Indian Statistical Institute, Kolkata

M. Math I year

End-Semester Examination-Autumn 2025

Algebra I

Full Marks: 50 *Date: 26.11.25* *Time Allotted: 3 h*

Instructions: Answer all questions. Your final marks = $\min\{\text{Total marks obtained}, 50\}$.

(1) Let $n = d_1 d_2$ and $N = \mathbb{Z}/n\mathbb{Z}$. Show that the sequence

$$0 \longrightarrow d_2 N \xrightarrow{\iota} N \xrightarrow{x \mapsto d_1 x} d_1 N \longrightarrow 0$$

is short exact, and that this sequence splits if and only if $\gcd(d_1, d_2) = 1$. (4 + 4)

(2) Let I and J be two ideals of a commutative ring R with unity.

(a) Show that

$$R/I \otimes_R R/J \cong R/(I + J).$$

(b) Compute the dimension of the $\mathbb{Q}$-vector space

$$\mathbb{Q}[x]/(x^2 - 2) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x^4 - 4).$$

(4 + 4)

(3) Let $M = \mathbb{Z}/6\mathbb{Z}$ and let $\varphi : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3$ be the homomorphism defined by

$$\varphi(x, y) = (\,2x,\ 3y,\ 2x + 3y\,).$$

Compute $M \otimes_{\mathbb{Z}} \mathrm{Tor}_{\mathbb{Z}}(\mathrm{Coker}(\varphi))$. (8)

(4) Let $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$ denote the multiplicative group of nonzero complex numbers, regarded as a $\mathbb{Z}$-module under via $n \cdot z = z^n$. Compute $\mathrm{Tor}_{\mathbb{Z}}(\mathbb{C}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q})$. (4)

(5) State **True** or **False** with justification.

(a) A Noetherian algebra over a Noetherian ring R is a finitely generated R-algebra.

(b) $\dim_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = 2$ (3 + 3)

(6) Let R be a commutative ring with unity such that

(i) for each maximal ideal $\mathfrak{m}$ of R, the local ring $R_{\mathfrak{m}}$ is Noetherian;

(ii) for each $x \neq 0$ in R, the set of maximal ideals of R containing x is finite.

Show that R is Noetherian. (6)

(7) Let R be a PID and M a finitely generated R-module.

(a) Show that M is free R-module if and only if $M \otimes_R R_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m} \subset R$. (3 + 5)

(b) Show that M is a torsion module if and only if $\mathrm{Hom}_R(M, R) = 0$. (2 + 2)

(8) Let $T \in \mathrm{SL}_2(\mathbb{R})$ be a linear map on $\mathbb{R}^2$ with $\det(T) = 1$. Using the trace(T), list all possible Jordan canonical forms of T up to conjugacy (*Hint: Characteristic polynomial*). (8)