$, the polar of B. It follows that if $B_1, B_2 \in \mathcal{B}$, then $h(B_1^*, B_2^*) = h(B_1, B_2)$.

Since in the dual of a reflexive space, any equivalent norm on X^* is a dual norm, we have the following observation:

Remark 5.3.4. If $\mathcal{B}_0$ is residual in $\mathcal{B}$ on a reflexive space, then the corresponding set $\mathcal{B}_0^*$ of dual norms is residual in the set of equivalent norm on X^*, and vice-versa.

Corollary 5.3.5. *If $(X, \|\cdot\|)$ is uniformly convex, then the set of equivalent uniformly convex norms is residual in the set of all equivalent norms on X.*

Corollary 5.3.6. *If X is super-reflexive, then there exists a norm on X which is both uniformly convex and uniformly smooth.*

5.4 Residuality of AUS norms

Following is an easy observation and we will skip its proof:

Lemma 5.4.1. *X has AHUMIP if and only if for every $\varepsilon > 0$, $0 < \alpha < \varepsilon$ and $M \geq 1$, there exists $K > 0$ such that for every $f \in S(X^*)$, there is $Z \in \mathcal{Z}$ containing f such that* *for any closed, convex set A with $A \subseteq B_Z[0, M]$ and* $\inf f(A) \geq \varepsilon$*, there exists* $z_0 \in X$ *and* $r_0 > 0$ *such that* $r_0 \leq K$*,* $A \subseteq B_Z[z_0, r_0]$ *and* $\inf f(B[z_0, r_0]) \geq \varepsilon - \alpha$*.*

We now come to the residuality of AUS norms.

Theorem 5.4.2. *If X has AHUMIP, then there exists a dense G_δ-subset $\mathcal{B}_0 \subseteq \mathcal{B}$ such that for every $B \in \mathcal{B}_0$, $(X, \|\cdot\|_B)$ has AHUMIP.*

Proof. For $n, k \in \mathbb{N}$ and $0 < \alpha < 1/2n$, consider the following object:

$$\begin{aligned}\mathcal{A}_{\alpha,n,k} \;:=\; & \{(A, f, Z) : A \subseteq X \text{ closed convex }, f \in S(X^*) \text{ and } Z \in \mathcal{Z} \\ & \text{containing } f \text{ such that there exists a closed ball } z_0 + r_0 B(X) \\ & \text{satisfying } r_0 \leq k, \sup_{a \in A} \|z_0 - a\|_Z \leq r_0 \text{ and} \\ & \inf f(z_0 + r_0 B(X)) \geq 1/n - \alpha\}.\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_{n,k} \;:=\; & \{B \in \mathcal{B} : \text{there exists } 0 < \alpha, \gamma < 1/2n, K \geq 1 \text{ such that for any} \\ & f \in S(X^*, \|\cdot\|_B^*), \text{ closed and convex } A \subseteq X \text{ and } Z \in \mathcal{Z} \\ & \text{containing } f \text{ with } (A, f/\|f\|, Z) \in \mathcal{A}_{\alpha,n,k}, \text{ there is a closed ball} \\ & z_1 + r_1 B, r_1 \leq K \text{ satisfying } \sup_{a \in A} \|a - z_1\|_{Z,B} \leq r_1 \text{ and} \\ & \inf f(z_1 + r_1 B) \geq \gamma\}.\end{aligned}$$

Let
$$\mathcal{B}_0 = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \mathcal{B}_{n,k}.$$

Claim 1: If $B \in B_0$, then $(X, \|\cdot\|_B)$ has AHUMIP.

Let $\varepsilon > 0$ and $M \geq 1$ be given. Let $h(B, B(X)) < \beta$. Choose $M_1 \geq M(1 + \beta)$ and $0 < \varepsilon_1 < \varepsilon/(1 + \beta)$.

Since $(X, \|\cdot\|)$ has AHUMIP, there exist $\gamma, K > 0$ that work for ε_1, M_1.

Let $n, k \in \mathbb{N}$ be such that $\gamma > 1/n$, $K \leq k$. Now, $B \in \mathcal{B}_0$. So, $B \in \mathcal{B}_{n,k}$.

Thus, there exists $0 < \alpha_1, \gamma_1 < 1/2n$, $K_1 > 0$ that satisfy the $\mathcal{B}_{n,k}$ conditions for B.

Let $f \in S(X^*, \|\cdot\|_B^*)$. Then, there exists $Z \in \mathcal{Z}$ containing f that works for $f/\|f\|$ in $(X, \|\cdot\|)$ with ε_1, M_1 and $\gamma, K > 0$.

Now, let A be a closed convex set with $\sup_{a\in A}\|a\|_{Z,B} \leq M$ and $\inf f(A) \geq \varepsilon$. So, $\sup_{a\in A}\|a\|_Z \leq M_1$ and $\inf(f/\|f\|)(A) \geq \varepsilon_1$.

So, there exists a closed ball $z_0 + r_0B(X)$ satisfying $\sup_{a\in A}\|a - z_0\|_Z \leq r_0$, $r_0 \leq K$ and $\inf(f/\|f\|)(z_0 + r_0B(X)) \geq \gamma > 1/n - \alpha_1$.

So, $(A, f/\|f\|, Z) \in \mathcal{A}_{\alpha_1,n,k}$. And the $\mathcal{B}_{n,k}$ conditions imply that there exists a closed ball $z_1 + r_1B$ satisfying $\sup_{a\in A}\|a - z_1\|_{Z,B} \leq r_1$, $r_1 \leq K_1$ and $\inf f(z_1 + r_1B) \geq \gamma_1$.

So, $(X, \|\cdot\|_B)$ has AHUMIP.

Claim 2: For each $n, k \in \mathbb{N}$, $\mathcal{B}_{n,k}$ is open in $\mathcal{B}$.

Let $n, k \in \mathbb{N}$ and $B_0 \in \mathcal{B}_{n,k}$.

There exists $0 < \alpha, \gamma < 1/2n$, $K > 0$ satisfying the $\mathcal{B}_{n,k}$ conditions for B_0. Clearly $\alpha, \gamma < 1/2$.

Let $B \in \mathcal{B}$ be such that $h(B, B_0) < \gamma/4K$. We will show that $B \in \mathcal{B}_{n,k}$. Let $K_1 = K + 1$ and $\gamma_1 = \frac{\gamma}{(4(1+\gamma/4K))}$.

Let $f \in S(X^*, \|\cdot\|_B^*)$, A be closed convex set and $Z \in \mathcal{Z}$ containing f such that $(A, f/\|f\|, Z) \in \mathcal{A}_{\alpha,n,k}$. Now, $B_0 \in \mathcal{B}_{n,k}$. Thus, there exists a closed ball $z_1 + r_1B_0$ with $r_1 \leq K$, $\sup_{a\in A}\|a - z_1\|_{Z,B_0} \leq r_1$ and $\inf(f/\|f\|_{B_0})(z_1 + r_1B_0) \geq \gamma$.

So, $\inf f(z_1) \geq \frac{r_1+\gamma}{1+\gamma/4K}$.

Now, $z_1 + r_1B_0 \subseteq z_1 + r_1(1 + \gamma/4K)B$. Also, $\sup_{a\in A}\|a - z_1\|_{Z,B} \leq r_1(1 + \gamma/4K)$. We have, $r_1(1 + \gamma/4K) \leq K + 1 = K_1$.

And,

$$\begin{aligned}
\inf f(z_1 + r_1(1 + \gamma/4K)B) &= f(z_1) - r_1(1 + \gamma/4K) \\
&\geq \frac{r_1 + \gamma}{1 + \gamma/4K} - r_1(1 + \gamma/4K) \\
&\geq \frac{\gamma}{1 + \gamma/4K} - K\left(1 + \gamma/4K - \frac{1}{1 + \gamma/4K}\right) \\
&\geq \frac{\gamma}{(4(1 + \gamma/4K))} = \gamma_1.
\end{aligned}$$

So, $B \in \mathcal{B}_{n,k}$.

Claim 3: For each $n, k \in \mathbb{N}$, $B_{n,k}$ is dense in $\mathcal{B}$.

Let $B_0 \in \mathcal{B}$ and $\varepsilon > 0$ be given. So, we need to show that there exists $B \in \mathcal{B}_{n,k}$ such that $h(B, B_0) < \varepsilon$.

Let $\lambda > h(B_0, B(X))$ and let $\delta = \varepsilon/2(1+\lambda)$. Consider

$$B = \overline{B_0 + \delta B(X)}.$$

It follows that

$$B_0 \subseteq B = \overline{B_0 + \delta B(X)} \subseteq B_0 + \delta(1+\lambda)B_0 = (1+\varepsilon/2)B_0$$

and hence,

$$h(B_0, B) \leq \varepsilon/2 < \varepsilon.$$

Also, $h(B, B(X)) < \varepsilon + \lambda = \beta$ (say). So, $\frac{1}{1+\beta}\|\cdot\|_B \leq \|\cdot\| \leq (1+\beta)\|\cdot\|_B$.

We claim that $B \in \mathcal{B}_{n,k}$.

Let $\alpha = 1/2n$, $\gamma = 1/4n(1+\beta)$, $K = k\left(1 + \beta + \frac{1}{\delta}\right)$.

Let $f \in S(X^*, \|\cdot\|_B^*)$, $A \subseteq X$ be closed convex and $Z \in \mathcal{Z}$ such that $(A, f/\|f\|, Z) \in \mathcal{A}_{\alpha,n,k}$. So, there exists a closed ball $z_0 + r_0 B(X)$ such that $r_0 \leq k$, $\sup_{a \in A} \|a - z_0\|_Z \leq r_0$ and $\inf(f/\|f\|)(z_0 + r_0 B(X)) \geq 1/2n$.

So, $\inf f(z_0 + r_0 B(X)) \geq 1/2n(1+\beta)$.

Let $x \in B_0$ be such that

$$\frac{K}{\delta}(f(x) + \|f\|_{B_0}) < 1/4n(1+\beta).$$

Then,

$$z_0 + r_0 B(X) = z_0 - \frac{r_0}{\delta}x + \frac{r_0}{\delta}(x + \delta B(X)) \subseteq z_0 - \frac{r_0}{\delta}x + \frac{r_0}{\delta}B$$

Now, $\|a - z_0 + (r_0/\delta)x\|_{Z,B} \leq r_0(1+\beta) + r_0/\delta \leq K$. Also,

$$\begin{aligned} \inf f(z_0 - \frac{r_0}{\delta}x + \frac{r_0}{\delta}B) &= \inf f(z_0 + r_0 B(X)) + \frac{r_0}{\delta}\inf f(B_0) - \frac{r_0}{\delta}f(x) \\ &\geq \frac{1}{2n(1+\beta)} - \frac{r_0}{\delta}\|f\|_{B_0} - \frac{r_0}{\delta}f(x) \\ &\geq \frac{1}{2n(1+\beta)} - \frac{K}{\delta}(\|f\|_{B_0} + f(x)) \\ &> \frac{1}{2n(1+\beta)} - \frac{1}{4n(1+\beta)} = \frac{1}{4n(1+\beta)} \end{aligned}$$

So, $B \in \mathcal{B}_{n,k}$. □

Corollary 5.4.3. *If $(X, \|\cdot\|)$ is AUS, then the set of equivalent norms which are AUS is residual in the set of all equivalent norms.*

Now, by the arguments in the proof of [16, Remark 4.5], we have residuality of AUC norms. Hence, we obtain the following result as a corollary which extends the result in [16] about the existence of an equivalent norm with both AUC and AUS to all Banach spaces and not merely reflexive spaces.

Corollary 5.4.4. *If X has an equivalent AUC norm and an equivalent AUS norm, then there is an equivalent norm which is both AUC and AUS.*

Chapter 6

Miscellaneous Results

6.1 Stability of the UMIP under ℓ_p sum

In this section, we discuss stability conditions for UMIP under ℓ_p-sum for $1 < p < \infty$. The conditions are quite similar to the stability of uniform convexity in the similar context. This gives another application of the moduli of UMIP discussed earlier. This material appeares in [24].

Let $1 < p, q < \infty$ be such that $1/p + 1/q = 1$. Let $\{X_i\}$ be a sequence of Banach spaces, and

$$X_p = \left(\bigoplus_{i=1}^{\infty} X_i \right)_p := \left\{ (x_i) : x_i \in X_i \text{ such that } \sum_{i=1}^{\infty} \|x_i\|^p < \infty \right\}$$

with $\|x\|_p = \left(\sum_{i=1}^{\infty} \|x_i\|^p \right)^{1/p}$ as the norm on X_p.

If $X_i = X$ for all i, we write $\ell_p(X)$ instead of $(\bigoplus_{i=1}^{\infty} X_i)_p$.

It is well known that

$$(X_p)^* = X_q^* = \left(\bigoplus_{i=1}^{\infty} X_i^* \right)_q ,$$

with the identification via $f(x) = \sum_{i=1}^{\infty} f_i(x_i)$.

Let $\beta_i(t)$ be the modulus of UMIP for X_i for all $i \in \mathbb{N}$, as given in Section 3.3.

Theorem 6.1.1. *Let $1 < p < \infty$. X_p has the UMIP if and only if for all $i \in \mathbb{N}$, X_i has the UMIP and $\inf_{i\in\mathbb{N}} \beta_i(t) > 0$ for every $0 < t < 1$.*

Proof. Let X_p has UMIP. Fix $k \in \mathbb{N}$. We show that X_k has the UMIP.

We use the characterisation of UMIP from Theorem 2.2.9(c).

Let $\varepsilon > 0$. Since X_p has the UMIP, there exists $\delta > 0$ such that for any $f \in S(X_q^*)$, there is $x \in S(X_p)$ such that if $y \in S(X_p)$ and $\|x - y\|_p < \delta$, then $\|f - f_y\|_q < \varepsilon/2$ for all $f_y \in D(y)$.

Let $f_k \in S(X_k^*)$. Define $f = (f_i) \in S(X_q^*)$ by putting

$$f_i = \begin{cases} f_k & \text{if} \quad i = k \\ 0 & \text{otherwise} \end{cases}$$

By the above, there is $x \in S(X_p)$ such that if $y \in S(X_p)$ and $\|x - y\|_p < \delta$, then $\|f - f_y\|_q < \varepsilon/2$ for all $f_y \in D(y)$. In particular, if $g \in D(x)$ then $\|f - g\|_q < \varepsilon/2$.

Now, if $g = (g_i) \in D(x)$, it follows from Hölder's inequality that

$$1 = \sum_i^{\infty} g_i(x_i) \le \sum_i^{\infty} \|g_i\|\|x_i\| \le \left(\sum_i^{\infty} \|g_i\|^q\right)^{1/q} \left(\sum_i^{\infty} \|x_i\|^p\right)^{1/p} = 1.$$

Therefore, for each $i \in \mathbb{N}$, $g_i(x_i) = \|g_i\|\|x_i\|$ and $\|x_i\|^p = \|g_i\|^q$, as equality holds in Hölder's inequality. Combining the two, we obtain that if $z_i = x_i/\|x_i\|$ for all i such that $x_i \neq 0$, then g has the form

$$g_i = \begin{cases} \|x_i\|^{p/q} h_i & \text{if} \quad x_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $h_i \in D(z_i)$.

Let $\gamma_k = \sum_{i \neq k} \|x_i\|^p$. It follows that $1 = \|x\|_p^p = \|x_k\|^p + \gamma_k$. And

$$(\varepsilon/2)^q > \sum_{i=1}^{\infty} \|f_i - g_i\|^q = \left\|f_k - \|x_k\|^{p/q} h_k\right\|^q + \gamma_k \ge \left(1 - \|x_k\|^{p/q}\right)^q + \gamma_k.$$

It follows that $\gamma_k < (\varepsilon/2)^q$. Therefore, $\|x_k\|^p > 1 - (\varepsilon/2)^q$.

Let $y_k \in S(X_k)$ be such that $\|y_k - z_k\| < \delta$ and $h_k \in D(y_k)$. Let $u \in S(X_p)$ be defined as

$$u_i = \begin{cases} \|x_k\| y_k & \text{if} \quad i = k \\ x_i & \text{otherwise} \end{cases}$$

We have $u \in S(X_p)$ and $\|x - u\|_p = \|x_k\| \|y_k - z_k\| < \delta$. Consider $g = (g_i)$ where

$$g_i = \begin{cases} \|x_k\|^{p/q} h_k & \text{if} \quad i = k \\ \|x_i\|^{p/q} f_{z_i} & \text{otherwise} \end{cases}$$

Here $f_{z_i} \in D(z_i)$. Then $g \in D(u)$. So, as before

$$(\varepsilon/2)^q > \|f - g\|_q^q = \left\| f_k - \|x_k\|^{p/q} h_k \right\|^q + \gamma_k.$$

Now,

$$\begin{aligned} \|f_k - h_k\| &\leq \|f_k - \|x_k\|^{p/q} h_k\| + \left\| \|x_k\|^{p/q} h_k - h_k \right\| \\ &\leq \varepsilon/2 + (1 - \|x_k\|^{p/q}) \\ &< \varepsilon/2 + (1 - (1 - (\varepsilon/2)^q)^{1/q}) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence, X_k has the UMIP and the δ that works for $\varepsilon/2$ for X_p works for ε for X_k. By Theorem 2.2.9, this implies $\inf_{i \in \mathbb{N}} \beta_i(t) > 0$.

Conversely, for any $t > 0$, let $\beta_0(t) = \inf_i \beta_i(t)$. Clearly, $\beta_0(t) > 0$ for any $0 < t < 1$ and for $0 < t_1 < t_2$, $\beta_0(t_1) \leq \beta_0(t_2)$ (using Remark 3.3.5).

Let $f = (f_i) \in S(X_q^*)$. If $f_i \neq 0$ then $f_i / \|f_i\| \in S(X_i^*)$. Then there exists $x_i \in S(X_i)$ such that

$$h_i \in \{g_i \in B(X_i^*) : g_i(x_i) > 1 - \beta_0(t)\} \implies \left\| h_i - \frac{f_i}{\|f_i\|} \right\| < t.$$

Define $z = (z_i) \in S(X_p)$ by

$$z_i = \begin{cases} \|f_i\|^{q/p} x_i & \text{if} \quad f_i \neq 0 \\ 0 & \text{if} \quad f_i = 0 \end{cases}$$

Let $g = (g_i) \in B(X_q^*)$ be such that $\|g - f\|_q \geq t$.

First, we deal with the case $\|g_i\| = \|f_i\|$ for all $i \in \mathbb{N}$. Let $a_i = \|f_i - g_i\|$ and $b_i = \|f_i\| = \|g_i\|$. We follow the proof in [14, Theorem 3]. Clearly, $\|f_i - g_i\| \leq \|f_i\| + \|g_i\|$, i.e., $a_i \leq 2b_i$. Now, Let $A := \{i \in \mathbb{N} : a_i/b_i > t/4\}$ and $B := \mathbb{N} \setminus A$. Then,

$$1 = \sum_{i=1}^{\infty} b_i^q \geq \sum_{i \in B} b_i^q \geq (4/t)^q \sum_{i \in B} a_i^q.$$

That is,

$$\sum_{i \in B} a_i^q \leq \left(t/4\right)^q.$$

Thus,

$$\sum_{i \in A} a_i^q = \sum_{i \in \mathbb{N}} a_i^q - \sum_{i \in B} a_i^q \geq t^q - (t/4)^q \geq \left(3t/4\right)^q.$$

Hence,

$$\alpha_0 = \sum_{i \in A} b_i^q \geq (3t/8)^q.$$

Since $\|f_i - g_i\| = a_i$ and $\|f_i\| = b_i$, we have $g_i(x_i) \leq (1 - \beta_0(a_i/b_i))b_i$. Therefore,

$$\begin{aligned}
g(z) &= \sum_{i=1}^{\infty} \|f_i\|^{q/p} g_i(x_i) \leq \sum_{i=1}^{\infty} b_i^q (1 - \beta_0(a_i/b_i)) \\
&= \sum_{i \in A} b_i^q (1 - \beta_0(a_i/b_i)) + \sum_{i \in B} b_i^q (1 - \beta_0(a_i/b_i)) \\
&\leq (1 - \beta_0(t/4)) \sum_{i \in A} b_i^q + \sum_{i \in B} b_i^q \text{ (follows from Remark 3.3.5)} \\
&= (1 - \beta_0(t/4))\alpha_0 + (1 - \alpha_0) = 1 - \alpha_0 \beta_0(t/4) \leq 1 - (3t/8)^q \beta_0(t/4).
\end{aligned}$$

It follows that for any $f \in S(X_q^*)$, there is $z \in S(X_p)$ with the property that, if $g \in B(X_q^*)$ is such that $\|g_i\| = \|f_i\|$ for all $i \in \mathbb{N}$ and $g(z) > 1 - (3t/8)\beta_0(t/4)$, then $\|f - g\|_q < t$.

Now, we move to the general case. We know that ℓ_q is uniformly convex. Let $\delta_q(t)$ for $0 < t \leq 2$ be the modulus of convexity for ℓ_q.

Choose α such that $0 < \alpha < t/2$ and $\delta_q(\alpha/2) + \alpha < (3t/16)\beta_0(t/8)$.

Let $g \in B(X_q^*)$ be such that $g(z) > 1 - \delta_q(\alpha/2)$. Then,

$$\sum_{i=1}^{\infty} \|f_i\|^{q/p}\|g_i\| \geq \sum_{i=1}^{\infty} \|f_i\|^{q/p}|g_i(x_i)| > 1 - \delta_q(\alpha/2).$$

Let $F := (\|f_i\|) \in S(\ell_q)$, $G := (\|g_i\|) \in B(\ell_q)$ and $u := (\|f_i\|^{q/p}) \in S(\ell_p)$. Then,

$$F(u) = \sum_{i=1}^{\infty} \|f_i\|\|f_i\|^{q/p} = 1 \quad \text{and} \quad G(u) > 1 - \delta_q(\alpha/2).$$

Now, by Remark 3.3.7,

$$\|F - G\|_q = \left(\sum_{i=1}^{\infty} \Big| \|g_i\| - \|f_i\| \Big|^q \right)^{1/q} < \alpha.$$

Let $h = (h_i)$, where $h_i = \frac{g_i\|f_i\|}{\|g_i\|}$. Clearly, $\|h_i\| = \|f_i\|$ for each i. And

$$\|g - h\|_q = \left(\sum_{i=1}^{\infty} \Big| \|g_i\| - \|f_i\| \Big|^q \right)^{1/q} < \alpha.$$

Since $g(z) > 1 - \delta_q(\alpha/2)$, we have $h(z) \geq g(z) - \|g - h\|_q > 1 - \delta_q(\alpha/2) - \alpha > 1 - (3t/16)\beta_0(t/8)$, by the choice of α. By the first case,

$$\|f - h\|_q < t/2.$$

Hence,

$$\|f - g\|_q \leq \|f - h\|_q + \|h - g\|_q < t.$$

□

Corollary 6.1.2. *Let $1 < p < \infty$. Then $\ell_p(X)$ has the UMIP if and only if X has the UMIP.*

6.2 Hereditary MIP does not imply Fréchet smoothness

As discussed in the introduction of this thesis, hereditary MIP implies smoothness. In this section, we show that this doesn't hold for Fréchet smoothness. That is, we show that hereditary MIP does not imply Fréchet smoothness.

To understand the involved aspects, we need the following definition.

Definition 6.2.1. A Banach space X is called Weakly Compactly Generated (WCG) if there is weakly compact subset K of X such that $\overline{span}(K) = X$.

For example, a reflexive Banach space or a separable Banach space is WCG.

Let X be an infinite dimensional WCG Asplund space. Borwein and Fabian in [8, Theorem 4] proved that X admits an equivalent Gâteaux smooth norm which is not Fréchet differentiable exactly at each $x \in \mathbb{R}x_0$, for some $x_0 \in S(X)$.

Suppose X is endowed with such a norm. Then,

Theorem 6.2.2. *X has hereditary MIP, i.e., every closed subspace of X has the MIP.*

Proof. Let Y be a closed subspace of X.

If $x_0 \notin Y$, then $Y \cap \mathbb{R}x_0 = \{0\}$. So, Y is Fréchet differentiable. Hence, Y has the MIP.

So, let $x_0 \in S(Y)$.

Let $e > 0$ and $f \in S(Y^*)$. Since Y is smooth (since, smoothness is inherited by closed subspaces), $D(x_0)$ is a singleton. So, by Bishop-Phelps theorem, there exists $f_{y_1} \in D(y_1)$ where $y_1 \neq x_0, -x_0$ such that $\|f_{y_1} - f\| < \varepsilon/2$. Since $y_1 \neq x_0, -x_0$, y_1 is a point of Fréchet differentiability, and hence, there exists $\delta > 0$ such that

$$S(B(Y^*), y_1, \delta) < \varepsilon/2.$$

Also, $f_{y_1} \in S(B(X^*), y_1, \delta)$. Hence,

$$S(B(Y^*), y_1, \delta) \subseteq B(f, \varepsilon).$$

So, Y has the MIP. $\square$

So, we conclude that X has hereditary MIP but it is not Fréchet differentiable.

The analogous question for UMIP, whether *hereditary UMIP implies uniform smoothness* seems open.

Another important connection between MIP and uniform smoothness is the following largely forgotten result by Sundaresan.

Theorem 6.2.3. *[44, Theorem 1] X is uniformly smooth if and only if X is super-MIP.*

In the above theorem, super-MIP refers to the property that every Banach space which is finitely representable in X (See [15, Definition 1.8]) has MIP.

Note that super-MIP implies hereditary MIP as well as reflexive. And since super properties are transitive, it also implies super-reflexive.

Chapter 7

Conclusion and Problems

7.1 Summary

In this thesis, we have been able to get a better understanding of the structure of a Banach space with UMIP. For instance, we proved a missing characterisation of UMIP in terms of uniformly w*-semidenting points. The proofs for UMIP characterisations in Chapter 2 simplifies the arguments in [47]. We also have a brief look at Fréchet smoothness and norm attaining points being w*-denting. Using this new characterisation and the idea of modulus of denting points, we defined a modulus of w*-semidenting points and thus, obtained the modulus of UMIP to understand it in quantitative terms. We then looked at the "Hyperplane" version of UMIP and characterised it in terms of uniform smoothness. Taking this study forward, we have been able to characterise AUS in terms of a ball separation property. Using these ball separation properties, we have obtained interesting residuality results for AUS, uniform smoothness and UMIP norms, thus adding to our understanding of these properties. We also improved upon the generalised MIP characterisations and thus been able to portray the missing picture connecting the generalised MIP and the generalised Asplund property. The discussion on the stability of UMIP under ℓ_p-sum helped us to realise that UMIP and uniform convexity have a strong quantitative analogy. Further, the counterexample to hereditary MIP implying Fréchet smoothness adds to our insight of the MIP structure. Hopefully, the contribution of this thesis will be considered significant in understanding UMIP structure of a Banach space and help in solving UMIP and super-reflexivity (uniform convex renorming) connection and the problems mentioned below.

7.2 Further Research

Of course, the most important open problem in this area remains:

Problem 1. *Does UMIP imply uniformly smooth renorming?*

Although, we haven't been able to solve it, this is the problem around which the work done in this thesis revolved. We hope that the work done in the thesis will prove to be helpful in analysing the issue at hand. For instance, the equivalence of uniform smoothness with H-UMIP brings us tantalisingly close to a solution. Similarly, the characterisation of uniform smoothness in terms of uniformly w*-denting points hints that the UR renorming technique suggested by Lancien in [33] may prove to be helpful in tacking the above problem. In [33, Theorem 7.7], Lancien proves that X has a uniformly convex renorming if and only if, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$D_\varepsilon^{n_0} = \phi,$$

where, $D_\varepsilon^0 = B(X)$ and given D_ε^n for $n \in N \cup \{0\}$, $D_\varepsilon^{n+1} = F_{D_\varepsilon^n,\varepsilon}$, where F is the derivation defined in Section 3.6. Similar techniques worked well in connection between denting points and LUR renorming (see [40]). So, we believe Lancien's techniques may prove to be fruitful in this direction (at least in separable Banach space as they are well behaved under renorming given that the space has the MIP). The difficulty that we encounter while applying Lancien's technique to UMIP case is that we are not able to say anything about D_ε^{*2}.

In fact, the following weaker version of the above problem also remains open.

Problem 2. *Does UMIP imply reflexivity?*

In separable spaces, MIP and Asplund property behave well with each other. But, in the non-separable case, this is no longer true. One of the major results concerning MIP is that every Banach space can be isometrically embedded in a Banach space with MIP. This result proves to be quite useful in disconnecting MIP from properties which are hereditary. For instance, in general, MIP does not imply reflexivity or Asplund or RNP. Problem 2 can be answered in negative if the following problem has an affirmative answer.

Problem 3. *Can any Banach space be isometrically embedded in a Banach space with the UMIP?*

In fact, if one wants to try a counterexample for Problem 1, then the following can be attempted.

Problem 4. *Does c_0 have a UMIP renorming?*

Returning to the search for a positive answer to Problem 1, consider the following :

Problem 5. *If X has both H-MIP and UMIP, does X have H-UMIP?*

We note that this is an isometric question and if the answer to Problem 5 is affirmative, Problem 1 also has a positive answer in separable Banach spaces.

We have seen that separable MIP spaces have separable dual, and hence, is Asplund. It follows that X^* has a dual LUR renorming. This, in turn, implies that X has H-MIP. It follows that a separable MIP space also has an equivalent H-MIP renorming.

Thus, by the residuality results discussed in this thesis, it follows that in a separable Banach space with UMIP, there is an equivalent norm on X that has both UMIP and H-MIP. Therefore, if the answer to Problem 5 is affirmative, there is an equivalent norm on X that has H-UMIP, and hence, is uniformly smooth. That is, Problem 1 also has a positive answer in separable case.

Regarding the hereditary MIP, we can ask the following question.

Problem 6. *Does hereditary UMIP imply uniform smoothness?*

Phelps proved that if w*-strongly exposed points of $B(X^*)$ are dense in $S(X^*)$, then X has the MIP.

Problem 7. *Does the converse hold true?*

For every $\varepsilon > 0$ and $f \in S(X^*)$, there exists $x \in M_\varepsilon$ such that

$$\|f - f_x\| < \varepsilon$$

for all $f_x \in D(x)$, where M_ε is the set of points of ε-Fréchet differentiability.

By Theorem 1.0.1(e), if X has MIP, then $S(X^*) = \bigcap_{\varepsilon>0} \overline{D(M_\varepsilon)}$. On the other hand, the set of all Fréchet smooth points $= \bigcap_{\varepsilon>0} M_\varepsilon$. This suggests the following problem:

Problem 8. *Does MIP imply existence of a point of Fréchet smoothness?*

In Chapter 5, we obtained residuality of AUS norms, uniformly smooth norms and norms with UMIP. An important tool in proving the residuality of these properties is the ball characterisation of the above mentioned properties. Also, we have similar ball separation characterisation of Fréchet smoothness discussed in Chapter 2. Considering the connection between residuality and ball separation properties, we can ask the following:

Problem 9. *If X has a Fréchet smooth norm, then is the set of equivalent norms which are Fréchet smooth residual in $\mathcal{B}$?*

If the answer to the above question is affirmative, then using the residuality of LUR norms, it's easy to answer the following open problem:

Problem 10. *If X has an equivalent Fréchet smooth norm and an equivalent LUR norm, then does there exist an equivalent norm which is both Fréchet smooth and LUR?*

The above ball separation characterisation of Fréchet smoothness in connection with residuality also give rise to the following conjecture.

Problem 11. *The set of equivalent norms preserving the set of norm attaining functionals is G_δ dense in the set of all equivalent norms.*

Let λ be the Lebesgue measure on $[0, 1]$ and

$$L^p(\lambda, X) := \{f : [0, 1] \to X \ : \int_0^1 \|f\|^p d\lambda < \infty\}$$

with $\|f\|_p := \left(\int_0^1 \|f\|^p d\lambda\right)^{1/p}$. The authors in [6, Corollary 12] show that $L^p(\lambda, X)$ has MIP if and only if X has MIP and X is Asplund.

This combined with the ℓ_p stability result in Corollary 6.1.2 brings us to the problem :

Problem 12. *Does $L^p(\lambda, X)$ have UMIP if and only if X has UMIP and X is Asplund?*

Following question raised in Chapter 4 about characterising $\mathcal{C}$-MIP in terms of strong $\mathcal{C}$-semidenting points seems crucial as strong $\mathcal{C}$-semidenting is a much natural and convenient generalisation of w*-semidenting notion.

Problem 13. *Can we characterise $\mathcal{C}$-MIP in terms of strong $\mathcal{C}$-semidenting points?*

We noted that separable spaces with $\mathcal{C}$-MIP are $\mathcal{C}$-Asplund in 4.5.4. Also, a separable Asplund space has a renorming with MIP. This suggests to look at the following problem.

Problem 14. *Does a separable $\mathcal{C}$-Asplund space have a renorming with (strong) $\mathcal{C}$-MIP?*

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