Restricted Mean Value Property on Riemannian manifolds and Carleson's problem on Damek-Ricci spaces

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Dedicated to my Mother

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Contents

Notations & Abbreviations

$\mathbb{R}$	the set of real numbers
$\mathbb{N}$	the set of positive integers
C, c or c'	positive constants whose values may change on each occurrence
$C_1, C_2, \ldots$ or $c_1, c_2, \ldots$	positive constants whose values are fixed throughout
$f_1 \lesssim f_2$	$f_1, f_2 \geq 0$ with $f_1 \leq C f_2$ for some $C > 0$.
$f_1 \lesssim_R f_2$	$f_1, f_2 \geq 0$ with $f_1 \leq C f_2$ for some $C > 0$, which depends on the choice of R.
$f_1 \gtrsim f_2$	$f_1, f_2 \geq 0$ with $f_1 \geq C f_2$ for some $C > 0$.
$f_1 \asymp f_2$	$f_1, f_2 \geq 0$ with $C f_1 \leq f_2 \leq C' f_1$ for some $C, C' > 0$.
χ_E	the indicator function on E.

Chapter 1

Introduction

One of the most key aspects of Harmonic Analysis is the study of the boundary behaviour of solutions of Partial Differential Equations. The central theme of this thesis is to study such problems for some suitable classes of Riemannian manifolds.

1.1 Restricted Mean Value Property

We recall that a continuous function in a domain D in $\mathbb{R}^n$ is harmonic if and only if it satisfies the spherical mean value property (with respect to the surface volume measure) for all spheres whose corresponding closed balls are contained in D. A classical problem which has been considered by various authors is whether harmonicity still holds if one assumes a much weaker version of the mean-value property. Namely, a continuous function on a domain D in $\mathbb{R}^n$ is said to satisfy the *restricted mean-value property* (RMVP for short) if for each point $x \in D$, there exists a sphere S of center x and some radius $\rho(x)$, such that the corresponding closed ball is contained in the domain D and $u(x)$ equals the mean value of u on the sphere S. One can then ask whether a function satisfying the restricted mean value property in D is harmonic in D.

For a bounded domain D, it turns out that the key factor in an answer to this question is the boundary behaviour of the function. Indeed, if one assumes that the function u extends continuously to the closure of the domain D, then Kellogg gave a simple argument to show that the function must be harmonic ([Ke34]). Without any assumptions on the boundary behaviour, there are counter-examples, which are unbounded (see page 22, [Li68]). In this context, Littlewood asked (in [Li68]) for $n = 2$, whether the unboundedness is the only obstruction. In other words, if the function satisfying the RMVP is also assumed to be bounded, then is it harmonic?

This is the classical 'one-circle problem' and in [HN94], Hansen and Nadirashvili showed that the above problem has a negative answer, that is, there exists a continuous, bounded function on the unit disk in $\mathbb{R}^2$ which satisfies RMVP but is not harmonic. The problem is still open for $n \geq 3$ however. Nevertheless, obtaining sufficient conditions for a function satisfying RMVP to be harmonic has received considerable attention over the years, in particular from Fenton and Hansen-Nadirashvili ([Fe76, Fe79, Fe87, HN93]).

While these authors considered this problem for domains in $\mathbb{R}^n$, in chapter 2, we consider this problem in the very general setting of a domain in a Riemannian manifold. The definition of the RMVP in this setting needs some explanation. In a Riemannian manifold, it is not true in general that harmonic functions satisfy the mean-value property with respect to mean-values over geodesic spheres when the mean-value is taken with respect to the surface volume measure induced by the Riemannian metric (although there is a special class of Riemannian manifolds for which this holds, namely the *harmonic manifolds*, see [Wi50]). The mean-value property does hold however, if one replaces the Riemannian measures on the geodesic spheres by the *harmonic measures* on the spheres. These are certain measures on the spheres (or more generally on the boundaries of precompact domains with smooth boundary in the manifold) which are mutually absolutely continuous with respect the Riemannian surface volume measures. For the definition of the harmonic measures we refer to section 2.2 of chapter 2.

The harmonic measures have the following property. For any harmonic function u on a domain D in a Riemannian manifold M and any geodesic ball B such that $\overline{B} \subset D$, one has the following mean value property

$$u(z) = \int_{\partial B} u(\xi) d\mu_{z,B}(\xi) \,, \tag{1.1.1}$$

where z is the center of B, and $\mu_{z,B}$ is the harmonic measure on ∂B with respect to z.

Thus harmonic functions on a Riemannian manifold do satisfy the mean-value property, if one takes mean-values with respect to harmonic measures. This leads us to the following definition:

Definition 1.1.1 (Restricted Mean-Value Property) *Let D be a domain in a Riemannian manifold M. A continuous function u on D is said to satisfy the Restricted Mean Value Property in D if for all $z \in D$, there exists $0 < \rho(z) < inj(z)$ (where $inj(z)$ is the injectivity radius of z) such that the closed ball $B = \overline{B(z, \rho(z))}$ is contained in D, and the equality (1.1.1) holds.*

We now let M be a Riemannian manifold, and let $D \subset M$ be any precompact domain in M with smooth boundary such that the distance to the boundary $d(z, \partial D)$ is smaller than the injectivity radius $inj(z)$, for all points z in D. In this setting we have Theorem 2.1.1 which states that if a bounded, real-valued, continuous function satisfying the RMVP in D also admits boundary limits on a full measure (with respect to the Riemannian measure) subset of the boundary ∂D, then the function is actually harmonic in D.

Our next result Theorem 2.1.2 does not assume boundedness nor any explicit boundary behaviour of u, but rather assumes a suitable rate of decay of the radius function $\rho(z)$ as z tends to the boundary of D in terms of the modulus of continuity of u on compacts in D.

Theorems 2.1.1 and 2.1.2 generalize [Fe76, Theorems 2 and 1] for the unit disk in $\mathbb{R}^2$, to the setting of smooth pre-compact domains in Riemannian manifolds.

We also have a result for when the boundary is "at infinity". Namely, if we take M to be a complete, simply connected Riemannian manifold of pinched negative curvature, then it has infinite injectivity radius at each point, and we can consider functions on the whole manifold M satisfying definition 1.1.1, where the domain $D = M$. In the previous two results where only bounded domains were considered, the radius function $z \mapsto \rho(z)$ was bounded. Instead if we take the domain D to be the whole manifold M, then it is natural to ask for an analogue of Theorem 2.1.1 for the whole manifold M, where the radius function is allowed to be unbounded.

In fact, we do have a result in this case too, with no restriction at all on the radius function (see Theorem 2.1.3). In this case however, in place of the boundary ∂D, one needs to consider the *Gromov boundary* ∂M of the Gromov hyperbolic space M, and the appropriate measure class on the Gromov boundary in this context is the harmonic measure class (see section 2.2 of chapter 2 for the definitions of Gromov boundary and harmonic measures on the Gromov boundary).

1.2 Carleson's problem

In the previous section, we noted that the boundary behaviour of the function under consideration was crucial in obtaining harmonicity. Following this general theme, we now look at the celebrated Carleson's problem regarding the regularity of the initial data and the pointwise convergence of its Schrödinger propagation.

In 1980, Carleson [Ca80] posed and solved the problem of determining the amount of regularity required on the initial data f, so that the solution of the one dimensional Schrödinger equation

$$\begin{cases} i\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,,\ (x,t) \in \mathbb{R} \times \mathbb{R} \\ u(0,\cdot) = f\,,\ \text{ on } \mathbb{R}\,, \end{cases}$$

converges pointwise to f, that is,

$$\lim_{t \to 0+} u(t,x) = f(x)\,,\ \text{for almost every } x \in \mathbb{R}\,.$$

More precisely, Carleson proved that the pointwise convergence does hold true if f is compactly supported and Hölder continuous of order $> 1/4$. On the other hand, Carleson also showed that there exists a compactly supported function f on $\mathbb{R}$ which is Hölder continuous of order $< 1/8$, for which the pointwise limit of the solution diverges almost everywhere. In 1982, Dahlberg and Kenig [DK82] noted that the only property of compactly supported Hölder continuous functions of order $> 1/4$, used by Carleson is that these functions belong to the inhomogeneous Sobolev space $H^{1/4}(\mathbb{R})$, where

$$H^\beta(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \left(1 + |\xi|^2\right)^\beta \left|\tilde{f}(\xi)\right|^2 d\xi < \infty \right\}\,.$$

Then they showed that the result of Carleson is actually sharp by constructing a counter-example of a function belonging to $H^\beta(\mathbb{R})$ for $\beta < 1/4$, for which the pointwise limit of the solution of the Schrödinger equation diverges almost everywhere. Subsequently they posed the question in higher dimensions $\mathbb{R}^n$, $n \geq 2$, to determine the optimal regularity of the initial condition in terms of its Sobolev index, so that the solution of the Schrödinger operator converges pointwise to its initial data almost everywhere. This is the celebrated Carleson's problem and for the last five decades, it has stimulated extensive research.

In 1983, Cowling [Co83] studied the problem for a general class of self-adjoint operators on $L^2(X)$ for a measure space X and obtained $\beta > 1$, to be a sufficient condition for the associated Schrödinger operator. For $\mathbb{R}^n$, in 1987, Sjölin [Sj87] improved the bound to $\beta > 1/2$, by means of a local smoothing effect. This improvement was also independently obtained by Vega [Ve88] in 1988. In 1990, Prestini [Pr90] solved the Carleson's problem while specializing to radial functions and obtained the dimension independent sharp bound $\beta \geq 1/4$. After continuous improvements made by several experts in the field, recently in 2017, the bound $\beta > 1/3$, was obtained for $\mathbb{R}^2$ by Du-Guth-Li [DGL17] using polynomial partitioning and decoupling. Then in 2019, Du-Zhang [DZ19] obtained the bound $\beta > n/2(n+1)$, for dimensions $n \geq 3$, by using decoupling and induction on scales. This bound is sharp except at the endpoint $\beta = n/2(n+1)$, due to a counterexample by Bourgain [Bo16]. Thus in the Euclidean setting, the Carleson's problem has been almost fully resolved.

In the case of a Riemannian manifold, where neither the doubling property holds nor a dilation structure is available, the above mentioned Euclidean machineries work no longer and thus it offers a fresh challenge. Recently in this regard, Wang and Zhang considered the problem in the non-Euclidean setting of the Real Hyperbolic spaces and proved that $\beta > 1/2$, is a sufficient condition for the pointwise convergence to hold true [WZ19, Theorem 1.1]. In [De25], we improved the regularity threshold from $\beta > 1/2$ to $\beta \geq 1/4$, while specializing to radial initial data, on the vastly general setting of Damek-Ricci spaces, generalizing Prestini's result in $\mathbb{R}^n$ [Pr90].

Real Hyperbolic spaces are the simplest examples of rank one Riemannian Symmetric spaces of noncompact type. On the other hand, the class of rank one Riemannian Symmetric spaces of noncompact type (barring the degenerate case of the Real Hyperbolic spaces) is contained in (and in fact accounts for a very small subclass of) the more general class of Damek-Ricci spaces [ADY96, p. 643]. They are also known as Harmonic NA groups. These spaces S are non-unimodular, solvable extensions of Heisenberg type groups N, obtained by letting $A = \mathbb{R}^+$ act on N by homogeneous dilations (we refer to section 3.2 in chapter 3 for details). On a Damek-Ricci space a function will be called *radial* if $f(x) = f(y)$ whenever the geodesic distances of x and y from the identity e are equal.

The above improvement in the regularity threshold for radial data in [De25] was obtained by proving a boundedness result for the local Maximal function associated to the Schrödinger equation. To explain this, we need to introduce some notations.

Let Δ be the Laplace-Beltrami operator on S corresponding to the left-invariant Riemannian metric. Its L^2-spectrum is the half line $(-\infty, -Q^2/4]$, where Q is the homogeneous dimension of N. The Schrödinger equation on S is given by

$$\begin{cases} i\frac{\partial u}{\partial t} = \Delta u \,, \quad (x,t) \in S \times \mathbb{R} \\ u(0,\cdot) = f \,, \quad \text{on } S \,. \end{cases} \tag{1.2.1}$$

Then for a radial function f belonging to the L^2-Schwartz class (see (3.2.7) for the definition),

$$S_t f(x) := \int_0^\infty \varphi_\lambda(x)\, e^{it\left(\lambda^2+\frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\,, \tag{1.2.2}$$

is the solution to (1.2.1), where φ_λ are the spherical functions, $\widehat{f}$ is the Spherical Fourier transform of f and $\mathbf{c}(\cdot)$ denotes the Harish-Chandra's $\mathbf{c}$-function.

To quantify the Carleson's problem on S, for $\beta \geq 0$, we recall the definition of the fractional L^2-Sobolev spaces on S for the special case of radial functions [APV15]:

$$H^\beta(S) := \left\{ f \in L^2(S) : \|f\|_{H^\beta(S)} := \left(\int_0^\infty \left(\lambda^2 + \frac{Q^2}{4}\right)^\beta |\widehat{f}(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{1/2} < \infty \right\}. \tag{1.2.3}$$

We note that by the Plancherel theorem, $\|f\|_{H^0(S)} = \|f\|_{L^2(S)}$.

Finally, we define the local maximal function associated to the solution (1.2.2),

$$S^* f(x) := \sup_{0<t<4/Q^2} |S_t f(x)|\ , \quad x \in S\,. \tag{1.2.4}$$

The main result in [De25] is the following L^1_{loc} boundedness for the maximal function: for any $R > 0$, on geodesic balls B_R centered at the identity with radius R, there exists a constant $C > 0$ (depending only on the dimension of S and R) such that

$$\|S^* f\|_{L^1(B_R)} \leq C\, \|f\|_{H^{1/4}(S)}\,, \tag{1.2.5}$$

holds for every radial L^2-Schwartz class function f on S.

In [De25, Theorem 1.4], we also addressed the optimal regularity of the initial data in the (degenerate) case of $\mathbb{H}^3 \cong SL(2,\mathbb{C})/SU(2)$, by showing that the boundedness property (1.2.5) fails whenever $\beta < 1/4$. The methods employed in [De25, Theorem 1.4] however, do not seem to have a straightforward generalization to the general setting of Damek-Ricci spaces, and thus obtaining the optimal regularity becomes an interesting question on its own.

This leads us to the natural question of obtaining the complete description of the pairs $(q,\beta) \in [1,\infty] \times [0,\infty)$ for which one has the following boundedness estimate for any $R > 0$, and every radial L^2-Schwartz class function f on S:

$$\|S^* f\|_{L^q(B_R)} \lesssim_R \|f\|_{H^\beta(S)}\,.$$

In the classical Euclidean setting, this problem was studied by Sjölin and he obtained the following [Sj97, Theorem 3]:

$$\begin{cases} \text{If } \beta < 1/4, \text{ then (3.1.1) holds for no } q. \\ \text{If } 1/4 \leq \beta < n/2, \text{ then (3.1.1) holds if and only if } q \leq 2n/(n-2\beta). \\ \text{If } \beta = n/2, \text{ then (3.1.1) holds if and only if } q < \infty. \\ \text{If } \beta > n/2, \text{ then (3.1.1)holds for all } q. \end{cases} \tag{1.2.6}$$

In the case of $\mathbb{H}^3$, using the explicit expression of the spherical functions φ_λ, the Plancherel measure and their close similarity with their Euclidean counterparts, proceeding as in the proof of [De25, Theorem 1.4], one can reduce matters to $\mathbb{R}^3$ to get the exact result of Sjölin (1.2.6). But in the general case of Damek-Ricci spaces (unlike $\mathbb{H}^3$), no explicit expression of φ_λ is known. Moreover, the weight of the Plancherel measure exhibits dichotomy in asymptotics for small and large frequencies (see [RS09, Lemma 4.8]):

$$|\mathbf{c}(\lambda)|^{-2} \asymp |\lambda|^2 (1+|\lambda|)^{n-3}, \tag{1.2.7}$$

(whre n denotes the dimension of S). The same estimate is also valid for the Real hyperbolic spaces $\mathbb{H}^n$, for $n \geq 2$. Thus the simple-minded approach of a straightforward reduction to the Euclidean case (like $\mathbb{H}^3$) breaks down. Nevertheless, the investigation conducted on $\mathbb{H}^3$, serves well to provide us with an interesting conjecture to explore the validity of the local mapping properties of the Schrödinger maximal function [Sj97, Theorem 3], in the setting of Damek-Ricci spaces. This we answer in the affirmative in Theorem 3.1.1. Consequently, we establish the sharpness of $\beta \geq 1/4$ for all Damek-Ricci spaces. We also obtain certain global estimates in Theorems 3.1.3 and 3.1.4 .

In this thesis, we include the case when N is abelian and $\mathfrak{n}$ coincides with its center in the definition of H-type groups, as a degenerate case, so that the Real hyperbolic spaces are also included in our discussion (see [CDKR98, pp. 209-210]).

1.3 Generalized Carleson's problem

As discussed earlier, in [De25], we have proved that if f is radial and belongs to the Sobolev space $H^\beta(S)$ for $\beta \geq 1/4$, then the solution to the Schrödinger equation with initial data f

converges to f, that is,

$$\lim_{t \to 0+} S_t f(x) = f(x) \,, \tag{1.3.1}$$

for almost every $x \in S$, with respect to the left Haar measure. Moreover, Theorem 3.1.1 establishes the sharpness of the endpoint $\beta = 1/4$. Interpreting the convergence (1.3.1) as convergence along vertical lines (in the product space $S \times (0, \infty)$), it is natural to ask what happens when the problem of (almost everywhere) pointwise convergence is considered along a wider approach region, for instance, the non-tangential convergence.

For $\mathbb{R}^n$, this question regarding non-tangential convergence has already been addressed. On $\mathbb{R}^n$, for $\beta > n/2$, by Sobolev imbedding and standard arguments, the non-tangential convergence to the initial data holds. However, Sjögren and Sjölin showed in [SS89] that the non-tangential convergence fails for $\beta \leq n/2$. Now as their counter-example is a non-radial function, it is natural to ask when the initial data is more symmetric, say radial, then can we have pointwise convergence along wider approach regions. In Theorem 4.1.1, we answer this question in the negative by constructing a radial initial data on the 3-dimensional Real Hyperbolic space $\mathbb{H}^3 \cong SL(2, \mathbb{C})/SU(2)$, whose Schrödinger propagation blows up on geodesic annuli along wide approach regions.

Theorem 4.1.1 shows that the solutions of the Schrödinger equation, unlike Harmonic functions or solutions of the Heat equation, do not admit a natural wide approach region. Thus it becomes interesting to study the pointwise convergence of solutions along appropriate curves and its connection with the regularity of the initial data. Such problems have received considerable interest in recent years: in dimension one, by Cho-Lee-Vargas [CLV12], Ding-Niu [DN17] and in higher dimensions, by Cao-Miao [CM23], Minguillón [Mi24].

In [CM23] and [Mi24], the authors have shown that on $\mathbb{R}^n$, the condition $\beta > n/2(n+1)$ is sufficient to obtain almost everywhere pointwise convergence along certain families of curves. Firstly, their arguments crucially use dilation and the explicit knowledge of its interaction with the Fourier transform. In the case of a Riemannian manifold, where the volume measure neither satisfies any doubling property nor does it admit dilation (such as Damek-Ricci spaces), the above mentioned Euclidean machineries work no longer and thus it offers a fresh challenge. Secondly, it is interesting to ask when the initial condition is more symmetric, for instance, radial, whether one can lower the above mentioned regularity threshold and yet have pointwise convergence.

We then study the Generalized Carleson's problem on pointwise convergence of solutions of the Schrödinger equation on Damek-Ricci spaces along general curves that satisfy certain Hölder conditions and bilipschitz conditions in the distance from the identity in Theorem 4.1.2 and Corollary 4.1.3. In the above results, we again obtain the sharp bound up to the endpoint $\beta \geq 1/4$. In Theorems 4.1.5 and 4.1.7, we also obtain certain Euclidean analogues (of Theorem 4.1.2) which can be interpreted as variations of results in [CM23] and [Mi24] (see Remark 4.1.8).

1.4 Organization of the thesis:

In **chapter 2:** in section 2.1, we state Theorems 2.1.1, 2.1.2 and 2.1.3 and outline the key ideas. In section 2.2, we discuss the relevant preliminaries regarding Gromov hyperbolic spaces, their boundaries, and harmonic measures. In section 2.3, we prove some crucial results on concentration of harmonic measures and convergence of Poisson integrals. In section 2.4, we construct an auxiliary subharmonic function which will play a central role in the proofs of our main results. In section 2.5, we prove some useful L^∞ maximum principles. In section 2.6, we give proofs of Theorems 2.1.1, 2.1.2 and 2.1.3. Results of this chapter have appeared in [BiD24].

In **chapter 3:** in section 3.1, we state Theorem 3.1.1 and outline the key ideas. In section 3.2, we recall certain aspects of Euclidean Fourier Analysis and the essential preliminaries about Damek-Ricci spaces and Spherical Fourier Analysis thereon. In section 3.3, we prove two auxiliary lemmata. The sufficient and necessary conditions of Theorem 3.1.1 are proved in sections 3.4 and 3.5 respectively. In section 3.6, Theorems 3.1.3 and Theorem 3.1.4 are proved. Some parts of the results of this chapter have appeared in [De25].

In **chapter 4:** in section 4.1, we state Theorems 4.1.1, 4.1.2, 4.1.5 and 4.1.7 and outline the key ideas. In section 4.2, we prove Theorem 4.1.1. In section 4.3, Theorem 4.1.2 is proved. In section 4.4, Theorems 4.1.5 and 4.1.7 are proved.

In **chapter 5:** we conclude by making some remarks and posing some new problems.

Chapter 2

Restricted Mean Value Property on Riemannian manifolds

2.1 Introduction

Let M be a Riemannian manifold, and let $D \subset M$ be any precompact domain in M with smooth boundary, such that the distance to the boundary $d(z, \partial D)$ is smaller than the injectivity radius $inj(z)$, for all points z in D. In this setting we have the following result:

Theorem 2.1.1 *If $u : D \subset M \to \mathbb{R}$ is bounded, continuous, satisfies the Restricted Mean Value Property in D and*

$$\lim_{\substack{z \to \xi \\ z \in D}} u(z) = u_\xi \text{ exists for almost every } \xi \in \partial D\,, \tag{2.1.1}$$

with respect to the Riemannian measure on ∂D, then u is harmonic in D.

Our next result does not assume boundedness nor any explicit boundary behaviour of u, but rather assumes a suitable rate of decay of the radius function $\rho(z)$ as z tends to the boundary of D in terms of the modulus of continuity of u on compacts in D. In order to state it, we introduce some notation.

For a pre-compact domain D define,

$$r_D := \sup\{d(x, \partial D) : x \in D\}\,. \tag{2.1.2}$$

Then for a continuous function u on D, given $\varepsilon > 0$ and given $r \in (0,1)$, let $\delta(r,\varepsilon;u)$ be the largest positive number such that

$$|u(z) - u(y)| \le \varepsilon$$

holds, provided

$$d(z,\partial D) \ge (1-r)r_D \text{ and } d(z,y) \le \delta(r,\varepsilon;u)\,.$$

Define a function r on D by $r(z) := 1 - (d\,(z,\partial D)\,/2\,r_D)$ for $z \in D$, and note that for any $z \in D$ we have $d(z,\partial D) \ge (1-r(z))r_D$.

We then have:

Theorem 2.1.2 *Let $\kappa \in (0,1), \tau \in (0,1)$ and let $\varepsilon : (0,(1-\tau)r_D] \to (0,\infty)$ be a continuous function such that $\varepsilon(r) \to 0$ as $r \to 0$. Let u be a continuous function on D such that u satisfies the Restricted Mean Value Property at each point $z \in D$ on a geodesic sphere of radius $\rho(z)$, where*

$$\rho(z) \le \begin{cases} \kappa\, d(z,\partial D) & \textit{if } d(z,\partial D) \ge (1-\tau)r_D \\ \min\left\{\kappa\, d(z,\partial D)\,, \delta(r(z),\varepsilon(d(z,\partial D));u)\right\} & \textit{if } d(z,\partial D) < (1-\tau)r_D\,. \end{cases}$$

Then u is harmonic in D.

Both results above follow the general scheme of the arguments in [Fe76], namely constructing subharmonic and superharmonic functions v and w respectively such that $w \le u \le v$, and then using the Maximum Principle to show that the nonnegative subharmonic function $v-w$ vanishes identically.

The two main ingredients required to carry out the above scheme are the following:

(1) Concentration of harmonic measures:

This states that if we have a sequence of points x_n in balls B_n, such that the balls B_n converge to a ball B and the points x_n converge to a point ξ on the boundary of B, then the harmonic measures μ_{x_n,B_n} on the boundaries ∂B_n, with respect to the basepoints x_n, converge weakly to the Dirac mass δ_ξ at ξ.

Note that if the sequence of balls is a constant sequence, $B_n = B$ for all n, then this fact follows immediately from the definition of harmonic measures in terms of solutions to the Dirichlet problem. For a general nonconstant sequence of converging balls this is somewhat non-trivial however. In Euclidean space this concentration of measure is easy to prove using the explicit formula for the Poisson kernels of the balls B_n. In a general Riemannian manifold, we are faced with the lack of any explicit formulae for the Poisson kernel of geodesic balls, and instead we use softer techniques, namely estimating harmonic measures by constructing superharmonic barriers.

(2) Convergence of Poisson integrals:

This states that if u is bounded and we have a sequence of balls B_n in D converging to a ball B in D, then the Poisson integrals of u on B_n converge pointwise on B to the Poisson integral of u on B.

Again, in Euclidean space this follows easily from the explicit formula for the Poisson kernel of a finite ball. In the case of a general Riemannian manifold, we prove this by writing the Poisson integral in terms of Brownian motion, and then applying the Dominated Convergence Theorem on the sample space of Brownian motion, i.e. the space of continuous paths in the manifold.

We also have a result for when the boundary is "at infinity". Namely, if we take M to be a complete, simply connected Riemannian manifold of pinched negative curvature, then it has infinite injectivity radius at each point, and we can consider functions on the whole manifold M satisfying definition 1.1.1, where the domain $D = M$. In the previous two results where only bounded domains were considered, the radius function $z \mapsto \rho(z)$ was bounded. Instead if we take the domain D to be the whole manifold M, then it is natural to ask for an analogue of Theorem 2.1.1 for the whole manifold M, where the radius function is allowed to be unbounded.

In fact, we do have a result in this case too, with no restriction at all on the radius function. In this case however, in place of the boundary ∂D, one needs to consider the *Gromov boundary* ∂M of the Gromov hyperbolic space M, and the appropriate measure class on the Gromov boundary in this context is the harmonic measure class (see section 2.2 for the definitions of Gromov boundary and harmonic measures on the Gromov boundary).

We then have the following:

Theorem 2.1.3 *Let M be a complete, simply connected Riemannian manifold of dimension $\geq$ 2, satisfying sectional curvature bounds $-b^2 \leq K_M \leq -a^2$, for some $0 < a \leq b$. Let $u : M \to \mathbb{R}$*

be a bounded, continuous function which satisfies the Restricted Mean Value Property. Suppose

$$\lim_{\substack{z \to \xi \\ z \in M}} u(z) = u_\xi \text{ , exists for almost every } \xi \in \partial M \text{ ,} \tag{2.1.3}$$

with respect to the harmonic measures on ∂M. Then u is harmonic in M.

We remark that any Riemannian symmetric space of noncompact type of rank one satisfies the conditions of Theorem 2.1.3.

The proof of the above Theorem involves some new ingredients related to the geometry of negatively curved manifolds. In particular, allowing the radius function ρ to be unbounded necessitates an understanding of horoballs and how geodesic balls converge to horoballs. We also need an L^∞ version of the maximum principle for subharmonic functions, concerning boundary values almost everywhere on the Gromov boundary with respect to harmonic measure, which we prove using Brownian motion.

This chapter is organised as follows. In section 2.2, we discuss the relevant preliminaries regarding Gromov hyperbolic spaces, their boundaries, and harmonic measures. In section 2.3, we prove the crucial results on concentration of harmonic measures and convergence of Poisson integrals. In section 2.4, we construct an auxiliary subharmonic function which will play a central role in the proofs of our main results. In section 2.5, we prove some useful L^∞ maximum principles. In section 2.6, we give proofs of Theorems 2.1.1, 2.1.2 and 2.1.3.

2.2 Preliminaries

We recall briefly some basic properties of Gromov hyperbolic spaces and CAT(k) spaces, for more details we refer to [BrH99].

A *geodesic* in a metric space X is an isometric embedding $\gamma : I \subset \mathbb{R} \to X$ of an interval into X. The metric space X is said to be *geodesic* if any two points in X can be joined by a geodesic. A geodesic metric space X is said to be *Gromov hyperbolic* if there is a $\delta \geq 0$ such that every geodesic triangle in X is δ-thin, i.e. each side is contained in the δ-neighbourhood of the union of the other two sides.

The *Gromov boundary* of a Gromov hyperbolic space X is defined to be the set ∂X of equivalence classes of geodesic rays in X. Here a geodesic ray is an isometric embedding γ :

$[0, \infty) \to X$ of a closed half-line into X, and two geodesic rays γ_1, γ_2 are said to be equivalent if the set $\{d(\gamma_1(t), \gamma_2(t)) \mid t \geq 0\}$ is bounded. The equivalence class of a geodesic ray γ is denoted by $\gamma(\infty) \in \partial X$.

A metric space is said to be *proper* if closed and bounded balls in the space are compact. Let X be a proper, geodesic, Gromov hyperbolic space. There is a natural topology on $\overline{X} := X \cup \partial X$, called the *cone topology* such that $\overline{X}$ is a compact metrizable space which is a compactification of X. In this case, for every geodesic ray γ, $\gamma(t) \to \gamma(\infty) \in \partial X$ as $t \to \infty$, and for any $x \in X, \xi \in \partial X$ there exists a geodesic ray γ such that $\gamma(0) = x, \gamma(\infty) = \xi$. We will refer to neighbourhoods of $\xi \in \partial X$ with respect to the cone topology as *cone neighbourhoods*.

For three points $x, y, z \in X$, the *Gromov inner product* of y, z with respect to x is defined to be

$$(y|z)_x := \frac{1}{2}(d(x, y) + d(x, z) - d(y, z)).$$

We now recall the definition of CAT(k) spaces for some $k < 0$. For $k < 0$, a geodesic space is called a CAT(k) space if all its geodesic triangles $\triangle$ satisfy the CAT(k) inequality: for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle} \subset \mathbb{H}^2(k)$, we have

$$d(x, y) \leq d(\overline{x}, \overline{y}) .$$

If the space X is in addition a CAT(k) space for some $k < 0$, then for any $x \in X$, the Gromov inner product $(.|.)_x : X \times X \to [0, +\infty)$ extends to a continuous function $(.|.)_x : \overline{X} \times \overline{X} \to [0, +\infty]$, such that $(\xi|\eta)_x = +\infty$ if and only if $\xi = \eta \in \partial X$. This holds in particular if X is a complete, simply connected Riemannian manifold of sectional curvature bounded above by k for some $k < 0$, as it is well known by a Theorem of Alexandrov that such a manifold is a CAT(k) space. In this case the compactification $\overline{X}$ is homeomorphic to the closed unit ball $\overline{\mathbb{B}} \subset \mathbb{R}^n$, and there is a homeomorphism $\phi : \overline{\mathbb{B}} \to \overline{X}$ such that the restriction to the open unit ball $\phi : \mathbb{B} \to X$ is a diffeomorphism.

We now assume that X is such a negatively curved manifold, i.e. X is complete, simply connected and with sectional curvature bounded above by $-a^2$ for some $a > 0$. The *Busemann cocycle* of X is the function $B : X \times X \times \partial X \to \mathbb{R}$ defined by

$$B(x, y, \xi) := \lim_{z \to \xi}(d(x, z) - d(y, z)).$$

The limit above exists, and the Busemann cocycle is a continuous function on $X \times X \times \partial X$. Fix a basepoint $o \in X$. Then for any $\xi \in \partial X$ and $r \in \mathbb{R}$, the *horoball based at* ξ *of radius* r is defined to be the set

$$H(\xi, r) := \{x \in X \mid B(x, o, \xi) < r \}.$$

The boundary of the horoball in X is given by the set

$$\partial H(\xi, r) = \{x \in X \mid B(x, o, \xi) = r \}$$

and is called a *horosphere based at* ξ *of radius* r. The closures of the horoball and the horosphere in $\overline{X}$ are given by the compact sets $H(\xi, r) \cup \partial H(\xi, r) \cup \{\xi\}$ and $\partial H(\xi, r) \cup \{\xi\}$ respectively.

From the Busemann cocycle, one can also define functions on X called *Busemann functions.* For this we fix a basepoint $o \in X$. Then for any $\xi \in \partial X$, we can define a function $B_\xi : X \to \mathbb{R}$, called the *Busemann function based at* ξ, which is given by

$$B_\xi(x) = B(x, o, \xi).$$

It is well-known that the Busemann functions B_ξ are C^2, and the gradient of the Busemann function has constant norm equal to one, $||\nabla B_\xi|| \equiv 1$. Note that the horospheres based at ξ, $\partial H(\xi, r), r \in \mathbb{R}$, are the level sets of the Busemann function B_ξ. Since the Busemann function is C^2 with nowhere vanishing gradient, it follows that the horospheres are C^2 submanifolds of X.

We will be needing some well-known results on the convergence of balls to horoballs and of horoballs to horoballs. For any $x \in X$, let $d_x : X \to \mathbb{R}$ denote the distance function from the point x, given by $d_x(y) = d(x, y), y \in X$. Then for any $\xi \in \partial X$, if $x_n \in X$ is a sequence such that $x_n \to \xi$ as $n \to \infty$, then the functions $d_{x_n} - d(o, x_n)$ converge in C^2 norm on compacts to the Busemann function B_ξ. Also, if $\xi_n \in \partial X$ is a sequence such that $\xi_n \to \xi$ as $n \to \infty$, then the Busemann functions B_{ξ_n} converge in C^2 norm on compacts to the Busemann function B_ξ.

It is well-known that if $x_n \in X$ is a sequence such that $x_n \to \xi \in \partial X$ as $n \to \infty$, and if $r_n > 0$ is a sequence such that $r_n - d(x_n, o) \to r \in \mathbb{R}$ as $n \to \infty$, then the closed balls $\overline{B}(x_n, r_n)$ converge to the closure of the horoball $H(\xi, r)$ in $\overline{X}$ with respect to the Hausdorff topology on compact subsets of $\overline{X}$. Similarly, if $\xi_n \in \partial X$ and $r_n \in \mathbb{R}$ are sequences such that $\xi_n \to \xi, r_n \to r$ as $n \to \infty$, then the closures in $\overline{X}$ of the horoballs $H(\xi_n, r_n)$ converge to the closure in $\overline{X}$ of the horoball $H(\xi, r)$, with respect to the Hausdorff topology on compact subsets of $\overline{X}$.

We also need to recall the definition of harmonic measures. These arise from the solution of the Dirichlet problem. Given a precompact domain D with smooth boundary in a Riemannian manifold, for any $x \in D$ the *harmonic measure on ∂D with respect to x* is the probability measure $\mu_{x,D}$ on ∂D defined by

$$\int_{\partial D} f \, d\mu_{x,D} = u_f(x)$$

for all continuous functions f on ∂D, where u_f is the solution of the Dirichlet problem in D with boundary value f. The harmonic measures $\mu_{x,D}$ are mutually absolutely continuous, in fact they are absolutely continuous with respect to the Riemannian measure of ∂D. The harmonic measure $\mu_{x,D}$ can also be described in terms of Brownian motion started at x. If $(B_t)_{t\geq 0}$ is a Brownian motion started at x, and τ is defined to be the first exit time from the domain D, i.e.

$$\tau = \inf\{\ t > 0 \mid B(t) \notin D\ \},$$

then we have

$$\int_{\partial D} f \, d\mu_{x,D} = \mathbb{E}(f(B_\tau))$$

for all continuous functions f on ∂D.

The harmonic measures also allow us to define the *Poisson integral* of any L^∞ function $f \in L^\infty(\partial D)$, which is a bounded harmonic function on D defined by

$$P[f](x) := \int_{\partial D} f \, d\mu_{x,D}\ ,\ x \in D.$$

For a complete, simply connected Riemannian manifold X of pinched negative curvature, i.e. with sectional curvature K satisfying $-b^2 \leq K_X \leq -a^2$ for some positive constants $b \geq a > 0$, it is well-known that the Dirichlet problem at infinity is solvable ([AS85], [Su83]), and so in this case one can also define a family of harmonic measures $\{\mu_x\}_{x\in X}$, which are probability measures on ∂X defined by

$$\int_{\partial X} f \, d\mu_x = u_f(x)$$

for any continuous function f on ∂X, where u_f is the solution of the Dirichlet problem at infinity with boundary value f, i.e. u_f is harmonic on X and $u_f(x) \to f(\xi)$ as $x \to \xi \in \partial X$, for any $\xi \in \partial X$. As in the case of a bounded domain, the harmonic measures μ_x are mutually absolutely continuous. As before, in this case also the harmonic measure μ_x can be described in terms of Brownian motion started at x. If $(B(t))_{t\geq 0}$ is a Brownian motion started at x, then it is known ([Su83]) that almost every sample path of the Brownian motion converges to a (random) point B_∞ in ∂X. This limiting point B_∞ is a random variable taking values in ∂X, whose distribution is precisely the harmonic measure μ_x; for any continuous function f on ∂X, we have

$$\int_{\partial D} f \ d\mu_{x,D} = \mathbb{E}(f(B_\infty)).$$

As in the case of a precompact domain, in this case too we can define the Poisson integral of any L^∞ function on ∂X, where the measure class on ∂X is that defined by the harmonic measures. As before, the Poisson integral of a function $f \in L^\infty(\partial X)$ is a bounded harmonic function on X defined by

$$P[f](x) := \int_{\partial X} f \ d\mu_x \ , \ x \in X.$$

Finally, it is also possible to define harmonic measures on the boundaries of horoballs. While any horoball $H = H(\xi, r)$ is not precompact in X, it is nevertheless precompact in $\overline{X}$, and its boundary in $\overline{X}$ is given by the compactified horosphere $S := \partial H \cup \{\xi\}$, which is a compact subset of $\overline{X}$. The Dirichlet problem for the horoball H can then be stated as follows: given a continuous function $f \in C(S)$, find a continuous extension $u \in C(H \cup S)$ such that u is harmonic in H. It is well-known from the Perron method that the Dirichlet problem is solvable if and only if there is a superharmonic barrier at each point of the boundary $S = \partial H \cup \{\xi\}$. For points $z \in \partial H$, the existence of a barrier follows from the fact that ∂H is C^2, while for the boundary point $\xi \in \partial X$ of the horoball, one may take as a barrier a function of the form $\phi(x) = \exp(-\delta d(o, x)), x \in X$, for any constant δ such that $0 < \delta < (n-1)a$. Such a function is known to be superharmonic on X, extends continuously to $H \cup S$, and has the correct boundary behaviour at ξ, namely $\phi(\xi) = 0$ and $\phi(x) > 0$ for all $x \in (H \cup S) \setminus \{\xi\}$.

Thus the Dirichlet problem is solvable for the horoball H, and then as before the solution of the Dirichlet problem gives rise to a family of harmonic measures $\mu_{x,H}$ on the boundary $S = \partial H \cup \{\xi\}$ of the horoball in $\overline{X}$.

2.3 Concentration of Harmonic measures and convergence of Poisson integrals

2.3.1 Concentration of Harmonic measures

In this subsection we prove the results on concentration of measures mentioned in the Introduction. We show that the harmonic measures of the complement of a neighborhood of a boundary point of a limiting domain go to zero as the basepoints in a sequence of domains converge to the boundary point. The following is our first result in this direction and uses the concept of local superharmonic barriers.

Throughout M denotes a Riemannian manifold.

Lemma 2.3.1 *Let $D \subset M$ be a domain with C^2 boundary. Let $\xi \in \partial D$, and let U be a precompact neighborhood of ξ such that $\overline{D} \cap U$ is C^2-diffeomorphic to a closed half-ball in $\mathbb{R}^m$. Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of precompact domains in M with C^2 boundary, and let $\{g_n\}_{n=1}^{\infty}$ be a sequence of C^2 local diffeomorphisms from U into M, with $U_n := g_n(U)$ satisfying*

(i) $g_n(U \cap \partial D) = U_n \cap \partial D_n$ and $g_n(U \cap D) = U_n \cap D_n$, for all $n \in \mathbb{N}$,

(ii) $g_n \to id$ in C^2 norm on U.

Then for $y_n \in D_n$ such that $y_n \to \xi$ as $n \to \infty$, we have,

$$\mu_{y_n,D_n}(\partial D_n \setminus U) \to 0 \text{ as } n \to \infty .$$

Proof: Let W_1, W_2 be two neighborhoods of ξ such that $\overline{W_2} \subset W_1$ and $\overline{W_1} \subset U$. Let $\varphi \in C^0(M)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 0$ on W_1 and $\varphi \equiv 1$ on $M \setminus U$. Then

$$\mu_{y_n,D_n}(\partial D_n \setminus U) \le \int_{\partial D_n} \varphi(\zeta)\, d\mu_{y_n,D_n}(\zeta) = u_n(y_n) , \tag{2.3.1}$$

where u_n is the harmonic extension of $\varphi|_{\partial D_n}$ to D_n.

Now by the regularity of $\overline{D} \cap U$, there exists a C^2 diffeomorphism

$$\psi : U \to V \subset \mathbb{R}^m ,$$

such that V is an open neighborhood of 0 , $\psi(\xi) = 0$,

$$\begin{aligned} \psi(\partial D \cap U) \quad &= \{x \in \mathbb{R}^m : \|x + e_1\| = 1\} \cap V , (\text{where } e_1 = (1, 0, \dots, 0)) \\ \psi(\overline{D} \cap U \setminus \{\xi\}) \quad &\subset \{x \in \mathbb{R}^m : \|x + e_1\| \le 1\} \cap V \setminus \{0\} . \end{aligned}$$

Let $\xi_n := g_n(\xi) \in U_n \cap \partial D_n$ for all $n \in \mathbb{N}$. Then consider the C^2 diffeomorphisms

$$\psi_n : U_n \to V \subset \mathbb{R}^m ,$$

defined by,

$$\psi_n = \psi \circ g_n^{-1} \,. \tag{2.3.2}$$

Then we note that

$$\begin{aligned} \psi_n(\xi_n) \quad & = 0\,, \\ \psi_n(\partial D_n \cap U_n) \quad & = \{x \in \mathbb{R}^m : \|x+e_1\| = 1\} \cap V\,, (\text{where } e_1 = (1,0,\ldots,0)) \\ \psi_n(\overline{D_n} \cap U_n \setminus \{\xi_n\}) \quad & \subset \{x \in \mathbb{R}^m : \|x+e_1\| \le 1\} \cap V \setminus \{0\}\,. \end{aligned}$$

Now as $g_n \to id$ in C^2 norm on U, we have $\xi_n \to \xi$. Moreover (2.3.2) implies that $\psi_n \to \psi$ in C^2 norm and hence, in particular $\left\{\|\psi_n\|_{C^2(\overline{D_n}\cap U_n)}\right\}_{n=1}^{\infty}$ is uniformly bounded.

For $h \in C^2(V)$, consider for $n \in \mathbb{N}$, $L_n h := (\Delta(h \circ \psi_n)) \circ \psi_n^{-1}$, which is a second order elliptic operator,

$$L_n = \sum_{i,j} a_{ij,n} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_{i,n} \frac{\partial}{\partial x_i}\,.$$

By uniform boundedness of C^2 norms of ψ_n, we get that for all $n \in \mathbb{N}$, there exists $\lambda_1, \lambda_2 > 0$ such that

$$\begin{aligned} & \sum_{i,j} a_{ij,n}(x)\, v_i v_j \ge \lambda_1 \|v\|^2 \text{ and} \\ & |b_{i,n}(x)| \le \lambda_2\,, \text{ for all } i\,, \text{ for all } x \in V \cap \psi_n(\overline{D_n} \cap U_n)\,,\ v = (v_1,\ldots,v_m) \in \mathbb{R}^m\,. \end{aligned}$$

Let $s(x) := 1 - e^{\alpha x_1}$ for $\alpha > 0$. Then $s(0) = 0$ and

$$s(\{x \in \mathbb{R}^m : \|x+e_1\| \le 1\} \setminus \{0\}) \subset (0,\infty)\,.$$

Also for all $x \in V \cap \psi_n(\overline{D_n} \cap U_n)$,

$$\begin{aligned} L_n\, s(x) \quad & = \quad -(\alpha^2\, a_{11,n} + \alpha\, b_{1,n}) e^{\alpha x_1} \\ & \le \quad -(\lambda_1\, \alpha^2 - \lambda_2\, \alpha) e^{\alpha x_1} \\ & \le \quad 0\,, \quad \text{for } \alpha \text{ sufficiently large}\,. \end{aligned}$$

It follows that $s_n := s \circ \psi_n$ is superharmonic on $D_n \cap U_n$, with $s_n(\xi_n) = 0$, for all n. Then by choosing α sufficiently large in the definition of s and replacing s by $2s$, we can ensure that

$$s_n \geq 1 \text{ on } (\overline{D_n} \cap U_n) \setminus W_2 ,$$

for large n.

We note due to hypothesis (ii), for large n, $\overline{W_2} \subset U_n$. Thus we extend s_n to all of $\overline{D_n}$ by,

$$\tilde{s_n}(z) = \begin{cases} \min\{1,\ s_n(z)\} & \text{if } z \in \overline{D_n} \cap W_2 \\ 1 & \text{if } z \in \overline{D_n} \setminus W_2 . \end{cases}$$

We note that $\tilde{s_n}$ is superharmonic, non-negative and for n sufficiently large, $\tilde{s_n} \geq \varphi$ on ∂D_n. The last fact is seen as follows.

On $\partial D_n \cap W_2$,

$$\varphi \equiv 0 \leq \tilde{s_n} ,$$

and on $\partial D_n \setminus W_2$,

$$\varphi \leq 1 \equiv \tilde{s_n} .$$

Hence by the maximum principle, it follows that for n large,

$$u_n(y_n) \leq \tilde{s_n}(y_n) \leq s_n(y_n) . \tag{2.3.3}$$

Now as $y_n \to \xi$ and $g_n \to id$ uniformly, it follows that $g_n^{-1}(y_n) \to \xi$. Hence for n large,

$$s_n(y_n) = s(\psi_n(y_n)) = s(\psi(g_n^{-1}(y_n))) \to s(0) = 0 \tag{2.3.4}$$

Hence, combining (2.3.1), (2.3.3) and (2.3.4), it follows that

$$\mu_{y_n,D_n}(\partial D_n \setminus U) \leq u_n(y_n) \leq s_n(y_n) \to 0 \text{ as } n \to \infty .$$

□

Note that the above Lemma applies in particular if we have a sequence of balls $B_n = B(x_n, r_n)$ converging to a ball $B = B(x, r)$ (meaning that $x_n \to x$ and $r_n \to r$ as $n \to \infty$) and $y_n \in B_n$

converges to $\xi \in \partial B$, or if we have a sequence of balls $B_n = B(x_n, r_n)$ converging to a horoball $H = H(\xi, r)$ (meaning that $x_n \to \xi$ and $r_n - d(x_n, o) \to r$ as $n \to \infty$) and $y_n \in B_n$ converges to $z \in \partial H$.

The next result can be viewed as a global analogue of Lemma 2.3.1 and uses the concept of global barriers. In the statement of the Lemma below, closures of domains in M are taken in the compactification $\overline{M}$.

Lemma 2.3.2 *Let M be a complete, simply connected Riemannian manifold, satisfying sectional curvature bounds $-b^2 \leq K_M \leq -a^2$, for some $0 < a \leq b$. Let W_n be a sequence of domains in M where W_n is either a ball, a horoball or the full space M such that $\overline{W_n}$ converges to $\overline{W}$ in the Hausdorff topology on compacts in $\overline{M}$, where W is either a horoball or the full space M.*

Let $z_n \in W_n$ be a sequence such that $z_n \to \xi$ as $n \to \infty$, for some $\xi \in \overline{W} \cap \partial M$. Then for any cone neighborhood U of ξ,

$$\mu_{z_n, W_n}(\partial W_n \setminus U) \to 0 \quad \text{as } n \to \infty .$$

Proof: Passing to a subsequence we may assume that either all $W_n = M$ or none of them is M.

In the former case, the result follows from the solution of the Dirichlet problem on ∂M as the harmonic measures μ_{z_n} converge weakly to the Dirac mass at ξ.

In the latter case, W_n is either a ball or a horoball. Depending on the limiting domain, this case has two subcases:

(i) the limiting domain is a horoball,

(ii) the limiting domain is all of M.

In case (i) let V be another cone neighborhood of ξ such that $\overline{V} \subset U$. Let $\psi \in C^0(\overline{M})$ be such that $0 \leq \psi \leq 1$, $\psi \equiv 0$ on V and $\psi \equiv 1$ on $\overline{M} \setminus U$. Then note that

$$\mu_{z_n, W_n}(\partial W_n \setminus U) \leq \int_{\partial W_n} \psi(y)\, d\mu_{z_n, W_n}(y) = u_n(z_n) , \tag{2.3.5}$$

where u_n is the harmonic extension of $\psi|_{\partial W_n}$ to W_n. Now note that for n large enough, $\partial W_n \setminus V$ is contained in a fixed ball $\tilde{B}$ centered at the origin.

Fix a $\delta \in (0, (m-1)a)$, where m is the dimension of M and $-a^2$ is the assumed upper bound on the sectional curvature. For such a δ, as shown in [AS85], the function

$$s(x) := \frac{1}{\varepsilon} e^{-\delta\, d(o,x)}\ ,\ x \in M\ ,$$

is superharmonic on M.

Now there exists $\varepsilon > 0$ such that

$$e^{-\delta\, d(o,x)} > \varepsilon\ ,\ \text{ for all } x \in \tilde{B}\ .$$

Then note that s is non-negative and $s \geq 1$ on $\tilde{B}$ and hence also on $\partial W_n \setminus V$, for n large enough. Thus for n sufficiently large, $\psi \leq s$ on ∂W_n. This is seen as follows.

On $\partial W_n \setminus V$

$$\psi \leq 1 \leq s\ ,$$

and on $\partial W_n \cap V$

$$\psi \equiv 0 \leq s\ .$$

Hence by the maximum principle, it follows that for n large enough,

$$u_n(z_n) \leq s(z_n)\ . \tag{2.3.6}$$

Then combining (2.3.5) and (2.3.6), it follows that

$$\mu_{z_n, W_n}(\partial W_n \setminus U) \leq u_n(z_n) \leq s(z_n) = \frac{1}{\varepsilon} e^{-\delta\, d(o,z_n)} \to 0 \text{ as } n \to \infty\ .$$

This completes the proof for the case (i).

Now for case (ii) we start by noting that the Dirichlet problem is solvable for a cone neighborhood U such that $\partial U \cap M$ is C^2 (we may assume without loss of generality that U satisfies this hypothesis). This is seen as follows. By the well-known Perron method for the solution of the Dirichlet problem, it is enough to show that there exists a barrier at each point of the boundary $\partial U \subset \overline{M}$. For each point $z \in \partial U \cap M$ on the finite part of the boundary, there exists a barrier at z by the C^2 regularity of $\partial U \cap M$. For a point $\eta \in \partial U \cap \partial M$ on the part of the boundary of U in ∂M, we can use the fact that the Dirichlet problem is solvable for M: for a barrier at ξ we take the restriction to $\overline{U}$ of the harmonic extension of a function $\psi \in C^0(\partial M)$ such that $\psi(\eta) = 0$ and $\psi(\zeta) > 0$ for all $\zeta \in \partial M - \{\eta\}$.

Then as the domains W_n converge to all of M and $z_n \to \xi$, we have for large n, $z_n \in U$ but $\overline{W_n} \setminus U$ is non-empty. Note that for any Brownian path which starts at z_n and first exits the domain W_n through $\partial W_n \setminus U$, the first exit point of this path from U must lie in the finite part $\partial U \cap M$ of ∂U. The probabilistic interpretation of harmonic measures then implies that

$$\mu_{z_n,W_n}\left(\partial W_n \setminus U\right) \leq \mu_{z_n,U}\left(\partial U \cap M\right) . \tag{2.3.7}$$

Now the solvability of the Dirichlet problem for U noted earlier implies that harmonic measures on U with respect to z_n converge weakly to the Dirac mass at ξ. This implies

$$\mu_{z_n,U}\left(\partial U \cap M\right) \to 0\,, \text{ as } n \to \infty\,. \tag{2.3.8}$$

from which it follows that

$$\mu_{z_n,W_n}\left(\partial W_n \setminus U\right) \to 0\,, \text{ as } n \to \infty\,.$$

This completes the proof of the Lemma. □

2.3.2 Convergence of Poisson integrals

In this subsection we show that convergence of domains implies convergence of the corresponding Poisson integrals of a function having nice boundary behaviour. We first prove some preliminary results.

Lemma 2.3.3 *Let B and D be pre-compact domains with smooth boundary such that $B \subset D$. Let $A \subset \partial B \cap \partial D$ be such that $\mu_{x,D}(A) = 0$ for all $x \in D$, then $\mu_{x,B}(A) = 0$ for all $x \in B$.*

Proof: Let $x \in B$. If the Brownian motion $(B_t)_{t\geq 0}$ starting at x first exits the domain B via the boundary portion A, then in particular it also first exits the domain D via the boundary portion A. If τ_1 and τ_2 are the first exit times for the Brownian motion from the domains B and D respectively, then we have

$$\mu_{x,B}(A) = Pr\{B_{\tau_1} \in A\} \leq Pr\{B_{\tau_2} \in A\} = \mu_{x,D}(A) = 0\,.$$

□

Lemma 2.3.4 *Let M be a complete, simply connected Riemannian manifold of dimension ≥ 2, satisfying sectional curvature bounds $-b^2 \leq K_M \leq -a^2$, for some $0 < a \leq b$. Let H be a horoball in M based at $\xi \in \partial M$. Let τ denote the first exit time of Brownian motion from H Then:*

(1) τ is finite almost surely.

(2) $\{\xi\}$ is not an atom for the harmonic measures on the compactified horosphere $\partial H \cup \{\xi\}$.

Proof: (1) Choose and fix an interior point $z \in H$. By taking limits of radii in Theorem 1.1 of [BH19] we get that the harmonic measure on ∂X with respect to z has no atoms. By the probabilistic interpretation of the harmonic measures, it follows that for Brownian motion starting from z, the limiting point B_∞ lies in $\partial X - \{\xi\}$ with probability 1. Since $\overline{H} \cap \partial M = \{\xi\}$, this implies that with probability 1 the Brownian motion starting from z first exits H at some finite boundary point of H (that is points lying on ∂H). Thus $\tau < \infty$ almost surely.

(2) The above means that

$$\mu_{z,H}\left(\partial H\right) = 1 \, ,$$

which implies

$$\mu_{z,H}\left(\{\xi\}\right) = 0 \, .$$

As z was an arbitrary interior point, the result follows. □

Lemma 2.3.5 *Let D be either of the following:*

(1) A precompact domain in a Riemannian manifold M, or

(2) $D = M$, a complete, simply connected Riemannian manifold of pinched negative curvature $-b^2 \leq K_M \leq -a^2$, for some $0 < a \leq b$. In this case $\overline{D}$ denotes the compactification $\overline{M}$.

Let $W_n \subset D$ be a sequence of domains such that $\overline{W_n} \to \overline{W}$ and $\partial W_n \to \partial W$ in the Hausdorff topology on compacts in $\overline{D}$, for some domain $W \subset D$.

Let $\gamma : [0, \infty) \to M$ be a continuous path starting from some $x \in W$. Let τ_n, τ be the first exit times of the path γ from W_n, W respectively.

Then $\tau_n \to \tau$ as $n \to \infty$.

Proof: Note that the hypothesis of the Lemma implies that $x \in W_n$ for all n large enough, we only consider such n. We consider two cases:

(1) $\tau < +\infty$:

In this case, given $\epsilon > 0$, by the definition of the first exit time there exists $t > 0$ with $\tau < t < \tau(\gamma) + \varepsilon$ such that $\gamma(t)$ lies outside an open neighbourhood U of $\overline{W}$. Then by the Hausdorff convergence hypothesis, we have $\overline{W_n} \subset U$ for all n large enough, thus $\gamma(t)$ lies outside of $\overline{W_n}$, and so $\tau_n \leq t$. Thus for n large,

$$\tau_n \leq t < \tau + \varepsilon \,.$$

On the other hand by definition of the first exit time, the portion of the path $\gamma([0, \tau - \varepsilon])$ is contained in $W - V$ for some open neighbourhood V of ∂W. By the Hausdorff convergence of ∂W_n to ∂W, for n large enough we have $\partial W_n \subset V$, thus $\gamma([0, \tau - \varepsilon])$ is contained in W_n, hence

$$\tau_n > \tau - \epsilon \,.$$

This completes the proof in the case $\tau < +\infty$.

(2) $\tau = +\infty$:

In this case, given $T > 0$, since $\tau = +\infty$ there exists a neighbourhood V of ∂W such that $\gamma([0, T]) \subset W - V$. Now as before, the Hausdorff convergence implies that for all n large enough we have $\partial W_n \subset V$, hence $\gamma([0, T]) \subset W_n$ and so $\tau_n \geq T$. It follows that $\tau_n \to +\infty = \tau$ as $n \to \infty$. □

Lemma 2.3.6 *Let D be either of the following:*

(1) A precompact domain in a Riemannian manifold M, or

(2) $D = M$, a complete, simply connected Riemannian manifold of pinched negative curvature $-b^2 \leq K_M \leq -a^2$, for some $0 < a \leq b$. In this case $\overline{D}$ denotes the compactification $\overline{M}$.

Let $W_n \subset D$ be a sequence of domains such that

- $\partial W_n \cap \partial D = \emptyset$,
- $\overline{W_n} \to \overline{W}$ *and* $\partial W_n \to \partial W$ *in the Hausdorff topology on compacts in* $\overline{D}$*, for some domain* $W \subset D$*.*

Moreover, in case (1) *we assume that* W_n *and* W *are balls, while in case* (2) *we assume that* W_n *are balls and* W *is either a ball, a horoball or all of* M*.*

Let u *be bounded, continuous on* D *and satisfy*

$$\lim_{\substack{z\to\xi \\ z\in D}} u(z) = f(\xi) \text{ exists for almost every } \xi \in \partial D\,, \tag{2.3.9}$$

with respect to the harmonic measures on ∂D*, where* f *is an* L^∞ *function on* ∂D*.*

Then the Poisson integrals of u *on* W_n *converge pointwise on* W *to the Poisson integral of* u *on* W*.*

Proof: First let us assume that D is as in (1) that is a bounded, pre-compact domain. Then W is either of the following:

(i) W is a ball in D such that $\partial W \cap \partial D$ has zero harmonic measure,

(ii) W is a ball in D such that $\partial W \cap \partial D$ has positive harmonic measure.

In both the cases (i) and (ii), by continuity, Lemma 2.3.3 and boundary behaviour (2.3.9), it follows that boundary limits of u exist for almost all points on ∂W (with respect to the Riemannian measure of ∂W) and hence u defines an L^∞ function on ∂W. Hence the Poisson integral of u on W is well-defined.

We choose and fix $x \in W$. By convergence of domains, $x \in W_n$ for all n sufficiently large. We consider the Brownian motion $(B_t)_{t\geq 0}$ starting at x. By convergence of first exit times in Lemma 2.3.5, we have

$$B_{\tau_n} \to B_\tau \text{ as } n \to \infty\,,$$

on almost all Brownian paths, where τ_n, τ are the first exit times of the Brownian motion (starting at x) from W_n, W respectively. Let A be the full harmonic measure set in ∂W where boundary limits of u exist. Now by the probabilistic interpretation of harmonic measures, almost all Brownian paths starting at x land in A. Consequently it follows that

$$u(B_{\tau_n}) \to u(B_\tau) \text{ almost surely as } n \to \infty\,.$$

Then by boundedness of u, the Dominated Convergence Theorem is applicable on the probability space of Brownian paths starting at x and it yields,

$$\mathbb{E}(u(B_{\tau_n})) \to \mathbb{E}(u(B_\tau)) \text{ as } n \to \infty \,,$$

that is,

$$\int_{\partial W_n} u(\xi)\, d\mu_{x,W_n}(\xi) \to \int_{\partial W} u(\xi)\, d\mu_{x,W}(\xi) \,,$$

which gives the result in this case.

Now for the case when D is as in (2), W is one of the following:

(i) a ball,

(ii) a horoball,

(iii) all of M.

In case (ii) it follows from Lemma 2.3.4 that u is defined almost everywhere with respect to the harmonic measure on ∂W, and so the Poisson integral of u on W is well-defined, while in case (iii) we take the Poisson integral of u to be the Poisson integral of the boundary values on ∂M given by (2.3.9), which are defined almost everywhere with respect to harmonic measure on ∂M. Thus in all cases the Poisson integral of u on W is well-defined.

Now for cases (i) and (ii), by Lemma 2.3.4, the first exit times are finite almost surely and hence arguments similar to that in the case of bounded domains yield the result.

Finally, for case (iii), we choose and fix $x \in M$. Then $x \in W_n$, for n sufficiently large. By Lemma 2.3.5, we have $\tau_n \to \infty$ almost surely as $n \to \infty$. Recalling from Sullivan's result [Su83] that the Brownian motion B_t converges almost surely as t tends to infinity to a random variable B_∞ taking values in ∂M, we have

$$B_{\tau_n} \to B_\infty \ \text{ almost surely as } n \to \infty.$$

Now the boundary limits of u exist for all points in a subset $A \subset \partial M$ of full harmonic measure. Since the harmonic measure on ∂M is the distribution of B_∞, we have $B_\infty \in A$ almost surely. It follows that

$$u(B_{\tau_n}) \to f(B_\infty) \ \text{ almost surely as } n \to \infty$$

Then by the Dominated Convergence Theorem on the probability space of paths, we get

$$\int_{\partial W_n} u(\xi) d\mu_{x,W_n}(\xi) = \mathbb{E}(u(B_{\tau_n})) \to \mathbb{E}(f(B_\infty)) = \int_{\partial M} f(\xi) d\mu_x(\xi) \, .$$

□

We now apply Lemma 2.3.6 to obtain the following more general version of convergence of Poisson integrals:

Lemma 2.3.7 *Let D be either of the following:*

(1) A precompact domain in a Riemannian manifold M, or

(2) $D = M$, a complete, simply connected Riemannian manifold of pinched negative curvature $-b^2 \leq K_M \leq -a^2$, for some $0 < a \leq b$. In this case $\overline{D}$ denotes the compactification $\overline{M}$.

Let $W_n \subset D$ be a sequence of domains such that $\overline{W_n} \to \overline{W}$ and $\partial W_n \to \partial W$ in the Hausdorff topology on compacts in $\overline{D}$, for some domain $W \subset D$.

Moreover we assume that in case (1) W_n and W are balls, while in case (2) W_n and W are either balls, horoballs or all of M.

Let u be bounded, continuous on D and satisfy

$$\lim_{\substack{z \to \xi \\ z \in D}} u(z) = f(\xi) \text{ exists for almost every } \xi \in \partial D \, ,$$

with respect to the harmonic measures on ∂D.

Then the Poisson integrals of u on W_n converge pointwise on W to the Poisson integral of u on W.

In fact the convergence is uniform on compacts in W.

Proof: We choose and fix $x \in W$. By convergence of domains, $x \in W_n$ for all n sufficiently large. For any such n, we can choose a ball $V_n \subset D$ such that

- $\partial V_n \cap \partial D = \emptyset$,
- $x \in V_n$,

- Hausdorff distances between the closures $\overline{V_n}$ and $\overline{W_n}$, as well as the boundaries ∂V_n and ∂W_n are sufficiently small so that by Lemma 2.3.6, we have

$$|P_{V_n}[u](x) - P_{W_n}[u](x)| < \frac{1}{n}\,, \tag{2.3.10}$$

where $P_{V_n}[u]$ and $P_{W_n}[u]$ are the Poisson integrals of u on V_n and W_n respectively, and

- $\overline{V_n} \to \overline{W}$ and $\partial V_n \to \partial W$ in the Hausdorff topology on compacts in $\overline{D}$.

Then Lemma 2.3.6 implies that given $\varepsilon > 0$, for n sufficiently large, we have

$$|P_{V_n}[u](x) - P_W[u](x)| < \frac{\varepsilon}{2}\,, \tag{2.3.11}$$

where $P_{V_n}[u]$ and $P_W[u]$ are the Poisson integrals of u on V_n and W respectively. Then by (2.3.10) and (2.3.11), for n large we have

$$|P_{W_n}[u](x) - P_W[u](x)| < \varepsilon\,.$$

This proves the pointwise convergence of the Poisson integrals. The uniform convergence on compacts of the Poisson integrals follows from this pointwise convergence and from the fact that the Poisson integrals are uniformly bounded harmonic functions. The uniform boundedness implies by classical results that the Poisson integrals are equicontinuous on compacts in W, and the pointwise convergence implies the uniqueness of the limit of any subsequence which converges uniformly on compacts, hence the whole sequence converges uniformly on compacts. □

2.4 Construction of an auxiliary subharmonic function

For $z \in D$, under the hypothesis of Theorem 2.1.1, we define $\rho_0(z) = d(z, \partial D)$ and let $\mathscr{F}_z$ be the collection of harmonic extensions of u on balls $B(x,r)$ where $x \in D, r \leq \rho_0(x), z \in B(x,r)$ and we have

$$u(x) = \int_{\partial B(x,r)} u(y)\, d\mu_{x,B(x,r)}(y)$$

(in the case where $r = \rho_0(x)$, we note that by Lemma 2.3.3, u is defined almost everywhere on $\partial B(x,r)$ with respect to the harmonic measure on $\partial B(x,r)$, and so by the harmonic extension of u on $B(x,r)$ we mean the Poisson integral of u on $B(x,r)$).

For $z \in D$, under the hypothesis of Theorem 2.1.2, define ρ_0 by,

$$\rho_0(z) = \begin{cases} \kappa\, d(z, \partial D) & \text{if } d(z, \partial D) \geq (1-\tau) r_D \\ \min\{\kappa\, d(z, \partial D)\ ,\ \delta(r(z), \varepsilon(d(z, \partial D)); u)\} & \text{if } d(z, \partial D) < (1-\tau) r_D\ , \end{cases} \tag{2.4.1}$$

and let $\mathscr{F}_z$ be the collection of harmonic extensions of u on balls $B(x, r)$ where $x \in D, r \leq \rho_0(x), z \in B(x, r)$ and

$$u(x) = \int_{\partial B(x,r)} u(y)\, d\mu_{x,B(x,r)}(y)\ .$$

Finally for $z \in M$, under the hypothesis of Theorem 2.1.3, we define $\mathscr{F}_z$ to be the collection of

- harmonic extensions of u on balls $B(x, r) \subset M$ where $z \in B(x, r)$ and

$$u(x) = \int_{\partial B(x,r)} u(y)\, d\mu_{x,B(x,r)}(y)\ ,$$

- harmonic extensions of u (i.e. the Poisson integral of the L^∞ function u on the horospheres, which is well-defined by Lemma 2.3.4) on horoballs that contain z,
- the harmonic extension of u (i.e. the Poisson integral of the L^∞ function u on ∂M) on M.

In the rest of this section D will be used to denote both a bounded, pre-compact domain (as in Theorems 2.1.1, 2.1.2) as well as a negatively curved Hadamard manifold (as in Theorem 2.1.3) according to context, with more details specified when required.

We now define,

$$v(z) := \sup_{h \in \mathscr{F}_z} h(z)\ ,\ \text{for all } z \in D. \tag{2.4.2}$$

Then note that, since u satisfies the RMVP, for any $z \in D$ there exists $h \in \mathcal{F}_z$ such that $u(z) = h(z)$, and hence

$$u(z) \leq v(z)\ ,\ \text{for all } z \in D. \tag{2.4.3}$$

In the rest of this section we will prove some important properties of v.

Lemma 2.4.1 *For all $z \in D$, there exists $h \in \mathscr{F}_z$ such that $v(z) = h(z)$.*

Proof: Choose and fix $z_0 \in D$. Let $\{h_n\}_{n=1}^{\infty} \in \mathscr{F}_{z_0}$ such that

$$h_n(z_0) \to v(z_0) \text{ as } n \to \infty \,.$$

For each $n \in \mathbb{N}$, let W_n be the domain such that h_n is the harmonic extension of u on W_n (i.e. the Poisson integrals of the L^∞ function u on ∂W_n).

Passing to a subsequence, we may assume that $\overline{W_n} \to \overline{W}$ and $\partial W_n \to \partial W$ (for some domain W such that $W \subset D$) in the Hausdorff topology on compacts in $\overline{D}$. As $z_0 \in W_n$ for all $n \in \mathbb{N}$, there are three cases:

(i) $W = \{z_0\}$,

(ii) W is non-degenerate and z_0 lies in the interior of W,

(iii) W is non-degenerate and $z_0 \in \partial W \cap D$.

In case (i), by continuity,

$$\begin{aligned} |h_n(z_0) - u(z_0)| &\leq \int_{\partial W_n} |u(y) - u(z_0)| \, d\mu_{z_0, W_n}(y) \\ &\leq \sup_{y \in \partial W_n} |u(y) - u(z_0)| \; \to 0 \text{ as } n \to \infty \,. \end{aligned}$$

Hence,

$$v(z_0) = u(z_0) = h(z_0) \,,$$

where h is the harmonic extension of u on the ball $B(z_0, \rho(z_0))$.

For case (ii), in the setting of Theorems 2.1.1 and 2.1.3, the hypothesis of Lemma 2.3.7 are satisfied. The same is also true in the setting of Theorem 2.1.2, as by the condition (2.4.1), all the domains W_n, W are contained in an open domain, say B such that $\overline{B} \subset D$. Then since u is continuous in $\overline{B}$, the hypotheses of Lemma 2.3.7 applied to the function u on the domain B are satisfied.

So now applying Lemma 2.3.7, we get

$$h_n(z_0) \to h(z_0) \,,$$

where h is the harmonic extension of u on the limiting domain W (i.e. the Poisson integral of the L^∞ function u on ∂W).

Finally for case (iii), we note that in the setting of Theorems 2.1.1, 2.1.2, W is a ball in D and in the setting of Theorem 2.1.3, W is either a ball or a horoball. Then by continuity, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|u(y) - u(z_0)| < \varepsilon/2\,, \text{ for all } y \in B(z_0, \delta) \subset D.$$

In the setting of Theorems 2.1.1, 2.1.3, set $C := \sup_D |u|$ and in the setting of Theorem 2.1.2, set $C := \sup_B |u|$, where B is a domain containing all W_n and W such that $\overline{B} \subset D$. Then by Lemma 2.3.1, we get for n large,

$$\begin{aligned} |h_n(z_0) - u(z_0)| &\leq \int_{\partial W_n \cap B(z_0,\delta)} |u(y) - u(z_0)|\, d\mu_{z_0,W_n}(y) \\ &+ \int_{\partial W_n \setminus B(z_0,\delta)} |u(y) - u(z_0)|\, d\mu_{z_0,W_n}(y) \\ &< \frac{\varepsilon}{2} + 2C\, \mu_{z_0,W_n}(\partial W_n \setminus B(z_0,\delta)) \\ &< \varepsilon\,. \end{aligned}$$

Hence,

$$v(z_0) = u(z_0) = h(z_0)\,,$$

where h is the harmonic extension of u on the ball $B(z_0, \rho(z_0))$. □

Lemma 2.4.2 *v is continuous and subharmonic in D.*

Proof: First we show that v is continuous. Choose and fix $z_0 \in D$. By Lemma 2.4.1 there exists $h \in \mathscr{F}_{z_0}$ such that $v(z_0) = h(z_0)$. Let W be the domain such that h is the harmonic extension of u on W (i.e. the Poisson integral of the L^∞ function u on ∂W). Note that $h \in \mathcal{F}_z$ for all $z \in W$. Then by definition of v (see (2.4.2)), we have

$$v(z) \geq h(z) \quad \text{for all } z \in W. \tag{2.4.4}$$

Hence by continuity of h,

$$\liminf_{z \to z_0} v(z) \geq \liminf_{z \to z_0} h(z) = h(z_0) = v(z_0)\,. \tag{2.4.5}$$

Now for the limsup, we consider $\{z_n\}_{n=1}^{\infty} \subset D$ such that $z_n \to z_0$ as $n \to \infty$. Let $h_n \in \mathscr{F}_{z_n}$ be the harmonic extensions of u on the domains W_n (i.e. the Poisson integrals of the L^∞ function

u on ∂W_n) such that $v(z_n) = h_n(z_n)$. Passing to a subsequence, we may assume that $\overline{W_n} \to \overline{W}$ and $\partial W_n \to \partial W$ (for some domain W such that $W \subset D$) in the Hausdorff topology on compacts in $\overline{D}$. Again as in the proof of Lemma 2.4.1, we consider three cases.

If $W = \{z_0\}$, then as in the proof of Lemma 2.4.1, continuity of u gives

$$v(z_n) = h_n(z_n) \to u(z_0) \leq v(z_0)\,.$$

If W is non-degenerate and z_0 lies in the interior of W, then for the harmonic extension of u on W, say h (i.e. the Poisson integral of the L^∞ function u on ∂W), we get, by Lemma 2.3.7,

$$v(z_n) = h_n(z_n) \to h(z_0) \leq v(z_0)\,.$$

In the third case, when W is non-degenerate and $z_0 \in \partial W \cap D$, then the same argument as in the proof of Lemma 2.4.1 using concentration of measure gives

$$v(z_n) = h_n(z_n) \to u(z_0) \leq v(z_0)\,.$$

Combining all the cases, we get

$$\limsup_{z \to z_0} v(z) \leq v(z_0)\,. \tag{2.4.6}$$

From (2.4.5) and (2.4.6), it follows that v is continuous at z_0. As z_0 was arbitrarily chosen, v is continuous in D.

To see that v is subharmonic in D, again choose and fix $z_0 \in D$. Let $h \in \mathscr{F}_{z_0}$ be the harmonic extension of u on a domain W (i.e. the Poisson integral of the L^∞ function u on ∂W) such that $v(z_0) = h(z_0)$. Then for all balls B compactly contained in W, by (2.4.4) we have

$$v(z_0) = h(z_0) = \int_{\partial B} h(y)\, d\mu_{z_0,B}(y) \leq \int_{\partial B} v(y)\, d\mu_{z_0,B}(y)\,.$$

Since subharmonicity is a local property, it follows that v is subharmonic in D. □

Our next Lemma shows that if u has nice boundary behaviour then so does v.

Lemma 2.4.3 *Under the hypothesis of Theorem 2.1.1 or Theorem 2.1.3, if $\xi \in \partial D$ such that*

$$\lim_{\substack{z \to \xi \\ z \in D}} u(z) = u_\xi\,, \textit{ then } \lim_{\substack{z \to \xi \\ z \in D}} v(z) = u_\xi\,.$$

Proof: Let $\xi \in \partial D$ such that u has the mentioned boundary behaviour at ξ. Let $\{z_n\}_{n=1}^{\infty} \subset D$ such that $z_n \to \xi$ as $n \to \infty$. We will show that

$$v(z_n) \to u_\xi \text{ as } n \to \infty\,. \tag{2.4.7}$$

This is seen as follows. By Lemma 2.4.1, there exist domains W_n such that $W_n \subset D$ and the harmonic extensions $h_n \in \mathscr{F}_{z_n}$ of u on W_n (i.e. the Poisson integrals of the L^∞ function u on ∂W_n) satisfy $v(z_n) = h_n(z_n)$. We note that to show (2.4.7), it is enough to show that each subsequence of $\{v(z_n)\}_{n=1}^{\infty}$ has a further subsequence which converges to u_ξ . So let $\{v(z_{n_k})\}_{k=1}^{\infty}$ be a given subsequence, which after relabelling we write as $\{v(z_n)\}_{n=1}^{\infty}$.

Now in the setting of Theorem 2.1.1, there are two cases:

(i) diameters of W_n converge to 0,

(ii) diameters of W_n do not converge to 0.

In case (i), the domains W_n are balls and $\overline{W_n} \to \{\xi\}$ in the Hausdorff topology on compacts in $\overline{D}$. Now as the radii of these balls are converging to 0, the second fundamental forms of the ∂W_n's are going to $+\infty$, whereas the second fundamental form of ∂D is bounded, hence for n large, ∂W_n can intersect ∂D at most at one point (which has zero harmonic measure). Then by the boundary behaviour of u at ξ, we have

$$\begin{aligned} |v(z_n) - u_\xi| &\leq \int_{\partial W_n} |u(y) - u_\xi|\, d\mu_{z_n,W_n}(y) \\ &\leq \sup_{y \in D \cap \overline{B(\xi,\epsilon_n)}} |u(y) - u_\xi| \to 0 \text{ as } n \to \infty\,. \end{aligned}$$

(where $\epsilon_n = d(z_n, \xi) + diam(W_n) \to 0$ as $n \to \infty$).

In case (ii) by passing to a subsequence, we may assume that $\overline{W_n} \to \overline{W}$ and $\partial W_n \to \partial W$ (for some non-degenerate domain W such that $W \subset D$) in the Hausdorff topology on compacts in $\overline{D}$. Then using the boundary behaviour of u at ξ, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|u(y) - u_\xi| < \varepsilon/2\,, \text{ for all } y \in D \cap B(\xi, \delta)\,. \tag{2.4.8}$$

Set $C := \sup_D |u|$. Now if for all n large,

$$\mu_{z_n,W_n}(\partial W_n \cap \partial D) = 0\,, \tag{2.4.9}$$

then for n large we have, by (2.4.8) and Lemma 2.3.1,

$$\begin{aligned} |v(z_n) - u_\xi| &= \left| \int_{\partial W_n \cap D} (u(y) - u_\xi)\ d\mu_{z_n, W_n}(y) \right| \\ &\leq \int_{\partial W_n \setminus B(\xi,\delta)} |u(y) - u_\xi|\, d\mu_{z_n, W_n}(y) \\ &+ \int_{\partial W_n \cap B(\xi,\delta) \cap D} |u(y) - u_\xi|\, d\mu_{z_n, W_n}(y) \\ &\leq 2C\mu_{z_n, W_n}(\partial W_n \setminus B(\xi,\delta)) + \frac{\varepsilon}{2} \\ &< \varepsilon\,. \end{aligned} \tag{2.4.10}$$

On the other hand if (2.4.9) is not true for W_n, then we can consider balls V_n such that

- $z_n \in V_n$,
- $\partial V_n \cap \partial D = \emptyset$,
- the Hausdorff distance between the closures $\overline{V_n}$ and $\overline{W_n}$, as well as the boundaries ∂V_n and ∂W_n are sufficiently small so that for $\tilde{h}_n$, the harmonic extensions of u on V_n, one has by Lemma 2.3.7,
$$|\tilde{h}_n(z_n) - h_n(z_n)| < \frac{1}{n}\,, \tag{2.4.11}$$
- $\overline{V_n} \to \overline{W}$ and $\partial V_n \to \partial W$ in the Hausdorff topology on compacts in $\overline{D}$.

Then proceeding as in (2.4.10) for the domains V_n, we get for n large enough,

$$|\tilde{h}_n(z_n) - u_\xi| < \varepsilon\,. \tag{2.4.12}$$

Finally, combining (2.4.11) and (2.4.12) we have for n large,

$$|v(z_n) - u_\xi| \leq |h_n(z_n) - \tilde{h}_n(z_n)| + |\tilde{h}_n(z_n) - u_\xi| < 2\varepsilon\,.$$

This completes the proof of (2.4.7) in the setting of Theorem 2.1.1.

Then in the setting of Theorem 2.1.3, there are again two cases:

(i) diameters of W_n are bounded,

(ii) diameters of W_n are unbounded.

In case (i), the domains W_n are balls and as their diameters are uniformly bounded it follows that $\overline{W_n} \to \{\xi\}$ in the cone topology. Then using the boundary behaviour of u at ξ and proceeding as in case (i) in the scenario of bounded domains, we get the result.

Similarly, for case (ii), using the boundary behaviour of u at ξ, Lemma 2.3.2, Lemma 2.3.7 and proceeding as in case (ii) in the scenario of bounded domains, we get the result. $\square$

Remark 2.4.4 One may also define $w(z) := \inf_{h \in \mathscr{F}_z} h(z)$, for all $z \in D$. Proceeding similarly as above it can be shown that

- for all $z \in D$, there exists $h \in \mathscr{F}_z$ such that $w(z) = h(z)$,
- w is continuous, superharmonic in D satisfying $w(z) \leq u(z)$, for all $z \in D$ and
- under the hypothesis of Theorem 2.1.1 or Theorem 2.1.3, if $\xi \in \partial D$ such that

$$\lim_{\substack{z \to \xi \\ z \in D}} u(z) = u_\xi \text{ , then } \lim_{\substack{z \to \xi \\ z \in D}} w(z) = u_\xi \; .$$

This w will be useful for us.

2.5 Some L^∞ maximum principles

In this section we prove some useful maximum principles. The first one is for pre-compact domains D.

Lemma 2.5.1 *Let φ be a bounded, continuous, real-valued subharmonic function on D such that*

$$\lim_{\substack{z \to \xi \\ z \in D}} \varphi(z) = f(\xi) \; ,$$

for almost every boundary point ξ, with respect to the harmonic measures on ∂D, where $f \in L^\infty(\partial D)$. Then

$$\varphi(x) \leq P[f](x) \; ,$$

for all $x \in D$, where $P[f]$ is the Poisson integral of f.

We also have an analogue of Lemma 2.5.1 for manifolds with pinched negative curvature.

Lemma 2.5.2 *Let φ be a bounded, continuous, real-valued subharmonic function on M (where M is as in Theorem 2.1.3) such that*

$$\lim_{\substack{z \to \xi \\ z \in M}} \varphi(z) = f(\xi)\,, \tag{2.5.1}$$

for almost every $\xi \in \partial M$, with respect to the harmonic measures on ∂M, where $f \in L^\infty(\partial M)$. Then

$$\varphi(x) \leq P[f](x)\,, \tag{2.5.2}$$

for all $x \in M$, where $P[f]$ is the Poisson integral of f.

The proofs of Lemmata 2.5.1 and 2.5.2 are similar. We only give the proof of Lemma 2.5.2.

Proof: [Proof of Lemma 2.5.2] Let $x \in M$, and for $R > 0$ let τ_R denote the first exit time of Brownian motion started at x from the ball $B(x, R)$. Since Brownian paths are continuous, it follows that

$$\tau_R \to \infty \text{ almost surely as } R \to \infty.$$

Recalling Sullivan's result ([Su83]) that almost surely the Brownian motion $(B_t)_{t \geq 0}$ converges to a point B_∞ on the boundary ∂M, it follows that $B_{\tau_R} \to B_\infty$ almost surely as $R \to \infty$.

Let $A \subset \partial M$ be a set of full harmonic measure, $\mu_x(A) = 1$, such that $\varphi(z) \to f(\xi)$ as $z \to \xi$ for all $\xi \in A$. Since the harmonic measure μ_x on ∂M is the distribution of the random variable B_∞, it follows that $B_\infty \in A$ almost surely. Hence almost surely we have $\varphi(B_{\tau_R}) \to f(B_\infty)$ as $R \to \infty$.

Since φ is bounded, the random variables $\{\varphi(B_{\tau_R})\}_{R>0}$ are uniformly bounded, thus by the Dominated convergence Theorem we have

$$\mathbb{E}(\varphi(B_{\tau_R})) \to \mathbb{E}(f(B_\infty)) \text{ as } R \to \infty. \tag{2.5.3}$$

On the other hand, since φ is subharmonic in M, we have the sub-mean value property for geodesic spheres, hence

$$\varphi(x) \leq \mathbb{E}(\varphi(B_{\tau_R}))\,, \text{ for all } R > 0\,. \tag{2.5.4}$$

It follows from (2.5.3) and (2.5.4) that

$$\varphi(x) \leq \mathbb{E}(f(B_\infty)) = \int_{\partial M} f(\xi) d\mu_o(\xi) = P[f](x)$$

as required. $\square$

2.6 Proofs of the main results

For the reader's convenience, let us recall the the two auxiliary functions constructed in Section 2.4: the subharmonic function v and the superharmonic function w (defined in (2.4.2) and Remark 2.4.4 respectively).

Proof: [Proof of Theorem 2.1.1] By Lemma 2.4.3,

$$\lim_{\substack{z \to \xi \\ z \in D}} v(z) = u_\xi \,, \text{ for almost every } \xi \in \partial D \,.$$

By remark 2.4.4 we also have,

$$\lim_{\substack{z \to \xi \\ z \in D}} w(z) = u_\xi \,, \text{ for almost every } \xi \in \partial D \,.$$

Let $\varphi := v - w$. Then as by construction, $w \leq u \leq v$ and by the maximum principle and boundedness of u, both v and w are bounded, it follows that φ is a bounded, non-negative, continuous, subharmonic function in D such that

$$\lim_{\substack{z \to \xi \\ z \in D}} \varphi(z) = 0 \,, \text{ for almost every } \xi \in \partial D \,.$$

Then by Lemma 2.5.1,

$$\varphi \leq 0 \text{ on } D \,.$$

So φ vanishes identically on D. Therefore u, v and w are identically equal to each other on D. Hence u is both subharmonic and superharmonic, that is, harmonic in D. $\square$

Proof: [Proof of Theorem 2.1.2] Let $R \in (\tau, 1)$ and let $z \in D$ be such that

$$d(z, \partial D) \leq (1-\kappa)(1-R)r_D \,.$$

Then by Lemma 2.4.1, there exists a ball $B(x,r) \subset D$ such that the harmonic extension $h \in \mathscr{F}_z$ of u on $B(x,r)$ satisfies $v(z) = h(z)$. Moreover by (2.4.1),

$$\begin{aligned} d(x, \partial D) &\leq d(z, \partial D) + d(z,x) \\ &\leq (1-\kappa)(1-R)r_D + \rho_0(x) \\ &\leq (1-\kappa)(1-R)r_D + \kappa\, d(x, \partial D) \end{aligned}$$

Then an elementary computation along with the fact that $\kappa \in (0,1)$ implies that

$$d(x, \partial D) \leq (1-R)r_D \,. \tag{2.6.1}$$

Now as $R \in (\tau, 1)$, from above it follows that

$$d(x, \partial D) \leq (1-R)r_D < (1-\tau)r_D \,,$$

and hence by (2.4.1) we also have

$$d(z,x) < r \leq \rho_0(x) \leq \delta(r(x), \varepsilon(d(x, \partial D)); u) \,. \tag{2.6.2}$$

Then using (2.6.1) and (2.6.2) we get,

$$\begin{aligned} 0 &\leq v(z) - u(z) \\ &\leq |h(z) - u(x)| + |u(x) - u(z)| \\ &\leq \int_{\partial B(x,r)} |u(y) - u(x)|\, d\mu_{z, B(x,r)}(y) + |u(x) - u(z)| \\ &\leq \sup_{y \in \partial B(x,r)} |u(y) - u(x)| + |u(x) - u(z)| \\ &\leq 2\, \varepsilon(d(x, \partial D)) \\ &\leq 2 \sup_{t \in (0, (1-R)r_D]} \varepsilon(t) \ \to 0 \,, \ \text{as } R \to 1 \,. \end{aligned}$$

Thus

$$(v-u)(z) \to 0 \text{ uniformly as } d(z, \partial D) \to 0 \,.$$

Using remark 2.4.4 and proceeding as above we also have,

$$(u - w)(z) \to 0 \text{ uniformly as } d(z, \partial D) \to 0\,.$$

Thus $\varphi := v - w$ is a non-negative, continuous subharmonic function in D such that

$$\varphi(z) \to 0 \text{ uniformly as } d(z, \partial D) \to 0\,.$$

This implies that φ is also bounded in D. Then by Lemma 2.5.1,

$$\varphi \le 0 \text{ on } D\,.$$

The remainder of the proof is as in the final paragraph of the proof of Theorem 2.1.1. □

Proof: [Proof of Theorem 2.1.3] By Lemma 2.4.3,

$$\lim_{\substack{z \to \xi \\ z \in M}} v(z) = u_\xi \text{ for almost every } \xi \in \partial M\,.$$

Then using remark 2.4.4 and proceeding as in the proof of Theorem 2.1.1, we get that $\varphi := v - w$, is a bounded, non-negative, continuous, subharmonic function in M such that

$$\lim_{\substack{z \to \xi \\ z \in M}} \varphi(z) = 0 \text{ for almost every } \xi \in \partial M\,.$$

Then by Lemma 2.5.2,

$$\varphi \le 0 \text{ on } M\,.$$

The remainder of the proof is as in the final paragraph of the proof of Theorem 2.1.1. □

Chapter 3

Mapping properties of the Schrödinger maximal function on Damek-Ricci spaces for radial initial data

3.1 Introduction

In this chapter, we first obtain the complete description of the pairs $(q, \beta) \in [1, \infty] \times [0, \infty)$ for which one has the following boundedness estimate for any $R > 0$, and every radial L^2-Schwartz class function f on Damek-Ricci spaces S:

$$\|S^* f\|_{L^q(B_R)} \lesssim_R \|f\|_{H^\beta(S)} , \tag{3.1.1}$$

where B_R is the geodesic ball on S with center e and radius R. The following is the exact analogue of Sjolin's result on $\mathbb{R}^n$ (see (1.2.6)) for S:

Theorem 3.1.1 *(i) If $\beta < 1/4$, then (3.1.1) holds for no $q \in [1, \infty]$.*

(ii) If $1/4 \leq \beta < n/2$, then (3.1.1) holds if and only if $1 \leq q \leq 2n/(n - 2\beta)$.

(iii) If $\beta = n/2$, then (3.1.1) holds if and only if $1 \leq q < \infty$.

(iv) If $\beta > n/2$, then (3.1.1) holds for all $q \in [1, \infty]$.

In the case of $\mathbb{R}^n$, Sjölin's proof of [Sj97, Theorem 3] hinges heavily on the small and large argument asymptotics of the Bessel functions and the oscillation afforded by them and the multiplier corresponding to the Schrödinger semigroup. Unlike in the Euclidean case, in the general setting of Damek-Ricci spaces however, no explicit expression of the spherical functions is known. In fact, it only admits certain series expansions (the Bessel series expansion and a series expansion similar to the classical Harish-Chandra series expansion) depending on the geodesic distance from the identity of the group. This motivates us to first impose suitable restrictions on both the frequency of the Spherical Fourier transform of the initial condition as well as on the spatial variable and then to study the boundedness of the linearized maximal function for each such piece separately. Then Theorem 3.1.1 is eventually obtained by means of a cumulative synthesis of these simpler components, which in a way, is much more illuminating.

More precisely, in order to study the sufficient conditions of Theorem 3.1.1, we first consider the cases when the Spherical Fourier transform of the initial condition is supported in a neighborhood of the origin and when it is supported away from the origin. In the first case, the linearized maximal function is exceptionally well-behaved and is in fact quite straightforward to control. The latter case however, is quite involved. In the large frequency regime, we decompose the geodesic ball B_R into a closed ball and an open annulus,

$$B_R = \overline{B}_{R_0} \sqcup A(R_0, R)\,, \quad \text{for some } R_0 > 0\,.$$

While estimating on the ball $\overline{B}_{R_0}$, we expand the spherical function in the Bessel series (see Lemma 3.2.2) and decompose the linearized maximal function accordingly. By repeated applications of the Abel transform, followed by introducing a new Schwartz multiplier corresponding to the ratio of the weights of the Plancherel measures on $\mathbb{R}^n$ and S, we are able to draw an Euclidean connection to bound the first piece. The second piece is then taken care of by pointwise estimates of certain error terms. This essentially builds up on the primitive ideas employed for $\mathbb{H}^3$ in [De25].

On the other hand, for the annulus $A(R_0, R)$, there is no Bessel series expansion at our disposal and hence the Euclidean connection mentioned above ceases to exist. Instead, we work with a variation of the classical Harish-Chandra series expansion (see Lemma 3.2.3) which has been recently obtained by Anker-Pierfelice-Vallarino [APV15]. Although this series expansion looks similar to the classical Harish-Chandra series, it has the rather remarkable property of being constituted by coefficients having decay in the spectral parameter λ (see (3.2.8)), compared to the mere boundedness in λ in the Harish-Chandra series. This extra decay makes our life easy in the higher regularity regime ($\beta > 1/2$). But in the lower regularity scenario (when $1/4 \leq \beta \leq 1/2$), the above mentioned pointwise decay alone is apparently not sufficient. The crux of the matter here is to invoke the oscillation afforded by φ_λ and the Schrödinger multiplier. The oscillation of φ_λ is realized by looking at the leading terms in the series expansion. The corresponding oscillatory parts of the linearized maximal function are taken care of by proceeding with an abstract TT^* argument, followed by an oscillatory integral estimate and then finally applying the Pitt's inequality. The remaining part of the linearized maximal function is then dealt with by the pointwise decay of certain error terms appearing from the series expansion due to Anker et. al., which turns out to be of critical importance.

Our study of the sufficient conditions for (3.1.1) gives rise to two critical endpoints $\beta = 1/4$ and $n/2$, on either sides of which the behavior of the local Schrödinger maximal function changes. This is where our 'piecewise' approach helps us to identify that counter-examples can only occur when the Spherical Fourier transform of the initial data is supported away from the origin. In each of the three counter-examples, we weave several ideas explored in the study of the sufficient conditions for different pieces of the linearized maximal function. In the process, we also obtain the failure of the delicate local borderline fractional Sobolev embedding, which can be viewed as a refinement of the corresponding global borderline fractional Sobolev embedding obtained by Cowling-Giulini-Meda [CGM93, Theorem 4.7] (see remark 3.5.1) in the special case of rank one Riemannian symmetric spaces of non-compact type.

Remark 3.1.2

1. Theorem 3.1.1 asserts the sharpness of $\beta \geq 1/4$ for the problem of pointwise convergence of the solution of the Schrödinger equation to its radial initial data almost everywhere, considered in [De25] and thus generalizing [De25, Theorem 1.4] to all Damek-Ricci spaces.

2. In the degenerate case of the Real hyperbolic spaces, the arguments presented in this chapter can be carried out verbatim to obtain the analogue of Theorem 3.1.1.

3. In the definition of the maximal function (1.2.4) the time has been localized in the interval $(0, 4/Q^2)$. It has been done as such, only to estimate some oscillatory integral (Lemma 3.3.2) more conveniently and in fact, it is easy to see that our results (Theorem 3.1.1) continue to hold true when the time is instead localized in $(0, c)$ for any $c > 0$.

We now turn to global maximal estimates. On $\mathbb{R}^n$, an interesting result in this direction is the following transference principle obtained by Rogers [Ro08, Theorem 3]: the local estimate

$$\left\| \sup_{0<t<1} \left| e^{it\Delta_{\mathbb{R}^n}} f \right| \right\|_{L^2(B(o,1))} \lesssim \|f\|_{H^\beta(\mathbb{R}^n)} ,$$

holds for $\beta > \beta_0$ if and only if the global estimate

$$\left\| \sup_{0<t<1} \left| e^{it\Delta_{\mathbb{R}^n}} f \right| \right\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{H^\beta(\mathbb{R}^n)} ,$$

holds for $\beta > 2\beta_0$. Then in view of Sjolin's result (see (1.2.6)), one may expect the following global estimate to hold true for radial initial data on $\mathbb{R}^n$:

$$\left\| \sup_{0<t<1} \left| e^{it\Delta_{\mathbb{R}^n}} f \right| \right\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{H^\beta(\mathbb{R}^n)} , \text{ for } \beta > 1/2 . \tag{3.1.2}$$

This was indeed proved by Sjölin [Sj97, Theorem 4] and moreover it was also proved that (3.1.2) fails for any $q < 2$. We have an analogue of this result for the special case of $\mathbb{H}^3 \cong SL(2,\mathbb{C})/SU(2)$:

Theorem 3.1.3 *For radial initial data f belonging to the L^2-Schwartz class on $\mathbb{H}^3$, the global estimate*

$$\|S^* f\|_{L^2(\mathbb{H}^3)} \lesssim \|f\|_{H^\beta(\mathbb{H}^3)} , \tag{3.1.3}$$

holds true for $\beta > 1/2$ and moreover, the estimate (3.1.3) fails for $q < 2$.

It would be interesting to see if it is possible to generalize Theorem 3.1.3 to the setting of Damek-Ricci spaces. For the time being however, we only have the following weak boundedness result:

Theorem 3.1.4 *For radial initial data f belonging to the L^2-Schwartz class on a Damek-Ricci space S, the global estimate*

$$\|S^* f\|_{L^{2,\infty}(S)} \lesssim \|f\|_{H^\beta(S)} , \tag{3.1.4}$$

holds true for $\beta > 1/2$.

This chapter is organized as follows. In section 3.2, we recall certain aspects of Euclidean Fourier Analysis and the essential preliminaries about Damek-Ricci spaces and Spherical Fourier Analysis thereon. In section 3.3, we prove two auxiliary lemmata. The sufficient and necessary conditions of Theorem 3.1.1 are proved in sections 3.4 and 3.5 respectively. In section 3.6, Theorems 3.1.3 and Theorem 3.1.4 are proved.

3.2 Preliminaries

In this section, we recall the relevant preliminaries and fix our notations.

3.2.1 Fourier Analysis on $\mathbb{R}^n$:

In this subsection, we recall some standard results from Euclidean Fourier Analysis, most of which can be found in [Gr09, SW90]. On $\mathbb{R}$, for "nice" functions f, the Fourier transform $\widetilde{f}$ is defined as

$$\widetilde{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-ix\xi}\, dx\,.$$

An important inequality in one-dimensional Fourier Analysis is the Pitt's inequality:

Lemma 3.2.1 *[St56, p. 489] For $\beta \geq 0$ and $p \in (1,2]$, one has the inequality*

$$\left(\int_{\mathbb{R}} \left|\widetilde{f}(\xi)\right|^2 |\xi|^{-2\beta} d\xi\right)^{1/2} \lesssim \left(\int_{\mathbb{R}} |f(x)|^p\, |x|^{\beta_1 p} dx\right)^{1/p},$$

where

$$\beta_1 = \beta + \frac{1}{2} - \frac{1}{p}, \qquad 0 \leq \beta_1 < 1 - \frac{1}{p}.$$

In the following,

$$\mathscr{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) \mid \sup_{x\in\mathbb{R}} (1+|x|)^N \left| \left(\frac{d}{dx}\right)^M f(x) \right| < \infty, M, N \in \mathbb{N}\cup\{0\} \right\},$$

will denote the Schwartz class functions on $\mathbb{R}$ and $\mathscr{S}(\mathbb{R})_e$ will denote the even functions in $\mathscr{S}(\mathbb{R})$.

For $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$ the Riesz potential of order α is the operator

$$I_\alpha = (-\Delta)^{-\alpha/2}\,,$$

where $\Delta = \frac{d^2}{dx^2}$, is the Laplacian on $\mathbb{R}$. I_α can be explicitly written as,

$$I_\alpha(f)(x) = C_\alpha \int_{\mathbb{R}} f(y)\, |x-y|^{\alpha-1}\, dy\,,$$

for some $C_\alpha > 0$, whenever $f \in \mathscr{S}(\mathbb{R})$. For $\alpha > 0$, the Fourier transform of the Riesz potential of $f \in \mathscr{S}(\mathbb{R})$ satisfies the following identity:

$$(I_\alpha(f))^{\sim}(\xi) = C_\alpha |\xi|^{-\alpha} \widetilde{f}(\xi)\,. \tag{3.2.1}$$

For a radial function f in $\mathbb{R}^n$, $n \geq 2$, the Euclidean spherical Fourier transform is defined as

$$\mathscr{F} f(\lambda) := \int_0^\infty f(r)\, \mathscr{J}_{\frac{n-2}{2}}(\lambda r)\, r^{n-1}\, dr\,, \qquad \lambda \in [0,\infty),$$

where for all $\mu \geq 0$,

$$\mathscr{J}_\mu(z) = 2^\mu\, \pi^{1/2}\, \Gamma\left(\mu + \frac{1}{2}\right) \frac{J_\mu(z)}{z^\mu},$$

and J_μ are the Bessel functions [SW90, p. 154].

If $C^\infty(\mathbb{R}^n)_o$ denotes the class of smooth radial functions on $\mathbb{R}^n$ then using polar coordinates, the class of radial Schwartz class functions on $\mathbb{R}^n$ can be defined as

$$\mathscr{S}(\mathbb{R}^n)_o = \left\{ f \in C^\infty(\mathbb{R}^n)_o \mid \sup_{r>0} (1+r)^N \left| \left(\frac{d}{dr}\right)^M f(r) \right| < \infty, M, N \in \mathbb{N} \cup \{0\} \right\},$$

where $r = \|x\|$, is the distance of $x \in \mathbb{R}^n$ from the origin. It is known that

$$\mathscr{F} : \mathscr{S}(\mathbb{R}^n)_o \to \mathscr{S}(\mathbb{R})_e,$$

defines a topological isomorphism.

For $f \in \mathscr{S}(\mathbb{R}^n)_o$, using the Fourier transform, the solution to the Schrödinger equation:

$$\begin{cases} i\frac{\partial u}{\partial t} = \Delta u\,,\ (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(0,\cdot) = f\,,\ \text{on } \mathbb{R}^n\,, \end{cases}$$

is given by,

$$u(\|x\|,t) = \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda \|x\|)\, e^{it\lambda^2}\, \mathscr{F} f(\lambda)\, \lambda^{n-1} d\lambda\,.$$

3.2.2 Spherical Fourier Analysis on Damek-Ricci spaces:

In this section, we will explain the notations and state relevant results from Spherical Fourier analysis on Damek-Ricci spaces. Most of these results can be found in [ADY96, APV15, As95].

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra equipped with an inner product $\langle , \rangle$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement. We say that $\mathfrak{n}$ is an H-type algebra if for every $Z \in \mathfrak{z}$ the map $J_Z : \mathfrak{v} \to \mathfrak{v}$ defined by

$$\langle J_z X, Y\rangle = \langle [X,Y], Z\rangle, \quad X, Y \in \mathfrak{v}$$

satisfies the condition $J_Z^2 = -|Z|^2 I_{\mathfrak{v}}$, $I_{\mathfrak{v}}$ being the identity operator on $\mathfrak{v}$. A connected and simply connected Lie group N is called an H-type group if its Lie algebra is H-type. Since $\mathfrak{n}$ is nilpotent, the exponential map is a diffeomorphism and hence we can parametrize the elements in $N = \exp \mathfrak{n}$ by (X, Z), for $X \in \mathfrak{v}, Z \in \mathfrak{z}$. It follows from the Baker-Campbell-Hausdorff formula that the group law in N is given by

$$(X, Z)\left(X', Z'\right) = \left(X + X', Z + Z' + \frac{1}{2}[X, X']\right), \quad X, X' \in \mathfrak{v};\ Z, Z' \in \mathfrak{z}.$$

The group $A = \mathbb{R}^+$ acts on an H-type group N by nonisotropic dilation: $(X, Z) \mapsto (\sqrt{a}X, aZ)$. Let $S = NA$ be the semidirect product of N and A under the above action. Thus the multiplication in S is given by

$$(X, Z, a)\left(X', Z', a'\right) = \left(X + \sqrt{a}X', Z + aZ' + \frac{\sqrt{a}}{2}[X, X'], aa'\right),$$

for $X, X' \in \mathfrak{v};\ Z, Z' \in \mathfrak{z}; a, a' \in \mathbb{R}^+$. Then S is a solvable, connected and simply connected Lie group having Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$ with Lie bracket

$$\left[(X, Z, l), (X', Z', l')\right] = \left(\frac{1}{2}lX' - \frac{1}{2}l'X, lZ' - lZ + [X, X'], 0\right).$$

We write $na = (X, Z, a)$ for the element $\exp(X + Z)a, X \in \mathfrak{v}, Z \in \mathfrak{z}, a \in A$. We note that for any $Z \in \mathfrak{z}$ with

$$|Z| = 1, \quad J_Z^2 = -I_{\mathfrak{v}}.$$

So, J_Z defines a complex structure on $\mathfrak{v}$ and hence $\mathfrak{v}$ is even dimensional. $m_{\mathfrak{v}}$ and m_z will denote the dimension of $\mathfrak{v}$ and $\mathfrak{z}$ respectively. Let n and Q denote dimension and the homogenous dimension of S respectively:

$$n = m_{\mathfrak{v}} + m_{\mathfrak{z}} + 1 \quad \textit{and} \quad Q = \frac{m_{\mathfrak{v}}}{2} + m_{\mathfrak{z}}.$$

The group S is equipped with the left-invariant Riemannian metric induced by

$$\langle (X, Z, l), (X', Z', l') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + ll' \tag{3.2.2}$$

on $\mathfrak{s}$. The corresponding inner metric is denoted by $d(\cdot, \cdot)$. For $x \in S$, we denote by $s = d(e, x)$, the geodesic distance of x from the identity e. Then the left Haar measure dx of the group S may be normalized so that

$$dx = A(s)\, ds\, d\sigma(\omega)\ ,$$

where the density function A is given by [ADY96, Eqn. (1.16), p. 647],

$$A(s) = 2^{m_{\mathfrak{v}}+m_{\mathfrak{z}}} \left(\sinh(s/2)\right)^{m_{\mathfrak{v}}+m_{\mathfrak{z}}} \left(\cosh(s/2)\right)^{m_{\mathfrak{z}}}\ ,$$

and $d\sigma$ is the surface measure of the unit sphere. Using well-known estimates of hyperbolic functions, it follows that

$$A(s) \asymp \begin{cases} s^{n-1}, & 0 < s < R < \infty\ , \\ e^{Qs}, & R < s < \infty. \end{cases} \tag{3.2.3}$$

The above expression shows that S is of exponential volume growth.

A function $f : S \to \mathbb{C}$ is said to be radial if, for all x in S, $f(x)$ depends only on the geodesic distance of x from the identity e. If f is radial, then it follows from the expression of the left Haar measure given above that

$$\int_S f(x)\ dx = \int_0^\infty f(s)\ A(s)\ ds\ .$$

We now recall the spherical functions on Damek-Ricci spaces. The spherical functions φ_λ on S, for $\lambda \in \mathbb{C}$ are the radial eigenfunctions of the Laplace-Beltrami operator Δ, satisfying the following normalization criterion

$$\begin{cases} \Delta\varphi_\lambda = -\left(\lambda^2 + \frac{Q^2}{4}\right)\varphi_\lambda \\ \varphi_\lambda(e) = 1\,. \end{cases}$$

For all $\lambda \in \mathbb{R}$ and $x \in S$, the spherical functions satisfy the relation

$$\varphi_\lambda(x) = \varphi_\lambda(s) = \varphi_{-\lambda}(s)\,.$$

Moreover, for all $\lambda \in \mathbb{R}$ and all $s \geq 0$:

$$|\varphi_\lambda(s)| \leq 1\,. \tag{3.2.4}$$

For "nice" radial functions f, defined on S, the spherical Fourier transform of f is defined by the following formula:

$$\widehat{f}(\lambda) := \int_S f(x)\varphi_\lambda(x)dx = \int_0^\infty f(s)\varphi_\lambda(s)A(s)ds, \qquad \lambda \in (0,\infty)\,.$$

For "nice" radial functions f we have the following Fourier inversion formula:

$$f(x) = C\int_0^\infty \widehat{f}(\lambda)\varphi_\lambda(x)|\mathbf{c}(\lambda)|^{-2}d\lambda\,,$$

where C depends only on $m_{\mathfrak{v}}$ and $m_{\mathfrak{z}}$. Moreover, the Spherical Fourier transform extends to an isometry from the space of radial L^2 functions on S onto $L^2\left((0,\infty), C|\mathbf{c}(\lambda)|^{-2}d\lambda\right)$. Here, $\mathbf{c}$ denotes Harish-Chandra $\mathbf{c}$-function which is given by the formula

$$\mathbf{c}(\lambda) = \frac{2^{(Q-2i\lambda)}\Gamma(2i\lambda)}{\Gamma\left(\frac{Q+2i\lambda}{2}\right)}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m_{\mathfrak{v}}+4i\lambda+2}{4}\right)}\,, \tag{3.2.5}$$

for all $\lambda \in \mathbb{R}\setminus\{0\}$. From the above formula one can deduce that for $j \in \mathbb{N}\cup\{0\}$, we have the following derivative estimates ([As95, Lemma 4.2]):

$$\left|\frac{d^j}{d\lambda^j}|\mathbf{c}(\lambda)|^{-2}\right| \lesssim_j (1+|\lambda|)^{n-1-j}\,,\ \ \lambda \in \mathbb{R}. \tag{3.2.6}$$

Next, we need to introduce the notion of Schwartz class. Since S is of exponential volume growth the requirement of L^p containment implies that the definition of Schwartz class should involve p. In this paper it will be sufficient to deal with the case $p = 2$. We therefore define

$\mathscr{S}^2(S)_o$, the class of radial L^2-Schwartz class functions on S by

$$\begin{aligned}&\mathscr{S}^2(S)_o\\ =\ &\left\{f \in C^\infty(S)_o \mid \sup_{s>0}(1+s)^N e^{\frac{Q}{2}s}\left|\left(\frac{d}{ds}\right)^M f(s)\right| < \infty, \{M,N\} \subset \mathbb{N}\cup\{0\}\right\},\end{aligned} \tag{3.2.7}$$

where $C^\infty(S)_o$ is the set of radial smooth functions on S (see [ADY96, p. 652]).

One important tool which relates the spherical Fourier transform on S with the Fourier transform on $\mathbb{R}$ is the so called Abel transform (see [ABCS, RS09]). Indeed, we have the following commutative diagram, where every map is a bijection:

$$\begin{tikzcd} \mathscr{S}^2(S)_o \arrow[r, "\mathscr{A}_{S,\mathbb{R}}"] \arrow[dr, "\wedge"'] & \mathscr{S}(\mathbb{R})_e \arrow[d, "\sim"] \\ & \mathscr{S}(\mathbb{R})_e \end{tikzcd}$$

- $\mathscr{A}_{S,\mathbb{R}}$ is the Abel transform defined from $\mathscr{S}^2(S)_o$ to $\mathscr{S}(\mathbb{R})_e$.
- $\wedge$ denotes the Spherical Fourier transform from $\mathscr{S}^2(S)_o$ to $\mathscr{S}(\mathbb{R})_e$.
- $\sim$ denotes the 1-dimensional Euclidean Fourier transform from $\mathscr{S}(\mathbb{R})_e$ to itself.

For more details regarding the Abel transform, we refer the reader to [ADY96, p. 652-653]. These have been generalized to the setting of Chébli-Trimèche Hypergroups [BX98], whose simplest case is that of radial functions on $\mathbb{R}^n$.

Our next topic is a detailed description of the elementary spherical functions φ_λ. For the purpose of this article, we will need two different series expansions of the elementary spherical functions. The first one is an expansion of φ_λ in terms of Bessel functions for points near the identity. To describe it we need to define the following normalizing constant in terms of the Gamma functions,

$$c_0 = 2^{m_{\mathfrak{z}}}\,\pi^{-1/2}\,\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\,.$$

The first series expansion mentioned above is given by the following lemma.

Lemma 3.2.2 *[As95, Theorem 3.1] There exist $R_0, 2 < R_0 < 2R_1$, such that for any $0 \leq s \leq R_0$, and any integer $M \geq 0$, and all $\lambda \in \mathbb{R}$, we have*

$$\varphi_\lambda(s) = c_0 \left(\frac{s^{n-1}}{A(s)} \right)^{1/2} \sum_{l=0}^{M} a_l(s) \mathscr{J}_{\frac{n-2}{2}+l}(\lambda s) s^{2l} + E_{M+1}(\lambda, s) \,,$$

where

$$a_0 \equiv 1 \,, \; |a_l(s)| \leq C(4R_1)^{-l} \,,$$

and the error term has the following behaviour

$$|E_{M+1}(\lambda, s)| \lesssim_M \begin{cases} s^{2(M+1)} & \text{if } |\lambda s| \leq 1 \\ s^{2(M+1)} |\lambda s|^{-\left(\frac{n-1}{2}+M+1\right)} & \text{if } |\lambda s| > 1 \,. \end{cases}$$

Moreover, for every $0 \leq s < 2$, the series

$$\varphi_\lambda(s) = c_0 \left(\frac{s^{n-1}}{A(s)} \right)^{1/2} \sum_{l=0}^{\infty} a_l(s) \mathscr{J}_{\frac{n-2}{2}+l}(\lambda s) s^{2l} \,,$$

is absolutely convergent.

For the asymptotic behaviour of φ_λ away from the identity, we have the following series expansion which is analogous to the classical Harish-Chandra series (but with better behaving coefficients):

Lemma 3.2.3 *[APV15, pp. 735-736, Lemma 1] For $\lambda \in \mathbb{R} \setminus \{0\}$, the series*

$$\varphi_\lambda(s) = 2^{-m_{\mathfrak{z}}/2} A(s)^{-1/2} \left\{ \mathbf{c}(\lambda) \sum_{\mu=0}^{\infty} \Gamma_\mu(\lambda) e^{(i\lambda-\mu)s} + \mathbf{c}(-\lambda) \sum_{\mu=0}^{\infty} \Gamma_\mu(-\lambda) e^{-(i\lambda+\mu)s} \right\} ,$$

converges uniformly on compacts not containing the group identity, where $\Gamma_0 \equiv 1$ and for $\mu \in \mathbb{N}$, one has the recursion formula,

$$(\mu^2 - 2i\mu\lambda)\Gamma_\mu(\lambda) = \sum_{j=0}^{\mu-1} \omega_{\mu-j} \Gamma_j(\lambda) \,.$$

The coefficients $\Gamma_\mu(\lambda)$ satisfy the following estimate: there exist constants $C > 0$, $d \geq 0$ such that

$$|\Gamma_\mu(\lambda)| \leq C\mu^d (1 + |\lambda|)^{-1} \,, \tag{3.2.8}$$

for all $\lambda \in \mathbb{R} \setminus \{0\}, \mu \in \mathbb{N}$. *Consequently, for* $\lambda \in \mathbb{R} \setminus \{0\}$ *and* s *belonging to compact sets in* $(0, \infty)$, *one can write* φ_λ *as*

$$\varphi_\lambda(s) = 2^{-m_\mathfrak{z}/2} A(s)^{-1/2} \left\{ \mathbf{c}(\lambda) e^{i\lambda s} + \mathbf{c}(-\lambda) e^{-i\lambda s} \right\} + E(\lambda, s) \,,$$

where

$$E(\lambda, s) = 2^{-m_\mathfrak{z}/2} A(s)^{-1/2} \left\{ \mathbf{c}(\lambda) \sum_{\mu=1}^{\infty} \Gamma_\mu(\lambda) e^{(i\lambda - \mu)s} + \mathbf{c}(-\lambda) \sum_{\mu=1}^{\infty} \Gamma_\mu(-\lambda) e^{-(i\lambda+\mu)s} \right\} \,,$$

and thus

$$|E(\lambda, s)| \lesssim A(s)^{-1/2} \left| \mathbf{c}(\lambda) \right| (1 + \lambda)^{-1} \,.$$

Remark 3.2.4 (1) As a consequence of Lemma 3.2.3, for any positive constant c, one has the following pointwise estimate of φ_λ:

$$|\varphi_\lambda(s)| \lesssim |\mathbf{c}(\lambda)| e^{-\frac{Q}{2}s}, \qquad \lambda > 0, \;\; s > c. \tag{3.2.9}$$

(2) For the degenerate case of the Real hyperbolic spaces $\mathbb{H}^n$, the relevant preliminaries and the series expansions of φ_λ can be found in [An91, ST78, AP14]. So the arguments of our main theorem can be carried out verbatim for $\mathbb{H}^n$, $n \geq 2$.

We will also have the occasion to talk about Lorentz spaces. In this regard, the space $L^{2,\infty}(S)$ is defined to be the collection of functions f defined on S such that

$$\|f\|_{L^{2,\infty}(S)} := \sup_{t>0} t \, d_f(t)^{1/2} < \infty \,, \tag{3.2.10}$$

where d_f is the distribution of f,

$$d_f(t) := |\{x \mid |f(x)| > t\}| \;.$$

We will also require the following information for $\mathbb{H}^3$. For details, we refer to [He00]. Let $G = SL(2, \mathbb{C})$ and K be its maximal compact subgroup $SU(2)$. Here,

$$A = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\} \,,$$

and Σ_+ consists of a single root α that occurs with multiplicity 2. We normalize α so that $\alpha(log\, a_t) = t$. Every $\lambda \in \mathbb{C}$ can be identified with an element in $\mathfrak{a}^*_{\mathbb{C}}$ by $\lambda = \lambda\alpha$. We see that in this identification (the half-sum of positive roots counted with multiplicity) $\rho = 1$, $Q = 2\rho = 2$ and

$$G/K \cong \mathbb{H}^3 ,$$

where $\mathbb{H}^3$ is the 3-dimensional Real Hyperbolic space with constant sectional curvature -1. Then by [He00, p. 432, Theorem 5.7] we get the following information for G/K:

- For $\lambda \in (0, \infty)$, the Spherical function φ_λ is given by,

$$\varphi_\lambda(s) = \frac{\sin(\lambda s)}{\lambda \sinh(s)} .$$

- The G-invariant Riemannian measure on G/K is $\sinh^2 s\, ds\, dk$, where ds and dk are the Lebesgue measure on $(0, \infty)$ and the Haar measure on $SU(2)$ respectively.
- Plancherel measure: $\lambda^2\, d\lambda$, where $d\lambda$ is the Lebesgue measure on $(0, \infty)$.

3.3 Two auxiliary lemmata

In this section, we prove two auxiliary lemmata. The first one is a one-one correspondence between radial Schwartz class functions on S and $\mathbb{R}^n$ whose Spherical Fourier transforms are supported away from the origin.

We start off by defining

$$\mathscr{S}(\mathbb{R})_e^\infty := \{g \in \mathscr{S}(\mathbb{R})_e \mid 0 \notin Supp(g)\}.$$

So, $\mathscr{S}(\mathbb{R})_e^\infty$ is the collection of all even Schwartz class functions on $\mathbb{R}$ which are supported outside an interval containing 0. Then we look at

$$\mathscr{S}(\mathbb{R}^n)_o^\infty := \mathscr{F}^{-1}\left(\mathscr{S}(\mathbb{R})_e^\infty\right) , \quad \text{and } \mathscr{S}^2(S)_o^\infty := \wedge^{-1}\left(\mathscr{S}(\mathbb{R})_e^\infty\right) .$$

We now present the following one-one correspondence:

Lemma 3.3.1 *For each $f \in \mathscr{S}^2(S)_o^\infty$, there exists a unique $g \in \mathscr{S}(\mathbb{R}^n)_o^\infty$ (and conversely) such that*

$$\lambda^{n-1}\mathscr{F}g(\lambda) = |\mathbf{c}(\lambda)|^{-2}\widehat{f}(\lambda)\,,$$

for all $\lambda \in (0,\infty)$.

Proof: By the properties of the Abel transform described before, we have the following commutative diagram,

$$\begin{tikzcd} \mathscr{S}^2(S)_o \arrow[r, "\mathscr{A}_{S,\mathbb{R}}"] \arrow[dr, "\wedge"'] & \mathscr{S}(\mathbb{R})_e \arrow[d, "\sim"] & \mathscr{S}(\mathbb{R}^n)_o \arrow[l, "\mathscr{A}_{\mathbb{R}^n,\mathbb{R}}"'] \arrow[dl, "\mathscr{F}"] \\ & \mathscr{S}(\mathbb{R})_e & \end{tikzcd}$$

where $\mathscr{A}_{\mathbb{R}^n,\mathbb{R}}$ is the Abel transform defined from $\mathscr{S}(\mathbb{R}^n)_o$ to $\mathscr{S}(\mathbb{R})_e$. As all the maps above are bijections, by defining

$$\mathscr{A} := \mathscr{A}^{-1}_{\mathbb{R}^n,\mathbb{R}} \circ \mathscr{A}_{S,\mathbb{R}},$$

we reduce matters to the following simplified commutative diagram:

$$\begin{tikzcd} \mathscr{S}^2(S)_o \arrow[r, "\mathscr{A}"] \arrow[dr, "\wedge"'] & \mathscr{S}(\mathbb{R}^n)_o \arrow[d, "\mathscr{F}"] \\ & \mathscr{S}(\mathbb{R})_e \end{tikzcd}$$

Then by the definition of the spaces $\mathscr{S}(\mathbb{R})_e^\infty$, $\mathscr{S}(\mathbb{R}^n)_o^\infty$ and $\mathscr{S}^2(S)_o^\infty$, the above commutative diagram descends to the following, where every arrow is again a bijection:

$$\begin{tikzcd} \mathscr{S}^2(S)_o^\infty \arrow[r, "\mathscr{A}"] \arrow[dr, "\wedge"'] & \mathscr{S}(\mathbb{R}^n)_o^\infty \arrow[d, "\mathscr{F}"] \\ & \mathscr{S}(\mathbb{R})_e^\infty \end{tikzcd}$$

Consequently, for $f \in \mathscr{S}^2(S)_o^\infty$, there exists a unique $\mathscr{A}f \in \mathscr{S}(\mathbb{R}^n)_o^\infty$ (and conversely) such that

$$\widehat{f}(\lambda) = \mathscr{F}(\mathscr{A}f)(\lambda)\,. \tag{3.3.1}$$

Next let us define a map $\mathfrak{m} : \mathscr{S}(\mathbb{R})_e^\infty \to \mathscr{S}(\mathbb{R})_e^\infty$, by the formula

$$\mathfrak{m}(\kappa)(\lambda) := \frac{|\mathbf{c}(\lambda)|^{-2}}{\lambda^{n-1}} \kappa(\lambda).$$

Because of the derivative estimates of $|\mathbf{c}(\lambda)|^{-2}$ (see (3.2.6)), the map $\mathfrak{m}$ is well defined and is a bijection with the inverse given by,

$$\mathfrak{m}^{-1}(\kappa)(\lambda) := \frac{\lambda^{n-1}}{|\mathbf{c}(\lambda)|^{-2}} \kappa(\lambda).$$

As the Euclidean Fourier transform $\mathscr{F}$ is a bijection between $\mathscr{S}(\mathbb{R}^n)_o^\infty$ and $\mathscr{S}(\mathbb{R})_e^\infty$, we get an induced map $\mathcal{M}$ obtained by conjugating $\mathfrak{m}$ with $\mathscr{F}$, which is a bijection from $\mathscr{S}(\mathbb{R}^n)_o^\infty$ onto itself:

$$\begin{array}{ccc} \mathscr{S}(\mathbb{R}^n)_o^\infty & \xrightarrow{\mathcal{M}} & \mathscr{S}(\mathbb{R}^n)_o^\infty \\ \Big\downarrow{\scriptstyle \mathscr{F}} & & \Big\downarrow{\scriptstyle \mathscr{F}} \\ \mathscr{S}(\mathbb{R})_e^\infty & \xrightarrow{\mathfrak{m}} & \mathscr{S}(\mathbb{R})_e^\infty \end{array}$$

Thus by (3.3.1), we get the unique choice $\mathcal{M}(\mathscr{A} f) \in \mathscr{S}(\mathbb{R}^n)_o^\infty$ such that

$$\mathscr{F}(\mathcal{M}(\mathscr{A} f))(\lambda) = \frac{|\mathbf{c}(\lambda)|^{-2}}{\lambda^{n-1}} \mathscr{F}(\mathscr{A} f)(\lambda) = \frac{|\mathbf{c}(\lambda)|^{-2}}{\lambda^{n-1}} \widehat{f}(\lambda)\,. \tag{3.3.2}$$

We are thus through by defining $g = \mathcal{M}(\mathscr{A} f)$. $\square$

We now turn to an oscillatory integral estimate which will be crucial for the proof of Theorem 3.1.1. Here R_0 is the positive constant as in Lemma 3.2.2.

Lemma 3.3.2 *Let $t : (0, \infty) \to (0, 4/Q^2)$ be a measurable function and $R > R_0$. Then for $s, s' \in (R_0, R)$,*

$$\left| \int_{\frac{1}{R_0}}^{\infty} \frac{e^{i\left\{\lambda(s'-s)+(t(s')-t(s))\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}}{\left(\lambda^2+\frac{Q^2}{4}\right)^{1/4}} d\lambda \right| \lesssim \left(\frac{1}{|s-s'|^{1/2}} + 1 \right) ,$$

where the implicit constant depends only on $m_{\mathfrak{v}}, m_{\mathfrak{z}}, R_0$ and R.

Proof: By [De25, Lemma 3.1], there exists a constant $B(n) > R_0$, depending only on the dimension n such that for all $s, s' \in (R_0, R)$,

$$\left| \int_{\max\left\{\frac{B(n)}{s}, \frac{B(n)}{s'}\right\}}^{\infty} \frac{e^{i\left\{\lambda(s'-s)+(t(s')-t(s))\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}}{\left(\lambda^2+\frac{Q^2}{4}\right)^{1/4}} d\lambda \right| \lesssim \frac{1}{|s-s'|^{1/2}}\,, \tag{3.3.3}$$

with the implicit constant depending only on $m_{\mathfrak{v}}, m_{\mathfrak{z}}$ and R.

Now as $s, s' \in (R_0, R)$, it follows that

$$\frac{B(n)}{R} < \max\left\{\frac{B(n)}{s}, \frac{B(n)}{s'}\right\} < \frac{B(n)}{R_0}\,.$$

Thus if $\max\left\{\frac{B(n)}{s}, \frac{B(n)}{s'}\right\} \leq \frac{1}{R_0}$, then

$$\begin{aligned}\left| \int_{\max\left\{\frac{B(n)}{s}, \frac{B(n)}{s'}\right\}}^{\frac{1}{R_0}} \frac{e^{i\left\{\lambda(s'-s)+(t(s')-t(s))\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}}{\left(\lambda^2+\frac{Q^2}{4}\right)^{1/4}} d\lambda \right| &\leq \int_{\max\left\{\frac{B(n)}{s}, \frac{B(n)}{s'}\right\}}^{\frac{1}{R_0}} \frac{d\lambda}{\lambda^{1/2}} \\ &\leq \int_{\frac{B(n)}{R}}^{\frac{1}{R_0}} \frac{d\lambda}{\lambda^{1/2}}\,. \end{aligned}\tag{3.3.4}$$

Thus it is justified to decompose the following integral as

$$\begin{aligned}&\int_{\frac{1}{R_0}}^{\infty} \frac{e^{i\left\{\lambda(s'-s)+(t(s')-t(s))\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}}{\left(\lambda^2+\frac{Q^2}{4}\right)^{1/4}} d\lambda \\ =\;& \int_{\max\left\{\frac{B(n)}{s}, \frac{B(n)}{s'}\right\}}^{\infty} \frac{e^{i\left\{\lambda(s'-s)+(t(s')-t(s))\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}}{\left(\lambda^2+\frac{Q^2}{4}\right)^{1/4}} d\lambda \\ &- \int_{\max\left\{\frac{B(n)}{s}, \frac{B(n)}{s'}\right\}}^{\frac{1}{R_0}} \frac{e^{i\left\{\lambda(s'-s)+(t(s')-t(s))\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}}{\left(\lambda^2+\frac{Q^2}{4}\right)^{1/4}} d\lambda\,.\end{aligned}$$

Hence by (3.3.3) and (3.3.4), we get that

$$\begin{aligned}
&\left|\int_{\frac{1}{R_0}}^{\infty} \frac{e^{i\left\{\lambda(s'-s)+(t(s')-t(s))\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}}{\left(\lambda^2+\frac{Q^2}{4}\right)^{1/4}} d\lambda\right| \\
\leq\ & \left|\int_{\max\left\{\frac{B(n)}{s},\frac{B(n)}{s'}\right\}}^{\infty} \frac{e^{i\left\{\lambda(s'-s)+(t(s')-t(s))\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}}{\left(\lambda^2+\frac{Q^2}{4}\right)^{1/4}} d\lambda\right| \\
&+\left|\int_{\max\left\{\frac{B(n)}{s},\frac{B(n)}{s'}\right\}}^{\frac{1}{R_0}} \frac{e^{i\left\{\lambda(s'-s)+(t(s')-t(s))\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}}{\left(\lambda^2+\frac{Q^2}{4}\right)^{1/4}} d\lambda\right| \\
\lesssim\ & \left(\frac{1}{|s-s'|^{1/2}}+1\right).
\end{aligned}$$

The case when $\max\left\{\frac{B(n)}{s},\frac{B(n)}{s'}\right\} > \frac{1}{R_0}$ can be argued similarly. $\square$

3.4 Boundedness of the maximal function

In this section, we prove the sufficient conditions of Theorem 3.1.1. In order to study the boundedness of the maximal function (1.2.4), it suffices to consider the linearized maximal function,

$$Tf(s) := S_{t(s)}f(s) = \int_0^{\infty} \varphi_\lambda(s)\, e^{it(s)\left(\lambda^2+\frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\,, \tag{3.4.1}$$

for $f \in \mathscr{S}^2(S)_o$ and prove

$$\|Tf\|_{L^q(B_R)} \lesssim \|f\|_{H^\beta(S)}\,. \tag{3.4.2}$$

In the above, $t(\cdot): S \to (0, 4/Q^2)$ is a radial measurable function on S.

We first work out the cases when the Spherical Fourier transform of f is either supported near the origin or away from the origin, in subsections 3.4.1 and 3.4.2 respectively. Then in subsection 3.4.3, we give the proof for the general case.

3.4.1 Initial condition with Spherical Fourier transform having small frequencies

In this subsection, we work under the assumption that $Supp(\widehat{f}) \subset [0, \Lambda)$ for some $\Lambda \in (0, \infty)$. In this case we have:

Lemma 3.4.1 *(3.4.2) holds for all $\beta \in [0,\infty)$ and all $q \in [1,\infty]$.*

Proof: The proof follows from the boundedness of φ_λ (3.2.4), an application of Cauchy-Schwarz inequality and the small frequency asymptotics of the density of the Planchrel measure given in (1.2.7). Indeed, for any $0 < s < R$,

$$\begin{aligned}
|Tf(s)| &\leq \int_0^\Lambda \left|\widehat{f}(\lambda)\right| |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&\leq \left(\int_0^\Lambda |\widehat{f}(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda\right)^{1/2} \left(\int_0^\Lambda |\mathbf{c}(\lambda)|^{-2} d\lambda\right)^{1/2} \\
&\lesssim \|f\|_{H^0(S)} \left(\int_0^\Lambda \lambda^2 d\lambda\right)^{1/2} \\
&= \frac{\Lambda^{3/2}}{\sqrt{3}} \|f\|_{H^0(S)}\,.
\end{aligned}$$

So we get the L^∞ estimate on B_R and hence the L^q estimates for all q. □

3.4.2 Initial condition with Spherical Fourier transform having large frequencies

In this subsection, we work under the standing assumption that $Supp(\widehat{f}) \subset [\frac{1}{R_0}, \infty)$, where R_0 is as in Lemma 3.2.2. In this case, we have:

Lemma 3.4.2 *(i) If $1/4 \leq \beta < n/2$, then (3.4.2) holds if $1 \leq q \leq 2n/(n-2\beta)$.*

(ii) If $\beta = n/2$, then (3.4.2) holds if $q \in [1,\infty)$.

(iii) If $\beta > n/2$, then (3.4.2) holds for all $q \in [1,\infty]$.

The key strategy in the proof of this Lemma, is to invoke the series expansions of φ_λ. This prompts us to decompose the ball B_R into a closed ball and an open annulus, $B_R = \overline{B}_{R_0} \cup A(R_0, R)$, where

$$\overline{B}_{R_0} = \{x \in S : d(e,x) \leq R_0\}, \quad A(R_0,R) = \{x \in S : R_0 < d(e,x) < R\}\,,$$

and study the boundedness of the linearized maximal function on the ball and the annulus separately.

3.4.2.1 Boundedness of T on the ball $\overline{B}_{R_0}$:

In this subsection, we look at inequalities of the form:

$$\|Tf\|_{L^q(\overline{B}_{R_0})} \lesssim \|f\|_{H^\beta(S)}\,. \tag{3.4.3}$$

Lemma 3.4.3 *(i) If $1/4 \leq \beta < n/2$, then (3.4.3) holds if $q \leq 2n/(n-2\beta)$.*

(ii) If $\beta = n/2$, then (3.4.3) holds if $q < \infty$.

(iii) If $\beta > n/2$, then (3.4.3) holds for all q.

To prove Lemma 3.4.3, we expand φ_λ using Lemma 3.2.2. For $s \in [0, R_0]$, putting $M = 0$ in Lemma 3.2.2, we get

$$\varphi_\lambda(s) = c_0 \left(\frac{s^{n-1}}{A(s)}\right)^{1/2} \mathscr{J}_{\frac{n-2}{2}}(\lambda s) + E_1(\lambda, s), \qquad \lambda \geq 1/R_0, \tag{3.4.4}$$

where

$$|E_1(\lambda, s)| \lesssim \begin{cases} s^2, & \text{if } \ \lambda s \leq 1 \\ s^2 (\lambda s)^{-\left(\frac{n+1}{2}\right)}, & \text{if } \ \lambda s > 1\,. \end{cases} \tag{3.4.5}$$

In accordance with (3.4.4), we decompose Tf as

$$Tf(s) = T_1 f(s) + T_2 f(s)$$

where

$$T_1 f(s) = c_0 \left(\frac{s^{n-1}}{A(s)}\right)^{1/2} \int_{\frac{1}{R_0}}^{\infty} \mathscr{J}_{\frac{n-2}{2}}(\lambda s)\, e^{it(s)\left(\lambda^2 + \frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda,$$

$$T_2 f(s) = \int_{\frac{1}{R_0}}^{\infty} E_1(\lambda, s)\, e^{it(s)\left(\lambda^2 + \frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda.$$

Now it suffices to look at inequalities of the form

$$\|T_j f\|_{L^q(\overline{B}_{R_0})} \lesssim \|f\|_{H^\beta(S)}, \quad j = 1, 2. \tag{3.4.6}$$

In other words, it is enough to prove the following lemmata regarding T_1 and T_2:

Lemma 3.4.4 *For T_1 we have,*

(i) If $1/4 \leq \beta < n/2$, then (3.4.6) holds if $q \leq 2n/(n-2\beta)$.

(ii) If $\beta = n/2$, then (3.4.6) holds if $q < \infty$.

(iii) If $\beta > n/2$, then (3.4.6) holds for all q.

Lemma 3.4.5 *For T_2 we have,*

(i) If $n \leq 4$, then (3.4.6) holds for all β and all q.

(ii) If $n \geq 5$, then

(a) If $\beta < \frac{n}{2} - 2$, then (3.4.6) holds if $q < 2n/(n-2\beta-4)$.

(b) If $\beta \geq \frac{n}{2} - 2$, then (3.4.6) holds for all q.

Remark 3.4.6 Since $2n/(n-2\beta-4)$ is larger than $2n/(n-2\beta)$, Lemma 3.4.5 shows that the operator T_2 is much better behaved compared to T_1. It is due to the behaviour of T_1 that the statement of our main result Theorem 3.1.1 is the same as the statement of the corresponding result of Sjölin [Sj97, Theorem 3].

Proof: [Proof of Lemma 3.4.4] As $f \in \mathscr{S}^2(S)_o$ with $Supp(\widehat{f}) \subset [\frac{1}{R_0}, \infty)$, by Lemma 3.3.1, we consider the unique function $\mathcal{M}(\mathscr{A}f) \in \mathscr{S}(\mathbb{R}^n)_o^\infty$ with the desired connection in terms of their Spherical Fourier transforms. We now compare suitable size estimates of f to their Euclidean counterparts corresponding to $\mathcal{M}(\mathscr{A}f)$. We first obtain a pointwise comparability of the Schrödinger propagation. In this regard let us consider the Euclidean countar-part T_0 of the operator T, defined on $\mathscr{S}(\mathbb{R}^n)_o$ and given by the formula

$$T_0\psi(s) = \int_0^\infty \mathscr{F}(\psi)(\lambda)\, e^{it(s)\lambda^2}\, \mathscr{J}_{\frac{n-2}{2}}(\lambda s)\, \lambda^{n-1} d\lambda, \qquad s \in (0,\infty), \quad \psi \in \mathscr{S}(\mathbb{R}^n)_o, \tag{3.4.7}$$

where t is the same measurable function on $(0,\infty)$, we had started with. Now, for $0 \leq s \leq R_0$, using the estimate of $A(s)$ (see (3.2.3)) and the relation (3.3.2), we have,

$$\begin{aligned} |T_1 f(s)| &\asymp \left| \int_{\frac{1}{R_0}}^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda s)\, e^{it(s)\lambda^2}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \right| \\ &= \left| \int_{\frac{1}{R_0}}^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda s)\, e^{it(s)\lambda^2}\, \mathscr{F}(\mathcal{M}(\mathscr{A}f))(\lambda)\, \lambda^{n-1}\, d\lambda \right| \\ &= \left| T_0 \mathcal{M}(\mathscr{A}f)(s) \right|. \end{aligned} \tag{3.4.8}$$

Again using the estimate (3.2.3) of $A(s)$, the above pointwise estimate implies that for all $q \in [1, \infty]$,

$$\|T_1 f\|_{L^q(\overline{B}_{R_0})} \asymp \|T_0 \mathcal{M}(\mathscr{A} f)\|_{L^q(\overline{B(o,R_0)})} , \tag{3.4.9}$$

where $B(o, R_0)$ is the Euclidean ball with center at the origin o and radius R_0. Next, we will compare the Sobolev norms of f and $\mathcal{M}(\mathscr{A} f)$. In this regard, again using the relation (3.3.2), we get that for any $\beta \geq 0$:

$$\begin{aligned}
\|f\|_{H^\beta(S)} &= \left(\int_{\frac{1}{R_0}}^{\infty} \left(\lambda^2 + \frac{Q^2}{4} \right)^\beta \left| \widehat{f}(\lambda) \right|^2 |\mathbf{c}(\lambda)|^{-2} \, d\lambda \right)^{1/2} \\
&\asymp \left(\int_{\frac{1}{R_0}}^{\infty} \left(\lambda^2 + 1 \right)^\beta |\mathscr{F}(\mathcal{M}(\mathscr{A} f))(\lambda)|^2 \frac{\lambda^{2(n-1)}}{|\mathbf{c}(\lambda)|^{-4}} |\mathbf{c}(\lambda)|^{-2} \, d\lambda \right)^{1/2} \\
&\asymp \left(\int_{\frac{1}{R_0}}^{\infty} \left(\lambda^2 + 1 \right)^\beta |\mathscr{F}(\mathcal{M}(\mathscr{A} f))(\lambda)|^2 \lambda^{n-1} \, d\lambda \right)^{1/2} \\
&= \|\mathcal{M}(\mathscr{A} f)\|_{H^\beta(\mathbb{R}^n)} .
\end{aligned} \tag{3.4.10}$$

Then combining (3.4.9), (3.4.10) and (1.2.6), we get Lemma 3.4.4. □

Proof: [Proof of Lemma 3.4.5] We recall that the operator T_2 was defined using the error term $E_1(\lambda, s)$ given in (3.4.4). By the estimate (3.4.5) of $E_1(\lambda, s)$ and the Cauchy-Schwarz inequality, we get that for $s \in (0, R_0]$,

$$\begin{aligned}
|T_2 f(s)| &\lesssim s^2 \int_{\frac{1}{R_0}}^{\frac{1}{s}} |\widehat{f}(\lambda)| |\mathbf{c}(\lambda)|^{-2} \, d\lambda + \frac{1}{s^{\left(\frac{n-3}{2}\right)}} \int_{\frac{1}{s}}^{\infty} \lambda^{-\left(\frac{n+1}{2}\right)} |\widehat{f}(\lambda)| |\mathbf{c}(\lambda)|^{-2} \, d\lambda \\
&\leq s^2 \left(\int_{\frac{1}{R_0}}^{\infty} \left(\lambda^2 + \frac{Q^2}{4} \right)^\beta |\widehat{f}(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{1/2} \left(\int_{\frac{1}{R_0}}^{\frac{1}{s}} \frac{|\mathbf{c}(\lambda)|^{-2} \, d\lambda}{\left(\lambda^2 + \frac{Q^2}{4} \right)^\beta} \right)^{1/2} \\
&\quad + \frac{1}{s^{\left(\frac{n-3}{2}\right)}} \left(\int_{\frac{1}{R_0}}^{\infty} \left(\lambda^2 + \frac{Q^2}{4} \right)^\beta |\widehat{f}(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{1/2} \left(\int_{\frac{1}{s}}^{\infty} \frac{|\mathbf{c}(\lambda)|^{-2} \, d\lambda}{\left(\lambda^2 + \frac{Q^2}{4} \right)^\beta \lambda^{n+1}} \right)^{1/2} \\
&= \|f\|_{H^\beta(S)} \left[s^2 \, I_1(s)^{1/2} + \frac{1}{s^{\left(\frac{n-3}{2}\right)}} \, I_2(s)^{1/2} \right] ,
\end{aligned} \tag{3.4.11}$$

where

$$I_1(s) = \int_{\frac{1}{R_0}}^{\frac{1}{s}} \frac{|\mathbf{c}(\lambda)|^{-2} \, d\lambda}{\left(\lambda^2 + \frac{Q^2}{4} \right)^\beta} , \qquad I_2(s) = \int_{\frac{1}{s}}^{\infty} \frac{|\mathbf{c}(\lambda)|^{-2} \, d\lambda}{\left(\lambda^2 + \frac{Q^2}{4} \right)^\beta \lambda^{n+1}} .$$

Using the estimate (1.2.7) of the function $|\mathbf{c}(\lambda)|^{-2}$, we see that

$$I_1(s) \lesssim \int_{\frac{1}{R_0}}^{\frac{1}{s}} \lambda^{n-1-2\beta}\, d\lambda \lesssim \begin{cases} \left(\frac{1}{s}\right)^{n-2\beta} - \left(\frac{1}{R_0}\right)^{n-2\beta}, & \text{if } \beta < \frac{n}{2} \\ \log\left(\frac{1}{s}\right) - \log\left(\frac{1}{R_0}\right), & \text{if } \beta = \frac{n}{2} \\ R_0^{2\beta-n} - s^{2\beta-n}, & \text{if } \beta > \frac{n}{2}, \end{cases} \tag{3.4.12}$$

and

$$I_2(s) \lesssim \int_{\frac{1}{s}}^{\infty} \frac{d\lambda}{\lambda^{2\beta+2}} = \frac{s^{2\beta+1}}{2\beta+1}\,. \tag{3.4.13}$$

Now, (i) and part (b) of (ii) follow easily from (3.4.11)-(3.4.13), for $q=\infty$. For part (a) of (ii), assuming $\beta < \frac{n}{2}-2$, combining (3.4.11)-(3.4.13) and using the estimate (3.2.3) of $A(s)$ we get that

$$\begin{aligned} \int_0^{R_0} |T_2 f(s)|^q\, A(s)\, ds &= \int_0^1 |T_2 f(s)|^q\, A(s)\, ds + \int_1^{R_0} |T_2 f(s)|^q\, A(s)\, ds \\ &\lesssim \|f\|_{H^\beta(S)}^q \left(\int_0^1 \frac{ds}{s^{q\left(\frac{n}{2}-\beta-2\right)-n+1}} + \int_1^{R_0} s^{n-1}\, ds \right). \end{aligned}$$

The first integral is convergent if and only if

$$q\left(\frac{n}{2}-\beta-2\right) - n + 1 < 1,$$

that is, $q < \frac{2n}{n-2\beta-4}$. This completes the proof of Lemma 3.4.5. $\square$

Using Lemmata 3.4.4 and 3.4.5, the proof of Lemma 3.4.3 follows.

3.4.2.2 Boundedness of T on the annulus $A(R_0, R)$:

In this subsection, we look at inequalities of the form:

$$\|Tf\|_{L^q(A(R_0,R))} \lesssim \|f\|_{H^\beta(S)}\,. \tag{3.4.14}$$

In our first lemma, we work in the higher regularity regime ($\beta > 1/2$) by only utilizing the pointwise decay of φ_λ.

Lemma 3.4.7 *If $\beta > 1/2$, then (3.4.14) holds for all $q \in [1, \infty]$.*

Proof: It suffices to prove the lemma for $q=\infty$. The proof will use the pointwise decay of φ_λ when both λ and s are large (3.2.9) and the Cauchy-Schwarz inequality. Indeed, for $\lambda \geq 1/R_0$ and $s \in (R_0, R)$,

$$\begin{aligned}
|Tf(s)| &\lesssim e^{-\frac{Q}{2}s} \int_{\frac{1}{R_0}}^{\infty} \lambda^{-\left(\frac{n-1}{2}\right)} \left|\widehat{f}(\lambda)\right| |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&\leq e^{-\frac{QR_0}{2}} \left(\int_{\frac{1}{R_0}}^{\infty} \left(\lambda^2+\frac{Q^2}{4}\right)^{\beta} |\widehat{f}(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda\right)^{1/2} \left(\int_{\frac{1}{R_0}}^{\infty} \frac{|\mathbf{c}(\lambda)|^{-2}\, d\lambda}{\left(\lambda^2+\frac{Q^2}{4}\right)^{\beta} \lambda^{n-1}}\right)^{1/2} \\
&\lesssim \|f\|_{H^\beta(S)} \left(\int_{\frac{1}{R_0}}^{\infty} \frac{d\lambda}{\lambda^{2\beta}}\right)^{1/2} .
\end{aligned}$$

The integral in the last line converges if and only if $\beta > 1/2$. This completes the proof. $\square$

For the lower regularity scenario ($1/4 \leq \beta \leq 1/2$), the analysis is more involved and the key idea is to make use of the oscillation afforded by both φ_λ and the Schrödinger multiplier.

Lemma 3.4.8 *If $1/4 \leq \beta \leq 1/2$, then (3.4.14) holds if $q \in [1,4]$.*

To prove the above Lemma we expand φ_λ using Lemma 3.2.3. For $R_0 < s < R$ and $\lambda \geq 1/R_0$, by Lemma 3.2.3, we write

$$\varphi_\lambda(s) = 2^{-m_{\mathfrak{z}}/2} A(s)^{-1/2} \left\{\mathbf{c}(\lambda) e^{i\lambda s} + \mathbf{c}(-\lambda) e^{-i\lambda s}\right\} + \mathscr{E}_2(\lambda, s)\,, \tag{3.4.15}$$

where

$$\mathscr{E}_2(\lambda, s) = 2^{-m_{\mathfrak{z}}/2} A(s)^{-1/2} \left\{ \mathbf{c}(\lambda) \sum_{\mu=1}^{\infty} \Gamma_\mu(\lambda) e^{(i\lambda-\mu)s} + \mathbf{c}(-\lambda) \sum_{\mu=1}^{\infty} \Gamma_\mu(-\lambda) e^{-(i\lambda+\mu)s} \right\} ,$$

and thus

$$|\mathscr{E}_2(\lambda, s)| \lesssim A(s)^{-1/2} |\mathbf{c}(\lambda)| (1+\lambda)^{-1} . \tag{3.4.16}$$

Hence for $R_0 < s < R$ and $\lambda \geq 1/R_0$, in accordance to (3.4.15), we decompose T as,

$$Tf(s) = T_3 f(s) + T_4 f(s) + T_5 f(s),$$

where

$$
\begin{aligned}
T_3f(s) &= 2^{-m_{\mathfrak{z}}/2}A(s)^{-1/2}\int_{\frac{1}{R_0}}^{\infty}\mathbf{c}(\lambda)\,e^{i\left\{\lambda s+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}\,\widehat{f}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,,\\
T_4f(s) &= 2^{-m_{\mathfrak{z}}/2}A(s)^{-1/2}\int_{\frac{1}{R_0}}^{\infty}\mathbf{c}(-\lambda)\,e^{i\left\{-\lambda s+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}\,\widehat{f}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,.
\end{aligned}
$$

$$
T_5f(s) = \int_{\frac{1}{R_0}}^{\infty}\mathscr{E}_2(\lambda,s)\,e^{it(s)\left(\lambda^2+\frac{Q^2}{4}\right)}\,\widehat{f}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,.
$$

Remark 3.4.9 We note that for the smallest value of β, that is, $\beta = 1/4$, the proposed largest value of q in Theorem 3.1.1 satisfies the inequality

$$
\frac{2n}{n-2\beta}\leq 4,\qquad n\geq 1. \tag{3.4.17}
$$

In view of Lemma 3.4.7 and (3.4.17), it is therefore enough to prove the following estimates at the end-points, to complete the proof of Lemma 3.4.8.

Lemma 3.4.10 *(a) For $j=3,4$, we have the inequalities*

$$
\|T_jf\|_{L^4(A(R_0,R))}\lesssim\|f\|_{H^{1/4}(S)}\,.
$$

(b) For T_5, we have the inequality

$$
\|T_5f\|_{L^\infty(A(R_0,R))}\lesssim\|f\|_{H^0(S)}\,.
$$

Proof: We first prove part (a). As the proofs for T_3 and T_4 are analogous in nature, we will concentrate only on T_3. We need to prove the inequality

$$
\left(\int_{R_0}^{R}|T_3f(s)|^4\,A(s)\,ds\right)^{1/4}\lesssim\left(\int_{\frac{1}{R_0}}^{\infty}\left(\lambda^2+\frac{Q^2}{4}\right)^{1/4}\left|\widehat{f}(\lambda)\right|^2|\mathbf{c}(\lambda)|^{-2}\,d\lambda\right)^{1/2}. \tag{3.4.18}
$$

Using the definition of T_3 we get

$$\begin{aligned}
& T_3 f(s) A(s)^{\frac{1}{4}} \\
= \;& 2^{-\frac{m_{\mathfrak{z}}}{2}} A(s)^{-\frac{1}{4}} \int_{\frac{1}{R_0}}^{\infty} \mathbf{c}(\lambda)\, e^{i\left\{\lambda s + t(s)\left(\lambda^2 + \frac{Q^2}{4}\right)\right\}}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
= \;& 2^{-\frac{m_{\mathfrak{z}}}{2}} A(s)^{-\frac{1}{4}} \int_{\frac{1}{R_0}}^{\infty} \mathbf{c}(\lambda)\, e^{i\left\{\lambda s + t(s)\left(\lambda^2 + \frac{Q^2}{4}\right)\right\}}\, g(\lambda) \left(\lambda^2 + \frac{Q^2}{4}\right)^{-\frac{1}{8}} |\mathbf{c}(\lambda)|^{-1}\, d\lambda\,,
\end{aligned}$$

where

$$g(\lambda) = \widehat{f}(\lambda) \left(\lambda^2 + \frac{Q^2}{4}\right)^{1/8} |\mathbf{c}(\lambda)|^{-1}\,.$$

We now define

$$Pg(s) := A(s)^{-1/4} \int_{\frac{1}{R_0}}^{\infty} \mathbf{c}(\lambda)\, e^{i\left\{\lambda s + t(s)\left(\lambda^2 + \frac{Q^2}{4}\right)\right\}}\, g(\lambda) \left(\lambda^2 + \frac{Q^2}{4}\right)^{-1/8} |\mathbf{c}(\lambda)|^{-1}\, d\lambda\,,$$

so that,

$$2^{m_{\mathfrak{z}}/2}\, T_3 f(s) A(s)^{1/4} = Pg(s)\,, \quad s \in (R_0, R)\,.$$

It follows that proving (3.4.18) is equivalent to proving

$$\left(\int_{R_0}^{R} |Pg(s)|^4\, ds\right)^{1/4} \lesssim \left(\int_{\frac{1}{R_0}}^{\infty} |g(\lambda)|^2\, d\lambda\right)^{1/2}\,. \tag{3.4.19}$$

Now for $\eta \in C_c^\infty(R_0, R)$ and $\lambda \in [1/R_0, \infty)$, setting

$$P^*\eta(\lambda) = \overline{\mathbf{c}(\lambda)} \left(\lambda^2 + \frac{Q^2}{4}\right)^{-\frac{1}{8}} |\mathbf{c}(\lambda)|^{-1} \int_{R_0}^{R} A(s)^{-1/4}\, e^{-i\left\{\lambda s + t(s)\left(\lambda^2 + \frac{Q^2}{4}\right)\right\}} \eta(s)\, ds\,,$$

it is easy to see that the relation

$$\int_{R_0}^{R} P\psi(s)\, \overline{\eta(s)}\, ds = \int_{\frac{1}{R_0}}^{\infty} \psi(\lambda)\, \overline{P^*\eta(\lambda)}\, d\lambda\,,$$

holds for all $\eta \in C_c^\infty(R_0, R)$ and $\psi \in L^2[1/R_0,\, \infty)$ having suitable decay at infinity. Using duality it follows that to prove (3.4.19), it suffices to prove the following inequality

$$\left(\int_{\frac{1}{R_0}}^{\infty} |P^* h(\lambda)|^2\, d\lambda\right)^{1/2} \lesssim \left(\int_{R_0}^{R} |h(s)|^{4/3}\, ds\right)^{3/4}\,, \tag{3.4.20}$$

for all $h \in C_c^\infty(R_0, R)$. Now,

$$\begin{aligned}
&\int_{\frac{1}{R_0}}^{\infty} |P^*h(\lambda)|^2 \, d\lambda \\
=\ &\int_{\frac{1}{R_0}}^{\infty} P^*h(\lambda) \, \overline{P^*h(\lambda)} \, d\lambda \\
=\ &\int_{\frac{1}{R_0}}^{\infty} \left[\overline{\mathbf{c}(\lambda)} \left(\lambda^2 + \frac{Q^2}{4}\right)^{-\frac{1}{8}} |\mathbf{c}(\lambda)|^{-1} \int_{R_0}^{R} A(s)^{-1/4} \, e^{-i\left\{\lambda s + t(s)\left(\lambda^2 + \frac{Q^2}{4}\right)\right\}} h(s) \, ds \right] \\
&\times \left[\mathbf{c}(\lambda) \left(\lambda^2 + \frac{Q^2}{4}\right)^{-\frac{1}{8}} |\mathbf{c}(\lambda)|^{-1} \int_{R_0}^{R} A(s')^{-1/4} \, e^{i\left\{\lambda s' + t(s')\left(\lambda^2 + \frac{Q^2}{4}\right)\right\}} \overline{h(s')} \, ds' \right] d\lambda \\
=\ &\int_{R_0}^{R} \int_{R_0}^{R} I(s, s') \, A(s)^{-\frac{1}{4}} \, A(s')^{-\frac{1}{4}} \, h(s) \, \overline{h(s')} \, ds \, ds' \,,
\end{aligned}$$

where

$$I(s, s') = \int_{\frac{1}{R_0}}^{\infty} \frac{e^{i\left\{\lambda(s'-s) + (t(s') - t(s))\left(\lambda^2 + \frac{Q^2}{4}\right)\right\}}}{\left(\lambda^2 + \frac{Q^2}{4}\right)^{\frac{1}{4}}} d\lambda \,.$$

So by Lemma 3.3.2, the estimate (3.2.3) of $A(s)$ and an application of Cauchy-Schwarz, it follows that

$$\begin{aligned}
&\int_{\frac{1}{R_0}}^{\infty} |P^*h(\lambda)|^2 \, d\lambda \\
\lesssim\ &\int_{R_0}^{R} \int_{R_0}^{R} \left(\frac{1}{|s - s'|^{\frac{1}{2}}} + 1 \right) |h(s)| \, |h(s')| \, s^{-\left(\frac{n-1}{4}\right)} \, (s')^{-\left(\frac{n-1}{4}\right)} \, ds \, ds' \\
\leq\ &\int_{R_0}^{R} \int_{R_0}^{R} \frac{|h(s)||h(s')|}{|s - s'|^{\frac{1}{2}}} \, s^{-\left(\frac{n-1}{4}\right)} \, (s')^{-\left(\frac{n-1}{4}\right)} \, ds \, ds' + \sqrt{R} R_0^{-\left(\frac{n-1}{2}\right)} \, \|h\|^2_{L^{4/3}(R_0, R)} \,.
\end{aligned}$$

As $h \in C_c^\infty(R_0, R)$, we can think of it as an even C_c^∞ function supported in $(-R, -R_0) \sqcup (R_0, R)$. We can therefore write the last integral as an one dimensional Riesz potential and apply (3.2.1) to get that for some positive number c,

$$\begin{aligned}
&\int_{R_0}^{R} \int_{R_0}^{R} \frac{|h(s)||h(s')|}{|s - s'|^{1/2}} \, s^{-\left(\frac{n-1}{4}\right)} \, (s')^{-\left(\frac{n-1}{4}\right)} \, ds \, ds' \\
=\ &\int_{0}^{\infty} \int_{0}^{\infty} \frac{|h(s)||h(s')|}{|s - s'|^{1/2}} \, s^{-\left(\frac{n-1}{4}\right)} \, (s')^{-\left(\frac{n-1}{4}\right)} \, ds \, ds' \\
=\ &c \int_{0}^{\infty} I_{1/2} \left((s')^{-\left(\frac{n-1}{4}\right)} |h| \right)(s) \, s^{-\left(\frac{n-1}{4}\right)} \, |h(s)| \, ds \\
=\ &c \int_{\mathbb{R}} |\xi|^{-\frac{1}{2}} \left| \left(s^{-\left(\frac{n-1}{4}\right)} |h| \right)^{\sim}(\xi) \right|^2 d\xi \,.
\end{aligned}$$

We now want to apply Pitt's inequality (see Lemma 3.2.1) to the function

$$x \mapsto |x|^{-\frac{n-1}{4}}|h(x)|, \qquad x \in \mathbb{R},$$

for $\beta = \frac{1}{4}$ and $p = \frac{4}{3}$. In this case it turns out that $\beta_1 = 0$. Hence by Pitt's inequality we obtain

$$\begin{aligned}
\int_{\mathbb{R}} |\xi|^{-\frac{1}{2}} \left| \left(s^{-\left(\frac{n-1}{4}\right)}|h|\right)^{\sim}(\xi)\right|^2 d\xi &\lesssim \left(\int_{\mathbb{R}} |h(x)|^{\frac{4}{3}}\, |x|^{-\left(\frac{n-1}{3}\right)}\, dx\right)^{3/2} \\
&= c\left(\int_{R_0}^{R} |h(s)|^{\frac{4}{3}}\, s^{-\left(\frac{n-1}{3}\right)}\, ds\right)^{3/2} \\
&\leq c\, R_0^{-\left(\frac{n-1}{2}\right)}\, \|h\|^2_{L^{4/3}(R_0,R)}\,.
\end{aligned}$$

This completes the proof of part (a).

The proof of part (b) does not involve oscillation instead it only uses the error term estimate (3.4.16), which is of critical importance here. Indeed, using (3.4.16) along with the Cauchy-Schwarz inequality, it follows that for all $s \in (R_0, R)$,

$$\begin{aligned}
|T_5 f(s)| &\lesssim A(s)^{-1/2} \int_{\frac{1}{R_0}}^{\infty} \lambda^{-1} \left|\widehat{f}(\lambda)\right| |\mathbf{c}(\lambda)|^{-1}\, d\lambda \\
&\lesssim R_0^{-\left(\frac{n-1}{2}\right)}\, \|f\|_{H^0(S)} \left(\int_{\frac{1}{R_0}}^{\infty} \frac{d\lambda}{\lambda^2}\right)^{1/2} \\
&\lesssim \|f\|_{H^0(S)}\,.
\end{aligned}$$

This completes the proof of Lemma 3.4.10 and also of Lemma 3.4.8. □

The proof of Lemma 3.4.2 finally follows from that of Lemmata 3.4.3, 3.4.7 and 3.4.8.

3.4.3 Proof of the sufficient conditions of Theorem 3.1.1:

In this subsection, we complete the proof of the sufficient conditions of Theorem 3.1.1.

Proof: Let $f \in \mathscr{S}^2(S)_o$. Then $\widehat{f} \in \mathscr{S}(\mathbb{R})_e$. Now let us choose an auxiliary non-negative even function $\psi \in C_c^{\infty}(\mathbb{R})$ such that $Supp(\psi) \subset \{\xi : \frac{1}{2} < |\xi| < 2\}$ and

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k}\xi) = 1\,,\; \xi \neq 0\,.$$

We set

$$\psi_1(\xi) := \sum_{k=-\infty}^{0} \psi(2^{-k}\xi)\,, \quad \psi_2(\xi) := \sum_{k=1}^{\infty} \psi(2^{-k}\xi)\,.$$

Then both ψ_1 and ψ_2 are even non-negative smooth functions with $Supp(\psi_1) \subset (-2, 2)$, $Supp(\psi_2) \subset \mathbb{R} \setminus (-1, 1)$ and $\psi_1 + \psi_2 \equiv 1$. Then we consider,

$$\widehat{f} = \widehat{f}\psi_1 + \widehat{f}\psi_2\,.$$

As $\widehat{f}\psi_j$, for $j = 1, 2$, are even Schwartz class functions on $\mathbb{R}$, by the Schwartz isomorphism theorem, there exist $\{f_1, f_2\} \subset \mathscr{S}^2(S)_o$ such that $\widehat{f_j} = \widehat{f}\psi_j$ for $j = 1, 2$. Now if we can prove the estimates

$$\|Tf_j\|_{L^{q_j}(B_R)} \lesssim \|f_j\|_{H^{\beta_j}(S)}\,, \quad j = 1, 2\,,$$

then it follows that

$$\begin{aligned}
\|Tf\|_{L^{\min\{q_1,q_2\}}(B_R)} &\lesssim \|Tf_1\|_{L^{q_1}(B_R)} + \|Tf_2\|_{L^{q_2}(B_R)} \\
&\lesssim \|f_1\|_{H^{\beta_1}(S)} + \|f_2\|_{H^{\beta_2}(S)} \\
&\leq 2\,\|f\|_{H^{\max\{\beta_1,\beta_2\}}(S)}\,. \qquad (3.4.21)
\end{aligned}$$

Then noting that $(1, \infty) \subset [\frac{1}{R_0}, \infty)$ as $R_0 > 2$, in view of (3.4.21) the sufficient conditions of Theorem 3.1.1 follow from Lemmata 3.4.1 and 3.4.2. □

3.5 Sharpness at the end-points

In this section, we prove the sharpness at the endpoints of Theorem 3.1.1. Here we will mainly work on B_{R_0} and hence the expression of φ_λ along with the estimates of the error term E_1 given in (3.4.4) and (3.4.5) will be repeatedly used.

Proof: [Proof of the necessary conditions of Theorem 3.1.1:] We divide the proof of necessity of the conditions in three different cases based on the values of the Sobolev exponent β.

Case 1: *If $\beta < 1/4$ then the local estimate (3.1.1) does not hold for $q \in [1, \infty]$.*

It suffices to prove that if $\beta < 1/4$, then the following inequality for the linearized maximal function T does not hold true

$$\|Tf\|_{L^1(B_1)} \lesssim \|f\|_{H^\beta(S)}\,.$$

We first recall the Euclidean counter-example formulated by Sjölin as it is relevant for us. In [Sj97, pp. 55-58], the author considered an even, non-negative C_c^∞ function ϕ on $\mathbb{R}$ with $Supp(\phi) \subset (-1,1)$ and then constructed $\{f_N\}_{N\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^n)_o$ such that,

$$\mathscr{F} f_N(\lambda) = N^{-1/2}\, \phi(-N^{-1/2}\lambda + N^{1/2})\, \lambda^{-\left(\frac{n-1}{2}\right)}, \qquad \lambda \in (0,\infty), \tag{3.5.1}$$

and then proved that

A) $Supp(\mathscr{F}(f_N)) \subset \left[N - \sqrt{N}, N + \sqrt{N}\right]$,

B) $\|f_N\|_{H^\beta(\mathbb{R}^n)} \lesssim N^{\beta - \frac{1}{4}}$ for all $N \in \mathbb{N}$,

C) For $\varepsilon > 0$, small and for all N sufficiently large, there exists a positive number c (independent of N) such that T_0 satisfies the inequality

$$|T_0 f_N(x)| \geq c\,, \qquad \varepsilon \leq \|x\| \leq 2\varepsilon. \tag{3.5.2}$$

In particular (A) above shows that for all $N \geq 2$, $f_N \in \mathscr{S}(\mathbb{R}^n)_o^\infty$. We now define a sequence of functions $\{g_N\}_{N\in\mathbb{N}} \subset \mathscr{S}^2(S)_o$ by

$$g_N := (\mathscr{A}^{-1} \circ \mathcal{M}^{-1}) f_N,$$

where $\mathscr{A}$ and $\mathcal{M}$ are as in the proof of Lemma 3.3.1. The key idea here is that since the Fourier transform of f_N is supported away from the origin and the spacial estimate (3.5.2) was done in a small annulus centered at the origin, as in the proof of Lemma 3.4.4, we can again implement the series expansion of φ_λ described in Lemma 3.2.2, the Abel transform and the map $\mathfrak{m}$ to form a connection between $\mathscr{S}(\mathbb{R}^n)_o^\infty$ and $\mathscr{S}^2(S)_o^\infty$.

Now, using the expression of φ_λ given in (3.4.4) and for $\varepsilon \leq s \leq 2\varepsilon < R_0$, we write

$$\begin{aligned} T g_N(s) &= c_0 \left(\frac{s^{n-1}}{A(s)}\right)^{1/2} \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda s)\, e^{it(s)\left(\lambda^2 + \frac{Q^2}{4}\right)}\, \widehat{g_N}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &\quad + \int_0^\infty E_1(\lambda, s)\, e^{it(s)\left(\lambda^2 + \frac{Q^2}{4}\right)}\, \widehat{g_N}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &= T_1 g_N(s) + T_2 g_N(s)\,. \end{aligned} \tag{3.5.3}$$

Then using (3.5.1) and proceeding exactly as in the proof of Lemma 3.4.4, we get

(a) $\widehat{g_N}(\lambda) = \frac{\lambda^{n-1}}{|\mathbf{c}(\lambda)|^{-2}} \mathscr{F} f_N(\lambda) = \frac{\lambda^{n-1}}{|\mathbf{c}(\lambda)|^{-2}} N^{-1/2} \phi(-N^{-1/2}\lambda + N^{1/2}) \lambda^{-\left(\frac{n-1}{2}\right)}$,

(b) $Supp(\widehat{g_N}) \subset \left[N - \sqrt{N}, N + \sqrt{N}\right]$,

(c) $\|g_N\|_{H^\beta(S)} \asymp \|f_N\|_{H^\beta(\mathbb{R}^n)} \lesssim N^{\beta - \frac{1}{4}}$,

(d) For $\varepsilon > 0$, small and for all $N \in \mathbb{N}$ sufficiently large, there exists a positive number c' (independent of N), such that

$$|T_1 g_N(s)| \geq c' \,, \qquad \text{for all } s \in [\varepsilon, 2\varepsilon].$$

We note that it follows from (c) above that if $\beta < 1/4$ then

$$\lim_{N \to \infty} \|g_N\|_{H^\beta(S)} = 0. \tag{3.5.4}$$

Next, we show that for all large $N \in \mathbb{N}$

$$|T_2 g_N(s)| \lesssim \frac{1}{N}, \qquad \text{for all } s \in [\varepsilon, 2\varepsilon]. \tag{3.5.5}$$

Then combining (3.5.3), (d) and (3.5.5), we get that for $\varepsilon > 0$, small and N sufficiently large, there exists $c'' > 0$, such that

$$|Tg_N(s)| \geq c'' \,, \qquad \text{for all } s \in [\varepsilon, 2\varepsilon]. \tag{3.5.6}$$

Then in view of (3.5.4) and (3.5.6) the claim of **Case 1** follows. It now remains to prove the inequality (3.5.5). Now, for the chosen small $\varepsilon > 0$, we can get $N_0 \in \mathbb{N}$ such that for all $N \geq N_0 \in \mathbb{N}$, $1/\varepsilon$ is strictly smaller than $N - \sqrt{N}$. Hence for all $s \in [\varepsilon, 2\varepsilon]$

$$s(N - \sqrt{N}) > 1, \quad N \geq N_0.$$

It follows from this and the estimate of the error term $E_1(\lambda, s)$ given in (3.4.5) that for $\lambda \in (N - \sqrt{N}, N + \sqrt{N})$,

$$|E_1(\lambda, s)| \lesssim s^2 (\lambda s)^{-\left(\frac{n+1}{2}\right)}, \quad s \in [\varepsilon, 2\varepsilon] \,,$$

as $\lambda s > 1$. It now follows from the formula for T_2 and (a), (b) that

$$\begin{aligned} |T_2\, g_N(s)| &\lesssim \int_{N-\sqrt{N}}^{N+\sqrt{N}} s^2 (\lambda s)^{-\left(\frac{n+1}{2}\right)} \, |\widehat{g_N}(\lambda)| \, |\mathbf{c}(\lambda)|^{-2} \, d\lambda \\ &\leq \frac{\|\phi\|_\infty}{s^{\left(\frac{n-3}{2}\right)}} \int_{N-\sqrt{N}}^{N+\sqrt{N}} \lambda^{-\left(\frac{n+1}{2}\right)} \, N^{-\frac{1}{2}} \, \lambda^{-\left(\frac{n-1}{2}\right)} \, \lambda^{n-1} \, d\lambda \\ &\lesssim C_\varepsilon N^{-\frac{1}{2}} \int_{N-\sqrt{N}}^{N+\sqrt{N}} \frac{d\lambda}{\lambda} \\ &\lesssim \frac{1}{N} \,, \end{aligned}$$

where the last inequality follows from the elementary estimate

$$\log\left(\frac{x+1}{x-1}\right) \lesssim \frac{1}{x} \,,$$

which is valid for all large x. This proves (3.5.5).

Case 2: *If $1/4 \leq \beta < n/2$, then the condition $q \leq 2n/(n-2\beta)$ is necessary for validity of the inequality (3.1.1).*

Let $\psi \in C_c^\infty(\mathbb{R})_e$ be (non-zero) non-negative with $Supp(\psi) \subset (1,2)$. Then for each $N \in \mathbb{N}$, there exists $f_N \in \mathscr{S}(\mathbb{R}^n)_o$ such that

$$\mathscr{F} f_N(\lambda) = \psi(\lambda/N) \,, \qquad \lambda \in [0,\infty) \,.$$

We note that

$$f_1(0) = \int_0^\infty \psi(\lambda) \lambda^{n-1} d\lambda = C_\psi > 0 \,, \tag{3.5.7}$$

and for $s \in [0,\infty)$, by the Euclidean Fourier inversion applied to radial functions

$$\begin{aligned} f_N(s) &= \int_0^\infty \psi\left(\frac{\lambda}{N}\right) \mathscr{J}_{\frac{n-2}{2}}(\lambda s) \lambda^{n-1} d\lambda \\ &= N^n \int_0^\infty \psi(\lambda) \mathscr{J}_{\frac{n-2}{2}}(\lambda N s) \lambda^{n-1} d\lambda = N^n f_1(Ns) \,. \end{aligned}$$

Therefore, by (3.5.7) there exists $\varepsilon \in (0,1/2)$ such that

$$|f_N(s)| > \frac{C_\psi}{2} N^n \,, \qquad \text{for all, } 0 \leq s < \frac{\varepsilon}{N}. \tag{3.5.8}$$

We now define as before $g_N = (\mathscr{A}^{-1} \circ \mathcal{M}^{-1}) f_N \in \mathscr{S}^2(S)^\infty_o$ and observe that

$$\begin{aligned} \|g_N\|_{H^\beta(S)} &= \left(\int_0^\infty \left(\lambda^2 + \frac{Q^2}{4} \right)^\beta \psi(\lambda/N)^2 \, \frac{\lambda^{2(n-1)}}{|\mathbf{c}(\lambda)|^{-4}} \, |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{1/2} \\ &\asymp N^{\beta+\frac{n}{2}} \left(\int_1^2 \psi(\lambda)^2 \, \lambda^{2\beta+n-1} \, d\lambda \right)^{1/2} \\ &\lesssim N^{\beta+\frac{n}{2}} \, . \end{aligned} \tag{3.5.9}$$

Next, for $s \in [0, \varepsilon/N)$, using the spherical Fourier inversion we write

$$\begin{aligned} S_0 g_N(s) = g_N(s) &= \int_0^\infty \widehat{g_N}(\lambda) \varphi_\lambda(s) \, |\mathbf{c}(\lambda)|^{-2} d\lambda \\ &= c \left(\frac{s^{n-1}}{A(s)} \right)^{1/2} \int_0^\infty \widehat{g_N}(\lambda) \mathscr{J}_{\frac{n-2}{2}}(\lambda s) \, |\mathbf{c}(\lambda)|^{-2} d\lambda \\ &\quad + \int_0^\infty \widehat{g_N}(\lambda) E_1(\lambda, s) \, |\mathbf{c}(\lambda)|^{-2} d\lambda = g_{N,1}(s) + g_{N,2}(s) \, . \end{aligned}$$

Using the properties of the operators $\mathcal{M}$ and $\mathscr{A}$ (as in the proof of Lemma 3.4.4) it follows that

$$|g_{N,1}(s)| \asymp |f_N(s)| \, , \quad \text{for all, } s \in [0, \varepsilon/N),$$

where the implicit constant is independent of N. Consequently, it follows from (3.5.8) that there exists a positive number c (independent of N) such that

$$|g_{N,1}(s)| \geq c N^n \, , \quad \text{for all, } s \in [0, \varepsilon/N). \tag{3.5.10}$$

We next show that the growth of $g_{N,2}$ is subordinated by $g_{N,1}$. As $s \in [0, \varepsilon/N)$ and $\varepsilon \in (0, 1/2)$ it follows that for $\lambda \in (1, 2)$ the quantity $\lambda N s$ is smaller than 1. Hence using the estimate (3.4.5) of the error term E_1 we get that for $s \in [0, \varepsilon/N)$

$$\begin{aligned} |g_{N,2}(s)| &\leq \int_0^\infty |E_1(\lambda, s)| \, \psi\left(\frac{\lambda}{N}\right) \, |\mathbf{c}(\lambda)|^{-2} d\lambda \\ &= N \int_1^2 |E_1(\lambda N, s)| \, \psi(\lambda) \, |\mathbf{c}(N\lambda)|^{-2} d\lambda \\ &\lesssim N \|\psi\|_\infty \int_1^2 s^2 (N\lambda)^{n-1} d\lambda \\ &< c' \varepsilon^2 N^{n-2} \, . \end{aligned}$$

Therefore, using triangle inequality and (3.5.10), it follows that for all $s \in [0, \varepsilon/N)$ and N sufficiently large,

$$\begin{aligned} |S_0 g_N(s)| &\geq |g_{N,1}(s)| - |g_{N,2}(s)| \\ &\geq cN^n - c'\varepsilon^2 N^{n-2} \geq c'' N^n . \end{aligned}$$

This implies that for all $x \in B_{\varepsilon/N}$,

$$S^* g_N(x) \geq c\, N^n , \tag{3.5.11}$$

with the constant c being independent of N. Now, if the estimate (3.1.1) holds, then by (3.5.9) and (3.5.11), it follows that

$$\left(\int_0^{\varepsilon/N} N^{nq}\, s^{n-1}\, ds \right)^{1/q} \lesssim N^{\beta + \frac{n}{2}} ,$$

which implies that

$$N^{n\left(1-\frac{1}{q}\right)} \lesssim N^{\beta+\frac{n}{2}} .$$

Then letting $N \to \infty$, it follows that

$$n\left(1 - \frac{1}{q}\right) \leq \beta + \frac{n}{2} ,$$

that is, $q \leq 2n/(n-2\beta)$.

Case 3: *If $\beta = n/2$ then (3.1.1) does not hold for $q = \infty$.*

In this case we need to show the failure of the inequality

$$\|S^* f\|_{L^\infty(B)} \leq C_B\, \|f\|_{H^{n/2}(S)} ,$$

for every ball centered at the identity e. We assume that the above inequality is true for some ball B_R of radius R centered at e with $R < R_0$, and then arrive at a contradiction. Now, if the above inequality is true then using the fact

$$f(x) = \lim_{t \to 0} S_t f(x), \qquad x \in B_R,\;\; f \in \mathscr{S}^2(S)_o ,$$

it follows that the following inequality also holds true for all $f \in \mathscr{S}^2(S)_o$,

$$\|f\|_{L^\infty(B_R)} \leq C_{B_R} \, \|f\|_{H^{n/2}(S)}. \tag{3.5.12}$$

Hence, in particular, (3.5.12) holds for all $f \in \mathscr{S}^2(S)_o$ with $supp(\widehat{f}) \subset (1, \infty)$. Let us choose and fix such a function f. By the spherical Fourier inversion and the expansion of φ_λ we have

$$\begin{aligned} f(s) &= c_0 \left(\frac{s^{n-1}}{A(s)} \right)^{1/2} \int_1^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda s) \, \widehat{f}(\lambda) \, |\mathbf{c}(\lambda)|^{-2} \, d\lambda \\ &\quad + \int_1^\infty E_1(\lambda, s) \, \widehat{f}(\lambda) \, |\mathbf{c}(\lambda)|^{-2} \, d\lambda \\ &= f_1(s) + f_2(s) \, . \end{aligned} \tag{3.5.13}$$

From the proof of Lemma 3.4.5, one gets the estimate

$$\|f_2\|_{L^\infty(B_R)} \leq C_{B_R} \, \|f\|_{H^{n/2}(S)} \, . \tag{3.5.14}$$

Thus by using (3.5.12)-(3.5.14), and the triangle inequality we have that

$$\|f_1\|_{L^\infty(B_R)} \leq C_{B_R} \, \|f\|_{H^{n/2}(S)} \, . \tag{3.5.15}$$

We again define $g := (\mathcal{M} \circ \mathscr{A})f$. It is clear that $\mathcal{M} \circ \mathscr{A}$ defines a bijection from the subspace of functions in $\mathscr{S}^2(S)_o$ whose Spherical Fourier transforms are supported in $(1, \infty)$ onto the subspace of functions in $\mathscr{S}(\mathbb{R}^n)_o$ whose Euclidean Spherical Fourier transforms are supported in $(1, \infty)$. Then proceeding as in the proof of Lemma 3.4.4, we get that

$$\|f_1\|_{L^\infty(B_R)} \asymp C_{B_R} \|g\|_{L^\infty(B(o,R))} \, , \qquad \|f\|_{H^{n/2}(S)} \asymp \|g\|_{H^{n/2}(\mathbb{R}^n)} \, .$$

Then combining the above with (3.5.15), we get that

$$\|g\|_{L^\infty(B(o,R))} \lesssim \|f_1\|_{L^\infty(B_R)} \lesssim \|f\|_{H^{n/2}(S)} \lesssim \|g\|_{H^{n/2}(\mathbb{R}^n)} \, ,$$

with the resultant implicit constant depending only on the ball $B(o, R)$. As f was arbitrarily chosen, it follows that the above inequality is true for all functions $g \in \mathscr{S}(\mathbb{R}^n)_o$ whose Euclidean Spherical Fourier transforms $\mathscr{F}g$ are supported in $(1, \infty)$. Now by an argument similar to the proof of Lemma 3.4.1, it follows that for all g in $\mathscr{S}(\mathbb{R}^n)_o$ whose Euclidean Spherical Fourier transforms $\mathscr{F}g$ are supported in $[0,2)$, we also have

$$\|g\|_{L^\infty(B(o,R))} \lesssim \|g\|_{H^{n/2}(\mathbb{R}^n)} .$$

Then an argument as in the proof of the sufficient conditions of Theorem 3.1.1 (subsection 3.4.3) involving smooth resolution of identity would give,

$$\|g\|_{L^\infty(B(o,R))} \lesssim \|g\|_{H^{n/2}(\mathbb{R}^n)} ,$$

for all g in $\mathscr{S}(\mathbb{R}^n)_o$. But this contradicts the failure of the local borderline fractional Sobolev embedding on $\mathbb{R}^n$ (for an explicit counter-example in the case of $\mathbb{R}^n$, we refer the reader to [Gr09, p. 9, Exercise 6.1.4]). This completes the proof. $\square$

Remark 3.5.1 In the special case of rank one Riemannian symmetric spaces of noncompact type G/K, one has the failure of the borderline fractional Sobolev embedding:

$$\|f\|_{L^\infty(G/K)} \lesssim \|f\|_{H^{n/2}(G/K)} , \tag{3.5.16}$$

by [CGM93, Theorem 4.7]. Our proof above indicates the failure of the more delicate local borderline fractional Sobolev embedding:

$$\|f\|_{L^\infty(B)} \lesssim \|f\|_{H^{n/2}(G/K)} ,$$

thus providing a refinement of (3.5.16).

3.6 Global estimates

In this section, we prove Theorems 3.1.3 and 3.1.4.

Proof: [Proof of Theorem 3.1.3] We first prove the positive result for $q = 2$. For $f \in \mathscr{S}^2(\mathbb{H}^3)_o$, we consider the linearized maximal function Tf (given by (3.4.1)) and its Euclidean counterpart

$T_0(\mathscr{A}f)$ (given by (3.4.7)) for $\mathscr{A}f \in \mathscr{S}(\mathbb{R}^3)_o$. For $0 < s < \infty$, we have

$$\begin{aligned}
Tf(s) &= \int_0^\infty \varphi_\lambda(s)\, e^{it(s)(\lambda^2+1)}\, \hat{f}(\lambda)\, \lambda^2\, d\lambda \\
&= \int_0^\infty \frac{\sin(\lambda s)}{\lambda \sinh(s)}\, e^{it(s)(\lambda^2+1)}\, \mathscr{F}(\mathscr{A}f)(\lambda)\, \lambda^2\, d\lambda \\
&= \left(\frac{e^{it(s)}s}{\sinh(s)}\right) \int_0^\infty \frac{\sin(\lambda s)}{\lambda s}\, e^{it(s)\lambda^2}\, \mathscr{F}(\mathscr{A}f)(\lambda)\, \lambda^2\, d\lambda \\
&= \left(\frac{e^{it(s)}s}{\sinh(s)}\right) \int_0^\infty \left(\sqrt{2}\,\Gamma\left(\frac{3}{2}\right) \frac{J_{\frac{1}{2}}(\lambda s)}{(\lambda s)^{\frac{1}{2}}}\right) e^{it(s)\lambda^2}\, \mathscr{F}(\mathscr{A}f)(\lambda)\, \frac{\lambda^2\, d\lambda}{\sqrt{2}\,\Gamma(\frac{3}{2})} \\
&= \left(\frac{e^{it(s)}s}{\sinh(s)}\right) T_0(\mathscr{A}f)(s)\,, \qquad (3.6.1)
\end{aligned}$$

and thus

$$\begin{aligned}
\|Tf\|_{L^2(\mathbb{H}^3)} = \left(\int_0^\infty |Tf(s)|^2\, \sinh^2 s\, ds\right)^{1/2} &= \left(\int_0^\infty |T_0(\mathscr{A}f)(s)|^2\, s^2\, ds\right)^{1/2} \\
&\asymp \|T_0(\mathscr{A}f)\|_{L^2(\mathbb{R}^3)}\,. \qquad (3.6.2)
\end{aligned}$$

Now the computations as in (3.4.10) yield,

$$\|f\|_{H^\beta(\mathbb{H}^3)} \asymp \|\mathscr{A}f\|_{H^\beta(\mathbb{R}^3)}\,,\ \ \beta \geq 0\,. \qquad (3.6.3)$$

Then combining (3.6.2), (3.6.3) and Sjölin's Euclidean result (3.1.2), we get the positive result on $\mathbb{H}^3$ for $q = 2$.

To get the failure for $q < 2$, using (3.6.1), we note

$$\begin{aligned}
\|Tf\|_{L^q(\mathbb{H}^3)} &= \left(\int_0^\infty |Tf(s)|^q\, \sinh^2 s\, ds\right)^{1/q} \\
&= \left(\int_0^\infty |T_0(\mathscr{A}f)(s)|^q\, s^2 \left(\frac{\sinh s}{s}\right)^{2-q} ds\right)^{1/q} \\
&\geq \left(\int_0^\infty |T_0(\mathscr{A}f)(s)|^q\, s^2\, ds\right)^{1/q} \\
&\gtrsim \|T_0(\mathscr{A}f)\|_{L^q(\mathbb{R}^3)}\,. \qquad (3.6.4)
\end{aligned}$$

Thus in view of (3.6.3) and (3.6.4), if

$$\|Tf\|_{L^q(\mathbb{H}^3)} \lesssim \|f\|_{H^\beta(\mathbb{H}^3)}\,,$$

is valid for $\beta > 1/2$ and $q < 2$, then it will also be true for $\mathbb{R}^3$ which contradicts [Sj97, Theorem 4] . This completes the proof of Theorem 3.1.3 . □

Proof: [Proof of Theorem 3.1.4] We decompose S as

$$S = \overline{B}_{R_0} \cup \left(S \setminus \overline{B}_{R_0}\right) =: B \cup A\,,$$

where R_0 is as in Lemma 3.2.2 and then study the boundedness properties of the linearized maximal function Tf (given by (3.4.1)) on B and A separately. On B, our local estimates given by Lemmata 3.4.1 and 3.4.3, along with the resolution of identity argument employed above yields,

$$\|Tf\|_{L^2(B)} \lesssim \|f\|_{H^{1/2}(S)}\,. \tag{3.6.5}$$

On A for $\lambda > 0$, using the estimate (3.2.9) and the Cauchy-Schwarz inequality, we get for $s \in (R_0, \infty)$,

$$\begin{aligned}
|Tf(s)| &\lesssim e^{-\frac{Q}{2}s} \int_0^\infty |\mathbf{c}(\lambda)| \left|\widehat{f}(\lambda)\right| |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&\leq e^{-\frac{Q}{2}s} \left(\int_0^\infty \left(\lambda^2 + \frac{Q^2}{4}\right)^\beta |\widehat{f}(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda\right)^{1/2} \left(\int_0^\infty \frac{d\lambda}{\left(\lambda^2 + \frac{Q^2}{4}\right)^\beta}\right)^{1/2} \\
&= e^{-\frac{Q}{2}s} \|f\|_{H^\beta(S)} \left(\int_0^\infty \frac{d\lambda}{\left(\lambda^2 + \frac{Q^2}{4}\right)^\beta}\right)^{1/2}.
\end{aligned}$$

Thus for $\beta > 1/2$, we get for $s \in (R_0, \infty)$,

$$|Tf(s)| \lesssim e^{-\frac{Q}{2}s} \|f\|_{H^\beta(S)}\,. \tag{3.6.6}$$

Hence in view of (3.6.5) and (3.6.6), to complete the proof of Theorem 3.1.4, it suffices to prove that the function $F(s) := e^{-\frac{Q}{2}s} \chi_A(s)$ belongs to $L^{2,\infty}(A)$. To see this, we note that

$$d_F(t) \begin{cases} = 0\,, & \text{if } t \geq 1\,, \\ \lesssim \frac{1}{t^2}\,, & \text{if } 0 < t < 1\,, \end{cases}$$

and hence,

$$\|F\|_{L^{2,\infty}(A)} = \sup_{t>0} t\, d_F(t)^{1/2} \lesssim 1\,.$$

This completes the proof of Theorem 3.1.4. □

Chapter 4

Generalized Carleson's problem for radial initial data

4.1 Introduction

In this chapter, we consider the problem of (almost everywhere) pointwise convergence of the solution of the Schrödinger equation to the initial data, along a wider approach region, for instance, the non-tangential convergence.

For $\mathbb{R}^n$, this question regarding non-tangential convergence has already been addressed. On $\mathbb{R}^n$, for $\beta > n/2$, by Sobolev imbedding and standard arguments, the non-tangential convergence to the initial data holds. However, Sjögren and Sjölin showed in [SS89] that the non-tangential convergence fails for $\beta \leq n/2$. As their counter-example is a non-radial function, it is natural to ask when the initial data is more symmetric, say radial, then can we have pointwise convergence along wider approach regions. In this chapter, we first construct a radial initial data on the 3-dimensional Real Hyperbolic space $\mathbb{H}^3 \cong SL(2,\mathbb{C})/SU(2)$, whose Schrödinger propagation blows up on geodesic annuli along wide approach regions:

Theorem 4.1.1 *Assume that $\gamma : [0,\infty) \to [0,\infty)$ is a strictly increasing continuous function with $\gamma(0) = 0$. Then given any compact geodesic annulus $\mathscr{A} \subset \mathbb{H}^3$, centered at the identity, there exists radial $f \in H^{1/2}(\mathbb{H}^3)$ such that its Schrödinger propagation u is continuous in $\mathscr{A} \times (0,\infty)$ and*

$$\limsup_{\substack{(y,t)\to(x,0)\\ d(x,y)<\gamma(t),\, t>0}} |u(y,t)| = +\infty\,, \quad \textit{for all } x \in \mathscr{A}\,.$$

Theorem 4.1.1 shows that the solutions of the Schrödinger equation, unlike Harmonic functions or solutions of the Heat equation, do not admit a natural wide approach region. Thus it becomes interesting to study the pointwise convergence of solutions along appropriate curves and its connection with the regularity of the initial data.

In this light, we commence our exploration on Damek-Ricci spaces by identifying a natural family of curves to study. For $x \in S$, we consider curves $\gamma_x : \mathbb{R} \to S$ such that $\gamma_x(0) = x$. Now as we will be exclusively dealing with radial initial data, it is both natural and customary to identify all the points x on a geodesic sphere centered at the identity e of S, solely by their geodesic distance, say s from e, that is, $s = d(e, x) \in [0, \infty)$. Here $d(\cdot, \cdot)$ is the inner metric on S corresponding to the left-invariant Riemannian metric (see 3.2.2). Then the solution of the Schrödinger equation with initial data $f \in \mathscr{S}^2(S)_o$, along a curve γ_s is given by,

$$S_t f(\gamma_s(t)) = S_t f(d(e, \gamma_s(t))) = \int_0^\infty \varphi_\lambda(d(e, \gamma_s(t)))\, e^{it\left(\lambda^2 + \frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\,.$$

Again keeping the notion of radiality in mind, we consider curves that satisfy the following natural analogues of the conditions considered in [CLV12]: there exist constants $C_j\, (j = 1, 2, 3)$, independent of $s, s' \in [0, \infty)$, $t, t' \in \mathbb{R}$ and $\alpha \geq 0$, such that

$$(\mathscr{H}_1) \qquad \left| \, d(e, \gamma_s(t)) - d(e, \gamma_s(t')) \, \right| \leq C_1 \, |t - t'|^{\alpha} \, ,$$

$$(\mathscr{H}_2) \qquad C_2 \, |s - s'| \leq | \, d(e, \gamma_s(t)) - d(e, \gamma_{s'}(t)) \, | \leq C_3 \, |s - s'| \, .$$

We note that by the triangle inequality, $(\mathscr{H}_1)$ is weaker than the condition:

$$d\left(\gamma_s(t), \gamma_s(t')\right) \leq C_1 \, |t - t'|^{\alpha} \, ,$$

and hence α is essentially the degree of tangential convergence.

Clearly, the above curves generalize the classical case of vertical lines $\gamma_s(t) = s$, studied in [De25] and thus one may expect to require more regularity on the initial data to guarantee pointwise convergence. Our result (Corollary 4.1.3) however, shows that the regularity threshold $\beta \geq 1/4$ (as in the case of vertical lines) is again sufficient to obtain almost everywhere pointwise convergence, even along such general curves. In fact, we prove more and this brings us to the definition of the homogeneous Sobolev spaces. For $\beta \geq 0$, the homogeneous Sobolev spaces for the special case of radial functions (corresponding to the shifted Laplace-Beltrami operator

$\tilde{\Delta} := \Delta + \frac{Q^2}{4}$) are defined as,

$$\dot{H}^{\beta}(S) := \left\{ f \in L^2(S) : \|f\|_{\dot{H}^{\beta}(S)} := \left(\int_0^{\infty} \lambda^{2\beta} \left|\widehat{f}(\lambda)\right|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{1/2} < \infty \right\} .$$

We now state our first result dealing with maximal estimates on annuli:

Theorem 4.1.2 *Let $0 < r_1 < r_2 < \infty$. Assume that γ satisfies $(\mathscr{H}_1)$ for $\alpha \in [\frac{1}{2}, 1]$ and $(\mathscr{H}_2)$ for $s, s' \in [0, r_2)$ and $t, t' \in [-T, T]$, for some fixed*

$$0 < T < \left(\frac{C_2 r_1}{2C_1} \right)^{1/\alpha} . \tag{4.1.1}$$

Then we have for all $f \in \mathscr{S}^2(S)_o$,

$$\left(\int_{r_1}^{r_2} \left(\sup_{t \in [-T,T]} |S_t f(d(e, \gamma_s(t)))| \right)^2 A(s)\, ds \right)^{1/2} \leq C \, \|f\|_{\dot{H}^{1/4}(S)} , \tag{4.1.2}$$

where the positive constant C depends only on r_1, r_2 and the dimension of S.

The key idea in the proof of Theorem 4.1.2 is to keep track of the oscillation afforded by both the spherical functions as well as the multiplier corresponding to the Schrödinger operator. But this becomes a challenging task as in the generality of Damek-Ricci spaces, no explicit expression of φ_λ is known. In this regard, we crucially use the series expansion of φ_λ stated in Lemma 3.2.3 which is only valid in compacts not containing the group identity. This forces us to consider specific time localizations (4.1.1) and moreover, to take the rather unorthodox approach of obtaining maximal estimates on annuli, instead of balls. The oscillation of φ_λ is realized by looking at the leading terms in the series expansion. The corresponding oscillatory parts of the linearized maximal function are taken care of by proceeding with an abstract TT^* argument, followed by an oscillatory integral estimate and then finally applying the Pitt's inequality. The remaining part of the linearized maximal function is then dealt with by the pointwise decay of certain error terms appearing from the series expansion in Lemma 3.2.3, which turns out to be of critical importance.

By standard arguments in the literature (for instance, see the proof of Theorem 5 of [Sj87]), Theorem 4.1.2 yields pointwise convergence for solutions of the Schrödinger operator on annuli,

$$A(r_1, r_2) := \{ x \in S \mid r_1 < d(e, x) < r_2 \} ,$$

for all $0 < r_1 < r_2 < \infty$. Then as

$$A\left(\frac{1}{N}, N\right) \uparrow S \setminus \{e\}\,, \text{ as } N \to \infty\,,$$

by a simple application of the Monotone Convergence Theorem, we get the desired result on pointwise convergence:

Corollary 4.1.3 *Let $\alpha \in [\frac{1}{2}, 1]$. Suppose that for every $s_0 \in [0,\infty)$, there exists a neighborhood $V \subset [0,\infty) \times \mathbb{R}$ of $(s_0, 0)$ such that $(\mathscr{H}_1)$ holds for $(s,t),(s,t') \in V$ and $(\mathscr{H}_2)$ holds for all $(s,t),(s',t) \in V$. Then for all $f \in \dot{H}^{1/4}(S)_o$ and hence for all $f \in H^{\beta}(S)_o$, with $\beta \geq 1/4$, we have*

$$\lim_{t\to 0} S_t f(\gamma_s(t)) = f(s)\,,$$

for almost every $s \in [0,\infty)$, which also means

$$\lim_{t\to 0} S_t f(\gamma_x(t)) = f(x)\,,$$

for almost every $x \in S$, with respect to the left Haar measure.

Remark 4.1.4 The sharpness of the regularity threshold $\beta = 1/4$ in (4.1.2), follows from the sharpness in the special case of vertical lines (see the proof of Theorem 3.1.1, statement (i)).

We now return to $\mathbb{R}^n$, to address the issue raised in the Introduction about lowering the required regularity threshold while specializing to radial initial data, for $n \geq 2$. For a radial function f belonging to the Schwartz class $\mathscr{S}(\mathbb{R}^n)_o$, the solution of the Schrödinger equation on $\mathbb{R}^n$,

$$\begin{cases} i\frac{\partial u}{\partial t} = \Delta_{\mathbb{R}^n} u\,,\ (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(0,\cdot) = f\,, \ \text{ on } \mathbb{R}^n\,, \end{cases} \tag{4.1.3}$$

is given by

$$\tilde{S}_t f(s) := \int_0^{\infty} \mathscr{J}_{\frac{n-2}{2}}(\lambda s)\, e^{it\lambda^2}\, \mathscr{F} f(\lambda)\, \lambda^{n-1}\, d\lambda\,, \tag{4.1.4}$$

where we recall that $\mathscr{J}_{\frac{n-2}{2}}$ is the modified Bessel function of order $\frac{n-2}{2}$, $\mathscr{F} f$ is the Euclidean Spherical Fourier transform of f and $s = \|x\|$, is the distance of x from the origin o. In this regard, we reformulate the conditions $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$, in terms of the Euclidean norm $\|\cdot\|$: there exist constants C_j $(j = 4,5,6)$, independent of $s, s' \in [0,\infty)$, $t, t' \in \mathbb{R}$ and $\alpha \geq 0$, such that

$$(\mathscr{H}_3) \qquad \big| \, \|\gamma_s(t)\| - \|\gamma_s(t')\| \, \big| \leq C_4 \, |t - t'|^{\alpha} \, ,$$

$$(\mathscr{H}_4) \qquad C_5 \, |s - s'| \leq | \, \|\gamma_s(t)\| - \|\gamma_{s'}(t)\| \, | \leq C_6 \, |s - s'| \, .$$

It is interesting to observe that the arguments in the proof of Theorem 4.1.2 can be carried out analogously for the special case of $\mathbb{R}^3$, but doing the same in the full generality of $\mathbb{R}^n$ seems to be a rather difficult task (see Remark 4.3.3 for more details). Nevertheless, taking a detour using local geometry of Riemannian manifolds and Fourier Analytic tools on Homogeneous spaces, we obtain the following analogue of Theorem 4.1.2 for small annuli:

Theorem 4.1.5 *Let $0 < \delta_1 < \delta_2$ be sufficiently small. Assume that γ satisfies $(\mathscr{H}_3)$ for $\alpha \in [\frac{1}{2}, 1]$ and $(\mathscr{H}_4)$ for $s, s' \in [0, \delta_2)$ and $t, t' \in [-T, T]$ for some fixed*

$$0 < T < \left(\frac{C_5 \delta_1}{2 C_4} \right)^{1/\alpha} .$$

Then we have for all $f \in \mathscr{S}(\mathbb{R}^n)_o$,

$$\left(\int_{\delta_1}^{\delta_2} \left(\sup_{t \in [-T, T]} \left| \tilde{S}_t f(\|\gamma_s(t)\|) \right| \right)^2 s^{n-1} \, ds \right)^{1/2} \leq C \, \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)} \, , \tag{4.1.5}$$

where the positive constant C depends only on δ_1, δ_2 and n.

For the proof of Theorem 4.1.5, we first consider the cases when the Fourier transform of f is supported in a neighborhood of the origin and when it is supported away from the origin. In the first case, the linearized maximal function is quite straightforward to control. The latter case however, is quite involved. In the large frequency regime, viewing $\mathbb{R}^n$ as the tangent space at the identity of a suitable Damek-Ricci space, we form a connection with Theorem 4.1.2:

- Using the fact that the Riemannian exponential map on a connected, simply-connected, complete, non-positively curved Riemannian manifold is a bijective local radial isometry, we can pushforward the curves in the small annuli (in $\mathbb{R}^n$) to the non-flat manifold, preserving the local geometric conditions.

- The connection between the initial data and their Schrödinger propagations for $\mathbb{R}^n$ and S, is then obtained by means of Lemma 3.3.1.

Then Theorem 4.1.5 follows by taking a resolution of identity.

Remark 4.1.6 The sharpness of the regularity threshold $\beta = 1/4$ in (4.1.5), follows from the sharpness in the special case of vertical lines [Sj97, pp. 55-58].

As an application of Theorem 4.1.5, we look at the family of curves whose norms satisfy,

$$\|\gamma_s(t)\| = s + C_7 t^{1/2}\,, \tag{4.1.6}$$

for some $C_7 \geq 0$, in some neighborhood of every $(s_0, 0)$. Suitable dilations of annuli and the invariance of the norms of the curves (4.1.6) under a certain parabolic dilation in space and time, make it possible to apply Theorem 4.1.5 to obtain the following result on pointwise convergence, almost everywhere on $\mathbb{R}^n$:

Theorem 4.1.7 *Suppose that for every $s_0 \in [0,\infty)$, there exists a neighborhood $V \subset [0,\infty)\times\mathbb{R}$ of $(s_0, 0)$ such that (4.1.6) holds for all $(s,t) \in V$. Then for all $f \in \dot{H}^{1/4}(\mathbb{R}^n)_o$ and hence for all $f \in H^\beta(\mathbb{R}^n)_o$, with $\beta \geq 1/4$, we have*

$$\lim_{t\to 0} S_t f(\gamma_s(t)) = f(s)\,,$$

for almost every $s \in [0,\infty)$, which also means

$$\lim_{t\to 0} S_t f(\gamma_x(t)) = f(x)\,,$$

for almost every $x \in \mathbb{R}^n$.

Remark 4.1.8 Our Euclidean results (Theorems 4.1.5 and 4.1.7) can be interpreted as variations of results in [CM23] and [Mi24], as imposing more symmetry on the intial data allows us to work with much lower regularity.

This chapter is organized as follows. In section 4.2, we prove Theorem 4.1.1. In section 4.3 Theorem 4.1.2 is proved. In section 4.4 Theorems 4.1.5 and 4.1.7 are proved.

4.2 A counter-example for wide approach regions

In this section, we present the proof of Theorem 4.1.1.

Proof: [Proof of Theorem 4.1.1] The compact geodesic annulus $\mathscr{A}$ is of the form,

$$\mathscr{A} = \left\{ x \in \mathbb{H}^3 \mid c_1 \leq d(e,x) \leq c_2 \right\} \,,$$

for some positive constants $c_1 < c_2$. Then depending on γ, c_1 and c_2, there exist positive constants c_3, c_4, c_5 such that $c_3 \in (0,1)$ and

$$0 < c_4 < c_1 < c_2 < c_5 \,,$$

so that

$$\bigcup_{x \in \mathscr{A}} \left\{ y \in \mathbb{H}^3 \mid d(x,y) < \gamma(t) \,,\; 0 < t < c_3 \right\} \subset \left\{ z \in \mathbb{H}^3 \mid c_4 \leq d(e,z) \leq c_5 \right\} =: \mathscr{A}' \,.$$

Let $\{x_j\}_{j=1}^{\infty}$ be a countable dense subset of $\mathscr{A}'$ and $\{t_j\}_{j=1}^{\infty}$ be any sequence of positive real numbers such that

$$c_3 > t_1 > t_2 > \cdots > 0 \,, \quad \text{and } t_j \downarrow 0 \,.$$

We claim that given any $x \in \mathscr{A}$, there exists a subsequence $\{(x_{j_l}, t_{j_l})\}_{l=1}^{\infty}$ such that

$$(x_{j_l}, t_{j_l}) \to (x, 0) \text{ as } l \to \infty \,, \text{ while } d(x, x_{j_l}) < \gamma(t_{j_l}) \,. \tag{4.2.1}$$

Indeed, fixing $x \in \mathscr{A}$, we set $t_{j_1} := t_1$ and then note that there exists $x_{j_1} (\neq x) \in B_{\gamma(t_{j_1})}(x)$, the geodesic ball with center x and radius $\gamma(t_{j_1})$. Then as γ is continuous and strictly increasing, there exists $t'_{j_1} \in (0, t_{j_1})$ such that $\gamma(t'_{j_1}) = d(x, x_{j_1})$. We now set $t_{j_2} := \max\{t_j < t'_{j_1}\}$ and again note that there exists $x_{j_2} (\neq x) \in B_{\gamma(t_{j_2})}(x)$. Then again as γ is continuous and strictly increasing, there exists $t'_{j_2} \in (0, t_{j_2})$ such that $\gamma(t'_{j_2}) = d(x, x_{j_2})$. Proceeding like this, we get a subsequence $\{(x_{j_l}, t_{j_l})\}_{l=1}^{\infty}$ such that

$$d(x, x_{j_l}) < \gamma(t_{j_l}) \,,$$

and combining it with the fact that $t_{j_l} \downarrow 0$, it follows that

$$(x_{j_l}, t_{j_l}) \to (x, 0) \text{ as } l \to \infty \,.$$

This completes the proof of the claim (4.2.1).

We next let $\{r_j\}_{j=1}^{\infty}$ and $\{R_j\}_{j=1}^{\infty}$ be two sequences such that

$$3 = r_1 < R_1 < r_2 < R_2 < \dots ,$$

and set

$$\Omega_j := \{\lambda \in (0,\infty) \mid r_j < \lambda < R_j\} .$$

Corresponding to the enumeration $\{x_j\}_{j=1}^{\infty}$, we consider radial $f \in L^2(\mathbb{H}^3)$, with the spherical Fourier transform

$$\widehat{f}(\lambda) = \sum_{j=1}^{\infty} \chi_{\Omega_j}(\lambda)\, \lambda^{-1}\, (\log \lambda)^{-3/4}\, \varphi_\lambda(s_j)\, e^{-it_j(\lambda^2+\rho^2)} ,$$

where $s_j = d(e, x_j)$. Then using the expression of φ_λ and the fact that $s_j \geq c_4$ (as $x_j \in \mathscr{A}'$), for all j, we get that

$$\|f\|^2_{H^{1/2}(\mathbb{H}^3)} = \int_0^{\infty} \left(\lambda^2+\rho^2\right)^{1/2} \left|\widehat{f}(\lambda)\right|^2 \lambda^2\, d\lambda \lesssim \int_3^{\infty} \frac{d\lambda}{\lambda(\log\lambda)^{3/2}} < +\infty .$$

Therefore $f \in H^{1/2}(\mathbb{H}^3)$. We now make our choice of $\{r_j\}_{j=1}^{\infty}$ and $\{R_j\}_{j=1}^{\infty}$ more precise. We first choose $r_1 = 3$, $R_1 = 5$. Given $r_1, R_1, \dots, r_{j-1}, R_{j-1}$, we then choose $r_j > R_{j-1}$ such that for $k < j$, one has for some $c_6 > 0$,

$$r_j^2 \geq \frac{c_6 2^j}{t_k - t_j} , \tag{4.2.2}$$

and

$$(t_k - t_j) r_j > s_j + s_k + 1 . \tag{4.2.3}$$

We also set $R_j = r_j^M$, where $M > 0$ is chosen large. For $m \in \mathbb{N}$, now let

$$u_m(s,t) := \int_3^{R_m} \varphi_\lambda(s)\, e^{it(\lambda^2+\rho^2)}\, \widehat{f}(\lambda)\, \lambda^2\, d\lambda .$$

Now using the explicit expression of φ_λ, we have for all $(x,t) \in \mathscr{A}' \times (0,\infty)$,

$$\begin{aligned}
u_m(x,t) = u_m(s,t) &= \sum_{j=1}^{m} \int_{\Omega_j} \lambda^{-1} (\log\lambda)^{-3/4} \varphi_\lambda(s)\, \varphi_\lambda(s_j)\, e^{i(t-t_j)(\lambda^2+\rho^2)} \lambda^2 \, d\lambda \\
&= -\frac{1}{4}\sum_{j=1}^{m} \frac{e^{i(t-t_j)\rho^2}}{\sinh(s)\sinh(s_j)} \int_{\Omega_j} \lambda^{-1} (\log\lambda)^{-3/4} e^{i\lambda(s+s_j)} e^{i(t-t_j)\lambda^2} d\lambda \\
&\quad +\frac{1}{4}\sum_{j=1}^{m} \frac{e^{i(t-t_j)\rho^2}}{\sinh(s)\sinh(s_j)} \int_{\Omega_j} \lambda^{-1} (\log\lambda)^{-3/4} e^{i\lambda(s-s_j)} e^{i(t-t_j)\lambda^2} d\lambda \\
&\quad +\frac{1}{4}\sum_{j=1}^{m} \frac{e^{i(t-t_j)\rho^2}}{\sinh(s)\sinh(s_j)} \int_{\Omega_j} \lambda^{-1} (\log\lambda)^{-3/4} e^{-i\lambda(s-s_j)} e^{i(t-t_j)\lambda^2} d\lambda \\
&\quad -\frac{1}{4}\sum_{j=1}^{m} \frac{e^{i(t-t_j)\rho^2}}{\sinh(s)\sinh(s_j)} \int_{\Omega_j} \lambda^{-1} (\log\lambda)^{-3/4} e^{-i\lambda(s+s_j)} e^{i(t-t_j)\lambda^2} d\lambda \\
&= u_{m,1}(s,t) + u_{m,2}(s,t) + u_{m,3}(s,t) + u_{m,4}(s,t)\,. \qquad (4.2.4)
\end{aligned}$$

For each $k \in \mathbb{N}$ and $m > k$, our goal is to estimate $|u_m(x_k,t_k)|$. We first focus on

$$\begin{aligned}
u_{m,1}(s_k,t_k) &= -\frac{1}{4}\sum_{j=1}^{m} \frac{e^{i(t_k-t_j)\rho^2}}{\sinh(s_k)\sinh(s_j)} A_j(s_k,t_k) \\
&= -\frac{1}{4}\sum_{j=1}^{k-1} \frac{e^{i(t_k-t_j)\rho^2}}{\sinh(s_k)\sinh(s_j)} A_j(s_k,t_k) - \frac{A_k(s_k,t_k)}{4\sinh(s_k)^2} \\
&\quad -\frac{1}{4}\sum_{j=k+1}^{m} \frac{e^{i(t_k-t_j)\rho^2}}{\sinh(s_k)\sinh(s_j)} A_j(s_k,t_k)\,,
\end{aligned}$$

where

$$A_j(s_k,t_k) = \int_{\Omega_j} \lambda^{-1} (\log\lambda)^{-3/4} e^{i\lambda(s_k+s_j)} e^{i(t_k-t_j)\lambda^2} d\lambda\,. \qquad (4.2.5)$$

As $s_j, s_k \geq c_4$, for all j,k, we first have

$$\begin{aligned}
\left|\sum_{j=1}^{k-1} \frac{e^{i(t_k-t_j)\rho^2}}{\sinh(s_k)\sinh(s_j)} A_j(s_k,t_k)\right| \lesssim \sum_{j=1}^{k-1}\int_{\Omega_j} \lambda^{-1} (\log\lambda)^{-3/4} d\lambda &\leq \int_3^{R_{k-1}} \lambda^{-1} (\log\lambda)^{-3/4} d\lambda \\
&\lesssim (\log r_k)^{1/4}\,.
\end{aligned}$$

Next, an integration by parts and an application of the Fundamental theorem of calculus along with the definition of Ω_k yield

$$\begin{aligned}
|A_k(s_k,t_k)| &= \left|\int_{\Omega_k} \lambda^{-1} (\log\lambda)^{-3/4} e^{2i\lambda s_k}\, d\lambda\right| \\
&\lesssim r_k^{-1} (\log r_k)^{-3/4} + \int_{\Omega_k} \left|\frac{d}{d\lambda}\left(\lambda^{-1} (\log\lambda)^{-3/4}\right)\right| d\lambda \\
&= 2r_k^{-1} (\log r_k)^{-3/4} - R_k^{-1} (\log R_k)^{-3/4} \\
&\lesssim r_k^{-1} (\log r_k)^{-3/4} .
\end{aligned}$$

For $j > k$, we note that

$$A_j(s_k,t_k) = \int_{\Omega_j} \lambda^{-1} (\log\lambda)^{-3/4} e^{iF_1(\lambda)}\, d\lambda ,$$

where,

$$F_1(\lambda) = (s_k + s_j)\lambda + (t_k - t_j)\lambda^2 .$$

Thus using the assumption that $\{t_j\}$ is monotonically decreasing, it follows that

$$F_1'(\lambda) = s_k + s_j + 2(t_k - t_j)\lambda \geq (t_k - t_j) r_j , \tag{4.2.6}$$

$$F_1''(\lambda) = 2(t_k - t_j) . \tag{4.2.7}$$

Now an integration by parts yields

$$\begin{aligned}
&\int_{\Omega_j} \lambda^{-1} (\log\lambda)^{-3/4} e^{iF_1(\lambda)}\, d\lambda \\
= &\int_{r_j}^{R_j} \frac{1}{\lambda (\log\lambda)^{3/4} iF_1'(\lambda)} iF_1'(\lambda)\, e^{iF_1(\lambda)}\, d\lambda \\
= &\left[\frac{1}{\lambda (\log\lambda)^{3/4} iF_1'(\lambda)}\, e^{iF_1(\lambda)}\right]_{r_j}^{R_j} - \int_{r_j}^{R_j} \frac{d}{d\lambda}\left(\frac{1}{\lambda (\log\lambda)^{3/4} iF_1'(\lambda)}\right) e^{iF_1(\lambda)}\, d\lambda \\
= &\ I - J .
\end{aligned}$$

Now invoking (4.2.2) and (4.2.6), it follows that

$$|I| \leq \frac{2}{(t_k - t_j) r_j^2} \leq \left(\frac{2}{c_6}\right) 2^{-j} .$$

Using (4.2.6) and (4.2.7), we also have

$$\left|\frac{d}{d\lambda}\left(\frac{1}{\lambda\,(\log\lambda)^{3/4} iF_1'(\lambda)}\right)\right| \le \frac{2}{\lambda^2\,|F_1'(\lambda)|} + \frac{|F_1''(\lambda)|}{\lambda|F_1'(\lambda)|^2} \le \frac{4}{(t_k-t_j)\lambda^3}\,.$$

Hence, again using (4.2.2)

$$|J| \le \frac{2}{(t_k-t_j)r_j^2} \le \left(\frac{2}{c_6}\right)2^{-j}\,.$$

Thus setting, $c_7 = 4/c_6$, we note that for $j > k$,

$$|A_j(s_k,t_k)| \le c_7 2^{-j}\,.$$

Therefore for $m > k$,

$$|u_{m,1}(s_k,t_k)| \lesssim (\log r_k)^{1/4} + r_k^{-1}\,(\log r_k)^{-3/4} + \sum_{j=k+1}^{m} 2^{-j} \lesssim (\log r_k)^{1/4}\,.$$

Similarly, we get that

$$|u_{m,4}(s_k,t_k)| \lesssim (\log r_k)^{1/4}\,.$$

We now consider

$$u_{m,2}(s_k,t_k) = \frac{1}{4}\sum_{j=1}^{m}\frac{e^{i(t_k-t_j)\rho^2}}{\sinh(s_k)\sinh(s_j)}B_j(s_k,t_k)\,,$$

where B_j is defined in an analogous fashion as A_j (see (4.2.5)). Then again proceeding as in the case of $u_{m,1}$, we get that

$$\left|\sum_{j=1}^{k-1}\frac{e^{i(t_k-t_j)\rho^2}}{\sinh(s_k)\sinh(s_j)}B_j(s_k,t_k)\right| \;\lesssim\; (\log r_k)^{1/4}\,.$$

Next, as $R_j = r_j^M$, we see that for $M > 0$ sufficiently large,

$$\begin{aligned}
B_k(s_k,t_k) &= \int_{r_k}^{R_k}\lambda^{-1}\,(\log\lambda)^{-3/4}\,d\lambda\\
&= 4\left((\log R_k)^{1/4} - (\log r_k)^{1/4}\right)\\
&= 4\left(1-\frac{1}{M^{1/4}}\right)(\log R_k)^{1/4}\\
&\gtrsim (\log R_k)^{1/4}\,.
\end{aligned}$$

For $j > k$, we note that

$$B_j(s_k, t_k) = \int_{\Omega_j} \lambda^{-1} (\log \lambda)^{-3/4} e^{iF_2(\lambda)} \, d\lambda \,,$$

where,

$$F_2(\lambda) = (s_k - s_j)\lambda + (t_k - t_j)\lambda^2 \,.$$

Thus

$$\begin{aligned} F_2'(\lambda) &= (s_k - s_j) + 2(t_k - t_j)\lambda \,, \\ F_2''(\lambda) &= 2(t_k - t_j) \,. \end{aligned}$$

So for $j > k$, either $s_k = s_j$ or $s_k \neq s_j$. But in either case using (4.2.3) and $\lambda \in \Omega_j$, we get

$$\left| F_2'(\lambda) \right| > |t_k - t_j|\lambda > |t_k - t_j| r_j \,.$$

Then again proceeding as before using the above inequality and (4.2.2), we get that

$$|B_j(s_k, t_k)| \leq c_7 2^{-j} \,.$$

Similarly, writing

$$u_{m,3}(s_k, t_k) = \frac{1}{4} \sum_{j=1}^{m} \frac{e^{i(t_k - t_j)\rho^2}}{\sinh(s_k) \sinh(s_j)} D_j(s_k, t_k) \,,$$

where D_j is defined in an analogous fashion as B_j, we would get that

$$\begin{aligned} &\left| \sum_{j=1}^{k-1} \frac{e^{i(t_k - t_j)\rho^2}}{\sinh(s_k) \sinh(s_j)} D_j(s_k, t_k) \right| \lesssim (\log r_k)^{1/4} \,, \\ &D_k(s_k, t_k) \gtrsim (\log R_k)^{1/4} \,, \\ &|D_j(s_k, t_k)| \leq c_7 2^{-j} \,, \text{ for } j > k \,. \end{aligned}$$

Thus for $m > k$ and $M > 0$ sufficiently large, using the decomposition (4.2.4), the fact that all $s_j, s_k \in [c_4, c_5]$ and the above estimates, we get that

$$\begin{aligned}
|u_m(x_k,t_k)| &\geq |u_{m,2}(s_k,t_k)+u_{m,3}(s_k,t_k)| - |u_{m,1}(s_k,t_k)| - |u_{m,4}(s_k,t_k)| \\
&\geq \frac{1}{4\sinh^2(s_k)}\left(B_k(s_k,t_k)+D_k(s_k,t_k)\right) - \frac{1}{4}\left|\sum_{j=1}^{k-1}\frac{e^{i(t_k-t_j)\rho^2}}{\sinh(s_k)\sinh(s_j)}B_j(s_k,t_k)\right| \\
&\quad -\frac{1}{4}\left|\sum_{j=1}^{k-1}\frac{e^{i(t_k-t_j)\rho^2}}{\sinh(s_k)\sinh(s_j)}D_j(s_k,t_k)\right| - \frac{1}{4}\left|\sum_{j=k+1}^{m}\frac{e^{i(t_k-t_j)\rho^2}}{\sinh(s_k)\sinh(s_j)}B_j(s_k,t_k)\right| \\
&\quad -\frac{1}{4}\left|\sum_{j=k+1}^{m}\frac{e^{i(t_k-t_j)\rho^2}}{\sinh(s_k)\sinh(s_j)}D_j(s_k,t_k)\right| - |u_{m,1}(s_k,t_k)| - |u_{m,4}(s_k,t_k)| \\
&\gtrsim (\log R_k)^{1/4} - (\log r_k)^{1/4} - 1 \\
&\gtrsim (\log R_k)^{1/4}\,.
\end{aligned} \tag{4.2.8}$$

We now show that for all $m \in \mathbb{N}$, u_m is continuous on $\mathscr{A}' \times (0,\infty)$. For $(s,t) \in [c_4,c_5] \times (0,\infty)$, let $(\varepsilon, h) \in \mathbb{R}^2$ be such that $s+\varepsilon \geq c_4/2$. We write,

$$|u_m(s+\varepsilon,t+h) - u_m(s,t)| \leq |u_m(s+\varepsilon,t+h) - u_m(s,t+h)| + |u_m(s,t+h) - u_m(s,t)|\ .$$

Then as $[3, R_m]$ is compact, by the Dominated Convergence Theorem, we get that

$$|u_m(s+\varepsilon,t+h) - u_m(s,t+h)| \leq \int_3^{R_m} |\varphi_\lambda(s+\varepsilon) - \varphi_\lambda(s)| \left|\widehat{f}(\lambda)\right| \lambda^2\, d\lambda \to 0 \text{ as } |\varepsilon| \to 0\,,$$

and also

$$|u_m(s,t+h) - u_m(s,t)| \leq \int_3^{R_m} \left|e^{ih(\lambda^2+\rho^2)} - 1\right| \left|\widehat{f}(\lambda)\right| \lambda^2\, d\lambda \to 0 \text{ as } |h| \to 0\,.$$

We now finally come to the Schrödinger propagation of f on $\mathbb{H}^3$, which we denote by u. Now if we can show that $u_m \to u$ locally uniformly on $\mathscr{A}' \times (0,\infty)$, then it follows that u is also continuous on $\mathscr{A}' \times (0,\infty)$. To see the local uniform convergence, let L be a compact subset of $\mathscr{A}' \times (0,\infty)$. We now claim that there exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$ and all $(x,t) \in L$,

$$r_j^2 \geq \frac{c_6 2^j}{|t-t_j|}\,, \tag{4.2.9}$$

and

$$|t-t_j| r_j > s_j + s + 1\,, \tag{4.2.10}$$

which basically means that (s_k, t_k) in (4.2.2) and (4.2.3) can be replaced by (s,t) corresponding to $(x,t) \in L$, when j is large enough. To see the above claims, we note that by compactness of L, there exists $j' \in \mathbb{N}$ such that for all $j \geq j'$,

$$t_j < \inf \{t | (x,t) \in L\} .$$

Then as the sequence $\{t_j\}_{j=1}^{\infty}$ is monotonically decreasing, combining it with (4.2.2), we get that for all $j > j'$ and all $(x,t) \in L$,

$$|t - t_j| > |t_{j'} - t_j| \geq \frac{c_6 2^j}{r_j^2} .$$

and also

$$\frac{|t - t_j|}{s_j + s + 1} > \frac{t - t_{j'}}{2c_5 + 1} .$$

Now as $r_j \uparrow \infty$, there exists $j'' \in \mathbb{N}$ such that for all $j > j''$,

$$\frac{t - t_{j'}}{2c_5 + 1} > \frac{1}{r_j} .$$

Thus setting $j_0 := \max\{j', j''\}$, the claims (4.2.9) and (4.2.10) are established. Then using (4.2.9) and (4.2.10), proceeding as before, we get that there exists $c_8 > 0$, such that for all $j > j_0$ and all $(x,t) \in L$,

$$\left| \int_{\Omega_j} \varphi_\lambda(s)\, e^{it(\lambda^2+\rho^2)}\, \widehat{f}(\lambda)\, \lambda^2\, d\lambda \right| \leq c_8 2^{-j} .$$

Thus for all $m > j_0$ and all $(x,t) \in L$,

$$\begin{aligned} |u_m(x,t) - u(x,t)| &= \left| \int_{R_m}^{\infty} \varphi_\lambda(s)\, e^{it(\lambda^2+\rho^2)}\, \widehat{f}(\lambda)\, \lambda^2\, d\lambda \right| \\ &\leq \sum_{j=m+1}^{\infty} \left| \int_{\Omega_j} \varphi_\lambda(s)\, e^{it(\lambda^2+\rho^2)}\, \widehat{f}(\lambda)\, \lambda^2\, d\lambda \right| \\ &\leq c_8 \sum_{j=m+1}^{\infty} 2^{-j} = c_8 2^{-m} . \end{aligned}$$

This gives the desired local uniform convergence and consequently, the continuity of u on $\mathscr{A}' \times (0,\infty)$. Finally, for any (x_k, t_k) and sufficiently large m_k, we have by the above inequality and (4.2.8),

$$|u(x_k,t_k)| \geq |u_{m_k}(x_k,t_k)| - c_8 \gtrsim (\log R_k)^{1/4} .$$

Now choose and fix $x \in \mathscr{A}$. We then consider the subsequence $\{(x_{j_l}, t_{j_l})\}_{l=1}^{\infty}$ satisfying the property (4.2.1) and obtain

$$\limsup_{\substack{(y,t)\to(x,0)\\ d(x,y)<\gamma(t),\, t>0}} |u(y,t)| \geq \lim_{l\to\infty} |u(x_{j_l}, t_{j_l})| \gtrsim \lim_{l\to\infty} (\log R_{j_l})^{1/4} = +\infty\,.$$

As $x \in \mathscr{A}$ was arbitrary, this completes the proof of Theorem 4.1.1. $\square$

4.3 Results on Damek-Ricci spaces

In this section, we aim to prove Theorem 4.1.2. We first prove the following oscillatory integral estimate:

Lemma 4.3.1 *Let $0 < \delta_1 < \delta_2 < \infty$. Let $\kappa(\cdot,\cdot) : [0,\infty) \times \mathbb{R} \to [0,\infty)$ satisfy*

$$|\kappa(s,t) - \kappa(s,t')| \lesssim |t - t'|^{\alpha}\,,$$
$$|\kappa(s,t) - \kappa(s',t)| \asymp |s - s'|\,,$$

for $\alpha \in [\frac{1}{2}, 1]$, $s, s' \in [0, \delta_2)$ and $t, t' \in [-T, T]$ with some fixed $T > 0$. Assume that $t(\cdot) : (\delta_1, \delta_2) \to [-T, T]$ is a measurable function. If $\mu \in C_c^{\infty}[0,\infty)$, then for any $C > 0$,

$$\left| \int_0^{\infty} e^{i\left\{\lambda(\kappa(s',t(s'))-\kappa(s,t(s)))+(t(s')-t(s))\left(\lambda^2+C\right)\right\}}\, \lambda^{-\frac{1}{2}}\, \mu\left(\frac{\lambda}{N}\right) d\lambda \right| \lesssim \frac{1}{|s-s'|^{1/2}}\,,$$

for all $s, s' \in (\delta_1, \delta_2)$ and $N \in \mathbb{N}$. The implicit constant in the conclusion depends only on μ.

Proof: For $s = s'$, the estimate is trivial. For $s \neq s'$, to prove the Lemma, it suffices to prove that

$$\left| \int_0^{\frac{1}{|s-s'|}} e^{i\left\{\lambda(\kappa(s',t(s'))-\kappa(s,t(s)))+(t(s')-t(s))\left(\lambda^2+C\right)\right\}}\, \lambda^{-\frac{1}{2}}\, \mu\left(\frac{\lambda}{N}\right) d\lambda \right| \lesssim \frac{1}{|s-s'|^{1/2}}\,, \tag{4.3.1}$$

and

$$\left| \int_{\frac{1}{|s-s'|}}^{\infty} e^{i\left\{\lambda(\kappa(s',t(s'))-\kappa(s,t(s)))+(t(s')-t(s))\left(\lambda^2+C\right)\right\}}\, \lambda^{-\frac{1}{2}}\, \mu\left(\frac{\lambda}{N}\right) d\lambda \right| \lesssim \frac{1}{|s-s'|^{1/2}}\,, \tag{4.3.2}$$

with the implicit constants possibly depending only on μ. The proof of (4.3.2) is given in [DN17, proof of Eqn. (3.6), pp. 233-239]. On the other hand, the proof of (4.3.1) is simple:

$$\begin{aligned}\left|\int_0^{\frac{1}{|s-s'|}} e^{i\left\{\lambda(\kappa(s',t(s'))-\kappa(s,t(s)))+(t(s')-t(s))\left(\lambda^2+C\right)\right\}}\,\lambda^{-\frac{1}{2}}\,\mu\left(\frac{\lambda}{N}\right)\,d\lambda\right| &\lesssim \int_0^{\frac{1}{|s-s'|}} \lambda^{-1/2}\,d\lambda \\ &= \frac{2}{|s-s'|^{1/2}}\,.\end{aligned}$$

This completes the proof of Lemma 4.3.1. □

We also require the following lemma:

Lemma 4.3.2 *Under the hypothesis of Theorem 4.1.2, we have for all $s \in (r_1, r_2)$ and $t \in [-T, T]$,*

$$\frac{C_2}{2}s \;<\; d(e,\gamma_s(t)) \;<\; \frac{3C_3}{2}s\,.$$

Proof: We first note by $(\mathscr{H}_2)$ that

$$C_2 s \;\le\; |d(e,\gamma_s(t)) - d(e,\gamma_0(t))| \;\le\; C_3 s\,. \tag{4.3.3}$$

Next by $(\mathscr{H}_1)$, we have

$$d(e,\gamma_0(t)) \;=\; |d(e,\gamma_0(t)) - d(e,\gamma_0(0))| \;\le\; C_1|t|^\alpha \;\le\; C_1 T^\alpha\,. \tag{4.3.4}$$

Therefore, combining (4.3.3) and (4.3.4), we get that

$$C_2 s - C_1 T^\alpha \;\le\; d(e,\gamma_s(t)) \;\le\; C_3 s + C_1 T^\alpha\,.$$

Now as $0 < T < \left(\frac{C_2 r_1}{2C_1}\right)^{1/\alpha}$, we see that

$$C_2 s - C_1 T^\alpha > C_2\left(s - \frac{r_1}{2}\right) > \frac{C_2}{2}s\,.$$

Similarly,

$$C_3 s + C_1 T^\alpha < C_3 s + C_2\left(\frac{r_1}{2}\right) \le C_3\left(s + \frac{r_1}{2}\right) < \frac{3C_3}{2}s\,.$$

This completes the proof of Lemma 4.3.2. □

We now present the proof of Theorem 4.1.2.

Proof: [Proof of Theorem 4.1.2] In order to prove the theorem, it suffices to prove the following estimate in terms of the linearized maximal function,

$$\left(\int_{r_1}^{r_2} |T_\gamma f(s)|^2 A(s)\, ds\right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(S)}\,, \tag{4.3.5}$$

where,

$$T_\gamma f(s) := \int_0^\infty \varphi_\lambda(d(e,\gamma_s(t(s))))\, e^{it(s)\left(\lambda^2+\frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\,, \tag{4.3.6}$$

and $t(\cdot) : (r_1, r_2) \to [-T, T]$ is a measurable function. Now by Lemma 4.3.2, we have for all $s \in (r_1, r_2)$,

$$\frac{C_2}{2} r_1 < \frac{C_2}{2} s < d(e,\gamma_s(t(s))) < \frac{3C_3}{2} s < \frac{3C_3}{2} r_2\,.$$

Then for $s \in (r_1, r_2)$ and $\lambda > 0$, expanding φ_λ by Lemma 3.2.3, we get that

$$\begin{aligned} \varphi_\lambda(d(e,\gamma_s(t(s)))) &= 2^{-m_{\mathfrak{z}}/2} A(d(e,\gamma_s(t(s))))^{-1/2} \left\{\mathbf{c}(\lambda) e^{i\lambda d(e,\gamma_s(t(s)))} + \mathbf{c}(-\lambda) e^{-i\lambda d(e,\gamma_s(t(s)))}\right\} \\ &\quad + \mathscr{E}(\lambda, d(e,\gamma_s(t(s))))\,, \end{aligned} \tag{4.3.7}$$

where

$$\begin{aligned} \mathscr{E}(\lambda, d(e,\gamma_s(t(s)))) &= 2^{-m_{\mathfrak{z}}/2} A(d(e,\gamma_s(t(s))))^{-1/2} \\ &\quad \times \left\{\mathbf{c}(\lambda)\sum_{\mu=1}^{\infty}\Gamma_\mu(\lambda) e^{(i\lambda-\mu)d(e,\gamma_s(t(s)))} + \mathbf{c}(-\lambda)\sum_{\mu=1}^{\infty}\Gamma_\mu(-\lambda) e^{-(i\lambda+\mu)d(e,\gamma_s(t(s)))}\right\}, \end{aligned}$$

and thus again using Lemma 4.3.2, the local growth asymptotics of the density function (3.2.3) and the estimate (3.2.8) on the coefficients Γ_μ, it follows that

$$|\mathscr{E}(\lambda, d(e,\gamma_s(t(s))))| \lesssim A(s)^{-1/2}\, |\mathbf{c}(\lambda)|\, (1+\lambda)^{-1}\,. \tag{4.3.8}$$

Hence for $s \in (r_1, r_2)$ and $\lambda > 0$, in accordance to (4.3.7), we decompose T_γ as,

$$\begin{aligned} & T_\gamma f(s) \\ = \; & 2^{-m_{\mathfrak{z}}/2} A(d(e,\gamma_s(t(s))))^{-1/2} \int_0^\infty \mathbf{c}(\lambda)\, e^{i\left\{\lambda d(e,\gamma_s(t(s))) + t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ & + 2^{-m_{\mathfrak{z}}/2} A(d(e,\gamma_s(t(s))))^{-1/2} \int_0^\infty \mathbf{c}(-\lambda)\, e^{i\left\{-\lambda d(e,\gamma_s(t(s))) + t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ & + \int_0^\infty \mathscr{E}(\lambda, d(e,\gamma_s(t(s))))\, e^{it(s)\left(\lambda^2+\frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ = \; & T_{\gamma,1} f(s) \,+\, T_{\gamma,2} f(s) \,+\, T_{\gamma,3} f(s)\,. \end{aligned}$$

The arguments for $T_{\gamma,1}$ and $T_{\gamma,2}$ are similar and hence we work out the details only for $T_{\gamma,1}$. We aim to show that

$$\left(\int_{r_1}^{r_2} |T_{\gamma,1}f(s)|^2 A(s)\, ds\right)^{1/2} \lesssim \left(\int_0^\infty \lambda^{1/2}\, |\widehat{f}(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda\right)^{1/2}. \tag{4.3.9}$$

Now

$$\begin{aligned}
& T_{\gamma,1}f(s)\, A(s)^{1/2} \\
= \;& 2^{-m_{\mathfrak{z}}/2} A(d(e,\gamma_s(t(s))))^{-1/2} A(s)^{1/2} \int_0^\infty \mathbf{c}(\lambda) e^{i\left\{\lambda d(e,\gamma_s(t(s)))+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}} \widehat{f}(\lambda) |\mathbf{c}(\lambda)|^{-2} d\lambda \\
= \;& 2^{-m_{\mathfrak{z}}/2} A(d(e,\gamma_s(t(s))))^{-1/2} A(s)^{1/2} \int_0^\infty \mathbf{c}(\lambda) e^{i\left\{\lambda d(e,\gamma_s(t(s)))+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}} g(\lambda) \lambda^{-1/4} |\mathbf{c}(\lambda)|^{-1} d\lambda,
\end{aligned}$$

where

$$g(\lambda) = \widehat{f}(\lambda)\, \lambda^{1/4}\, |\mathbf{c}(\lambda)|^{-1}\,.$$

Then setting,

$$P_\gamma g(s) := A(d(e,\gamma_s(t(s))))^{-1/2} A(s)^{1/2} \int_0^\infty \mathbf{c}(\lambda) e^{i\left\{\lambda d(e,\gamma_s(t(s))))+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}} g(\lambda) \lambda^{-1/4} |\mathbf{c}(\lambda)|^{-1} d\lambda\,,$$

that is,

$$2^{m_{\mathfrak{z}}/2}\, T_{\gamma,1}f(s)\, A(s)^{1/2} = P_\gamma g(s)\,,$$

we note that proving (4.3.9) is equivalent to proving

$$\left(\int_{r_1}^{r_2} |P_\gamma g(s)|^2\, ds\right)^{1/2} \lesssim \left(\int_0^\infty |g(\lambda)|^2\, d\lambda\right)^{1/2}. \tag{4.3.10}$$

We choose $\rho \in C_c^\infty[0,\infty)$ non-negative such that

$$\begin{aligned}
\rho(\lambda) &= 1\,,\quad \lambda \leq 1 \\
&= 0\,,\quad \lambda \geq 2\,.
\end{aligned}$$

For $N > 2$ and $s \in (r_1, r_2)$, we set

$$\begin{aligned}
P_{\gamma,N}g(s) \;:=\;& A(d(e,\gamma_s(t(s))))^{-1/2} A(s)^{1/2} \\
& \times \int_0^\infty \mathbf{c}(\lambda) e^{i\left\{\lambda d(e,\gamma_s(t(s))))+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}} \rho\left(\frac{\lambda}{N}\right) g(\lambda) \lambda^{-1/4} |\mathbf{c}(\lambda)|^{-1} d\lambda\,.
\end{aligned}$$

Now for $\eta \in C_c^\infty(r_1, r_2)$ and $\lambda > 0$, setting

$$\begin{aligned} P^*_{\gamma,N}\eta(\lambda) \quad &:= \quad \overline{\mathbf{c}(\lambda)}\, \rho\left(\frac{\lambda}{N}\right) \lambda^{-1/4} \left|\mathbf{c}(\lambda)\right|^{-1} \\ &\quad \times \int_{r_1}^{r_2} A(d(e,\gamma_s(t(s))))^{-1/2} A(s)^{1/2} e^{-i\left\{\lambda d(e,\gamma_s(t(s))) + t(s)\left(\lambda^2 + \frac{Q^2}{4}\right)\right\}} \eta(s) ds \,, \end{aligned}$$

it is easy to see that

$$\int_{r_1}^{r_2} P_{\gamma,N}\psi(s)\, \overline{\eta(s)}\, ds = \int_0^\infty \psi(\lambda)\, \overline{P^*_{\gamma,N}\eta(\lambda)}\, d\lambda \,,$$

holds for all $\eta \in C_c^\infty(r_1, r_2)$ and $\psi \in L^2(0,\infty)$ having suitable decay at infinity. Thus it suffices to prove that

$$\left(\int_0^\infty \left|P^*_{\gamma,N}h(\lambda)\right|^2 d\lambda\right)^{1/2} \lesssim \left(\int_{r_1}^{r_2} |h(s)|^2\, ds\right)^{1/2} , \tag{4.3.11}$$

for all $h \in C_c^\infty(r_1, r_2)$, with the implicit constant independent of N, as then letting $N \to \infty$, we can obtain the estimate (4.3.10). Now by Fubini's theorem,

$$\begin{aligned} &\int_0^\infty \left|P^*_{\gamma,N}h(\lambda)\right|^2 d\lambda \\ = &\int_{r_1}^{r_2}\int_{r_1}^{r_2} I_N(s,s') A(d(e,\gamma_s(t(s))))^{-\frac{1}{2}} A(s)^{\frac{1}{2}} A(d(e,\gamma_{s'}(t(s'))))^{-\frac{1}{2}} A(s')^{\frac{1}{2}} h(s)\overline{h(s')} ds\, ds', \end{aligned}$$

where

$$I_N(s,s') = \int_0^\infty e^{i\left\{\lambda(d(e,\gamma_{s'}(t(s'))) - d(e,\gamma_s(t(s)))) + (t(s') - t(s))\left(\lambda^2 + \frac{Q^2}{4}\right)\right\}} \lambda^{-1/2} \rho\left(\frac{\lambda}{N}\right)^2 d\lambda \,.$$

Then by Lemmata 4.3.1, 4.3.2 and the local growth asymptotics of the density function $A(s)$, we get that

$$\int_0^\infty \left|P^*_{\gamma,N}h(\lambda)\right|^2 d\lambda \lesssim \int_{r_1}^{r_2}\int_{r_1}^{r_2} \frac{1}{|s-s'|^{1/2}}\, |h(s)|\, |h(s')|\, ds\, ds',$$

with the implicit constant independent of N. As $h \in C_c^\infty(r_1, r_2)$, we can think of it as an even C_c^∞ function supported in $(-r_2, -r_1) \sqcup (r_1, r_2)$. We can therefore write the last integral as an one dimensional Riesz potential and apply (3.2.1) to get that for some positive number c,

$$\begin{aligned} \int_{r_1}^{r_2}\int_{r_1}^{r_2} \frac{1}{|s-s'|^{1/2}}\, |h(s)|\, |h(s')|\, ds\, ds' \quad &= \quad \int_0^\infty\int_0^\infty \frac{1}{|s-s'|^{1/2}}\, |h(s)|\, |h(s')|\, ds\, ds' \\ &= \quad c\int_0^\infty I_{1/2}(|h|)(s)\, |h(s)|\, ds \\ &= \quad c\int_{\mathbb{R}} |\xi|^{-\frac{1}{2}} \left|\tilde{h}(\xi)\right|^2 d\xi \,. \end{aligned}$$

Then by Pitt's inequality (as in our case $\beta = \frac{1}{4}, p = 2$ and hence $\beta_1 = \beta + \frac{1}{2} - \frac{1}{p} = \frac{1}{4}$, both the conditions of Lemma 3.2.1 are satisfied),

$$\begin{aligned}
\int_{\mathbb{R}} |\xi|^{-\frac{1}{2}} \left|\tilde{h}(\xi)\right|^2 d\xi &\lesssim \int_{\mathbb{R}} |h(x)|^2 \, |x|^{\frac{1}{2}} \, dx \\
&= c \int_{r_1}^{r_2} |h(s)|^2 \, s^{\frac{1}{2}} \, ds \\
&\leq c \, r_2^{1/2} \, \|h\|^2_{L^2(r_1,r_2)} \, .
\end{aligned}$$

Thus we get (4.3.9). Similar arguments lead to the following estimate for $T_{\gamma,2}$,

$$\left(\int_{r_1}^{r_2} |T_{\gamma,2} f(s)|^2 A(s) \, ds\right)^{1/2} \lesssim \left(\int_0^{\infty} \lambda^{1/2} \, |\widehat{f}(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda\right)^{1/2} . \tag{4.3.12}$$

Finally for $T_{\gamma,3}$, using the estimate on the error term (4.3.8) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|T_{\gamma,3} f(s)| &\lesssim A(s)^{-1/2} \int_0^{\infty} (1+\lambda)^{-1} \left|\widehat{f}(\lambda)\right| \, |\mathbf{c}(\lambda)|^{-1} \, d\lambda \\
&\leq A(s)^{-1/2} \left[\int_0^1 \left|\widehat{f}(\lambda)\right| \, |\mathbf{c}(\lambda)|^{-1} \, d\lambda + \int_1^{\infty} \lambda^{-1} \left|\widehat{f}(\lambda)\right| \, |\mathbf{c}(\lambda)|^{-1} \, d\lambda\right] \\
&\leq A(s)^{-1/2} \, \|f\|_{\dot{H}^{1/4}(S)} \left[\left(\int_0^1 \frac{d\lambda}{\lambda^{1/2}}\right)^{1/2} + \left(\int_1^{\infty} \frac{d\lambda}{\lambda^{5/2}}\right)^{1/2}\right] \\
&\lesssim A(s)^{-1/2} \, \|f\|_{\dot{H}^{1/4}(S)} \, .
\end{aligned}$$

Thus

$$\left(\int_{r_1}^{r_2} |T_{\gamma,3} f(s)|^2 A(s) \, ds\right)^{1/2} \lesssim (r_2 - r_1)^{1/2} \, \|f\|_{\dot{H}^{1/4}(S)} \, . \tag{4.3.13}$$

Then combining (4.3.9), (4.3.12) and (4.3.13), we complete the proof. □

Remark 4.3.3 In the special case of $\mathbb{H}^3$, the $SL(2,\mathbb{C})$-invariant Riemannian measure on $\mathbb{H}^3$ is $\sinh^2 s \, ds \, dk$, where ds and dk are the Lebesgue measure on $(0,\infty)$ and the Haar measure on $SU(2)$ respectively. As noted in section 4.2, in this case, the spherical function is given by the explicit expression:

$$\varphi_\lambda(s) = \frac{1}{2i \sinh(s)} \left(\frac{e^{i\lambda s}}{\lambda} + \frac{e^{-i\lambda s}}{\lambda}\right) ,$$

and hence the proof of Theorem 4.1.2 can be carried out with relative ease (as the corresponding error term is zero). The situation is exactly similar in the classical 3-dimensional Euclidean space, as the spherical function on $\mathbb{R}^3$ is given by,

$$\frac{\sin(\lambda s)}{\lambda s} = \frac{1}{2is}\left(\frac{e^{i\lambda s}}{\lambda} + \frac{e^{-i\lambda s}}{\lambda}\right) .$$

In the full generality of $\mathbb{R}^n$ however, the spherical function is somewhat lesser explicit and is given by $\mathscr{J}_{\frac{n-2}{2}}$, the modified Bessel function of order $\frac{n-2}{2}$. Although the oscillatory nature of $\mathscr{J}_{\frac{n-2}{2}}$ is well-studied, yet proceeding analogously as in the arguments of the proof of Theorem 4.1.2 does not seem to be an easy task.

Aiming to eventually obtain some Euclidean results, we now focus on 'small annuli'. Let

$$0 < \delta_1 < \delta_2 \leq (2R_0)/(3C_3) ,$$

where R_0 is as in Lemma 3.2.2. As in Theorem 4.1.2, we also assume that γ satisfies $(\mathscr{H}_1)$ for $\alpha \in [\frac{1}{2}, 1]$ and $(\mathscr{H}_2)$ for $s, s' \in [0, \delta_2)$ and $t, t' \in [-T, T]$, for some fixed $T > 0$, such that

$$T^\alpha < (C_2\delta_1)/(2C_1) .$$

Then by Lemma 4.3.2, we have for all $s \in (\delta_1, \delta_2)$ and $t \in [-T, T]$,

$$d(e, \gamma_s(t)) < \frac{3C_3}{2} s < \frac{3C_3}{2}\delta_2 \leq R_0 .$$

Thus for $f \in \mathscr{S}^2(S)_o$, we can also decompose the linearized maximal function (4.3.6) in terms of the Bessel series expansion of φ_λ (by taking $M = 0$ in Lemma 3.2.2),

$$\begin{aligned}
T_\gamma f(s) &= \int_0^\infty \varphi_\lambda(d(e, \gamma_s(t(s))))\, e^{it(s)\left(\lambda^2 + \frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&= c_0 \left(\frac{d(e, \gamma_s(t(s)))^{n-1}}{A(d(e, \gamma_s(t(s))))}\right)^{1/2} \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda d(e, \gamma_s(t(s))))\, e^{it(s)\left(\lambda^2 + \frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&\quad + \int_0^\infty E(\lambda, d(e, \gamma_s(t(s))))\, e^{it(s)\left(\lambda^2 + \frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&= T_{\gamma,4} f(s) + T_{\gamma,5} f(s) .
\end{aligned}$$

We obtain the following boundedness result for $T_{\gamma,4}$, which will be crucial for the proof of Theorem 4.1.5 in the next section:

Corollary 4.3.4 *Let $0 < \delta_1 < \delta_2 \leq (2R_0)/(3C_3)$, where R_0 is as in Lemma 3.2.2. Assume that γ satisfies $(\mathscr{H}_1)$ for $\alpha \in [\frac{1}{2}, 1]$ and $(\mathscr{H}_2)$ for $s, s' \in [0, \delta_2)$ and $t, t' \in [-T, T]$, for some fixed*

$T>0$, such that $T^\alpha < (C_2\delta_1)/(2C_1)$. Then we have for all $f \in \mathscr{S}^2(S)_o$,

$$\left(\int_{\delta_1}^{\delta_2} |T_{\gamma,4}f(s)|^2 A(s)\, ds\right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(S)}\,,$$

where the implicit constant depends only on δ_1, δ_2 and the dimension of S.

Proof: By Theorem 4.1.2, it suffices to prove that

$$\left(\int_{\delta_1}^{\delta_2} |T_{\gamma,5}f(s)|^2 A(s)\, ds\right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(S)}\,, \tag{4.3.14}$$

where the implicit constant depends only on δ_1, δ_2 and the dimension of S. For $\lambda \geq 0$, we recall the estimates on the error term in Lemma 3.2.2 (for $M=0$),

$$|E(\lambda, d(e,\gamma_s(t(s))))| \lesssim \begin{cases} d(e,\gamma_s(t(s)))^2 & \text{if } \lambda d(e,\gamma_s(t(s))) \leq 1 \\ d(e,\gamma_s(t(s)))^2 \left\{\lambda d(e,\gamma_s(t(s)))\right\}^{-\left(\frac{n+1}{2}\right)} & \text{if } \lambda d(e,\gamma_s(t(s))) > 1\,. \end{cases}$$

Then using the above pointwise bounds, Lemma 4.3.2 and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |T_{\gamma,5}f(s)| &\lesssim d(e,\gamma_s(t(s)))^2 \int_0^{1/d(e,\gamma_s(t(s)))} \left|\widehat{f}(\lambda)\right| |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &\quad + \frac{1}{(d(e,\gamma_s(t(s))))^{\left(\frac{n-3}{2}\right)}} \int_{1/d(e,\gamma_s(t(s)))}^{\infty} \lambda^{-\left(\frac{n+1}{2}\right)} \left|\widehat{f}(\lambda)\right| |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &\lesssim \|f\|_{\dot{H}^{1/4}(S)} \left[s^2 I_1(s)^{1/2} + \frac{1}{s^{\left(\frac{n-3}{2}\right)}} I_2(s)^{1/2}\right], \end{aligned}$$

where

$$I_1(s) = \int_0^{1/d(e,\gamma_s(t(s)))} \frac{|\mathbf{c}(\lambda)|^{-2}\, d\lambda}{\lambda^{1/2}}\,, \quad \text{and} \quad I_2(s) = \int_{1/d(e,\gamma_s(t(s)))}^{\infty} \frac{|\mathbf{c}(\lambda)|^{-2}\, d\lambda}{\lambda^{n+\frac{3}{2}}}\,.$$

Now first using the asymptotics of $|\mathbf{c}(\lambda)|^{-2}$ given in (1.2.7) for small frequency and Lemma 4.3.2, we get that

$$I_1(s) \lesssim \int_0^{1/d(e,\gamma_s(t(s)))} \lambda^{3/2}\, d\lambda \lesssim \frac{1}{s^{5/2}}\,.$$

Similarly the asymptotics of $|\mathbf{c}(\lambda)|^{-2}$ given in (1.2.7) for large frequency and Lemma 4.3.2 yield

$$I_2(s) \lesssim \int_{1/d(e,\gamma_s(t(s)))}^{\infty} \frac{d\lambda}{\lambda^{5/2}} \lesssim s^{3/2}\,.$$

Hence, we obtain

$$\int_{\delta_1}^{\delta_2} |T_{\gamma,5}f(s)|^2 A(s)\, ds \lesssim_{\delta_1,\delta_2} \|f\|^2_{\dot{H}^{1/4}(S)} .$$

This completes the proof of Corollary 4.3.4. □

4.4 Results on $\mathbb{R}^n$

Our aim in this section is to prove Theorems 4.1.5 and 4.1.7. For $f \in \mathscr{S}(\mathbb{R}^n)_o$, with $n \geq 2$, the linearization of the maximal function appearing in Theorem 4.1.5 is given by,

$$\tilde{T}_\gamma f(s) := \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda\|\gamma_s(t(s))\|)\, e^{it(s)\lambda^2}\, \mathscr{F}f(\lambda)\, \lambda^{n-1}\, d\lambda\,, \tag{4.4.1}$$

where $t(\cdot) : (\delta_1, \delta_2) \to [-T, T]$ is a measurable function. We first consider the case when $\mathscr{F}f$ is supported near the origin.

Lemma 4.4.1 *Under the hypothesis of Theorem 4.1.5, we have for all $f \in \mathscr{S}(\mathbb{R}^n)_o$ with Supp $(\mathscr{F}f) \subset [0, \Lambda]$ for some $\Lambda > 0$,*

$$\left(\int_{\delta_1}^{\delta_2} \left|\tilde{T}_\gamma f(s)\right|^2 s^{n-1}\, ds\right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,,$$

where the implicit constant depends only on $\Lambda, \delta_1, \delta_2$ and n.

Proof: The Lemma follows from the boundedness of the modified Bessel functions and Cauchy-Schwarz inequality. Indeed, for $s \in (\delta_1, \delta_2)$,

$$\left|\tilde{T}_\gamma f(s)\right| \lesssim \int_0^\Lambda |\mathscr{F}f(\lambda)|\, \lambda^{n-1}\, d\lambda \leq \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)} \left(\int_0^\Lambda \lambda^{n-\frac{3}{2}}\, d\lambda\right)^{1/2} \lesssim_\Lambda \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,.$$

Hence, the result follows. □

We now turn our attention to the situation when $\mathscr{F}f$ is supported away from the origin. In this case, we need to do some local geometry. For each $n \geq 2$, there exists at least one n-dimensional Damek-Ricci space S (in particular, the degenerate case of Real hyperbolic spaces).

We now consider the Riemannian exponential map at e, denoted by $\exp_e$ defined on T_eS, the tangent space of S at the identity e. As Damek-Ricci spaces are connected, simply-connected, complete Riemannian manifolds of non-positive sectional curvature, $\exp_e$ defines a global diffeomorphism from T_eS onto S. Now for a general Riemannian manifold, $\exp_e$ is always a local radial isometry. Hence, there exists $\varepsilon > 0$, such that $\exp_e$ maps the ball $\mathcal{B}(0,\varepsilon) \subset T_eS$ (with respect to the norm $\tilde{d}(\cdot)$ induced by the Riemannian metric) diffeomorphically onto the geodesic ball $B_\varepsilon \subset S$, with the property that for any $p \in \mathcal{B}(0,\varepsilon)$, we have

$$d(e, exp_e(p)) = \tilde{d}(p) \,, \tag{4.4.2}$$

(see [HRWW20, p. 636]). Moreover, since any two finite dimensional inner product spaces with the same dimension, are isometrically isomorphic, there exists a linear isometry $\mathcal{I}$ from $\mathbb{R}^n$ equipped with the flat metric onto T_eS equipped with the Riemannian metric. Hence for any $x \in \mathbb{R}^n$, we have

$$\|x\| = \tilde{d}\left(\mathcal{I}(x)\right) \,. \tag{4.4.3}$$

We now restrict to the Euclidean ball $B(o,\varepsilon)$ and consider the composition,

$$\mathcal{E} := \exp_e \circ \mathcal{I} \,.$$

Thus combining (4.4.2) and (4.4.3), it follows that $\mathcal{E}$ maps the ball $B(o,\varepsilon) \subset \mathbb{R}^n$ diffeomorphically onto the geodesic ball $B_\varepsilon \subset S$, with the property that for any $x \in B(o,\varepsilon)$, we have,

$$d\left(e, \mathcal{E}(x)\right) = \|x\| \,. \tag{4.4.4}$$

As we are only interested in radial functions, identifying all the points on a geodesic sphere centered at e, contained in the geodesic ball B_ε, solely by their distance from e and using (4.4.4), we make the abuse of notation,

$$\mathcal{E}(x) \sim d\left(e, \mathcal{E}(x)\right) = \|x\|, \quad \text{for } x \in B(o,\varepsilon). \tag{4.4.5}$$

Now under the hypothesis of Theorem 4.1.5, a computation similar to Lemma 4.3.2 using ($\mathscr{H}_3$) and ($\mathscr{H}_4$) yields that for all $s \in [0,\delta_2)$ and $t \in [-T,T]$,

$$\|\gamma_s(t)\| < \frac{3C_6}{2}\delta_2 \,.$$

Then furthermore assuming $\delta_2 \leq (2\varepsilon)/(3C_6)$, we get that $\|\gamma_s(t)\| < \varepsilon$, for all $s \in [0,\delta_2)$ and $t \in [-T,T]$. Hence it makes sense to consider the pushforwards (by the map $\mathcal{E}$) of the curves γ

for $s \in [0, \delta_2)$ and $t \in [-T, T]$, which we write by (4.4.5) as,

$$(\mathcal{E}_*\gamma)_s(t) := \mathcal{E}(\gamma_s(t)) \,.$$

We now summarize some important properties of these pushforwarded curves:

Lemma 4.4.2 *Under the hypothesis of Theorem 4.1.5, with $\delta_2 \leq (2\varepsilon)/(3C_6)$, where ε is as in (4.4.4), for $s, s' \in [0, \delta_2)$ and $t, t' \in [-T, T]$, we have for $\alpha \in [\frac{1}{2}, 1]$,*

$$(i) \left| d\left(e, (\mathcal{E}_*\gamma)_s(t)\right) - d\left(e, (\mathcal{E}_*\gamma)_s(t')\right) \right| \leq C_4 |t - t'|^{\alpha} \,,$$
$$(ii)\; C_5 |s - s'| \leq \left| d\left(e, (\mathcal{E}_*\gamma)_s(t)\right) - d\left(e, (\mathcal{E}_*\gamma)_{s'}(t)\right) \right| \leq C_6 |s - s'| \,.$$

Proof: Using (4.4.4) and ($\mathscr{H}_3$), property (i) is proved as follows,

$$\begin{aligned} \left| d\left(e, (\mathcal{E}_*\gamma)_s(t)\right) - d\left(e, (\mathcal{E}_*\gamma)_s(t')\right) \right| &= \left| d\left(e, \mathcal{E}(\gamma_s(t))\right) - d\left(e, \mathcal{E}(\gamma_s(t'))\right) \right| \\ &= \big| \|\gamma_s(t)\| - \|\gamma_s(t')\| \big| \\ &\leq C_4 |t - t'|^{\alpha} \,. \end{aligned}$$

Property (ii) can be verified similarly. □

We are now ready to present our result for the high frequency regime.

Lemma 4.4.3 *Under the hypothesis of Theorem 4.1.5, with $\delta_2 \leq \frac{2}{3C_6} \min\{\varepsilon, R_0\}$, where ε is as in (4.4.4) and R_0 is as in Lemma 3.2.2, we have for all $f \in \mathscr{S}(\mathbb{R}^n)_o$ with Supp $(\mathscr{F}(f)) \subset (1, \infty)$,*

$$\left(\int_{\delta_1}^{\delta_2} \left| \tilde{T}_\gamma f(s) \right|^2 s^{n-1} \, ds \right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)} \,,$$

where the implicit constant depends only on δ_1, δ_2 and n.

Proof: As $f \in \mathscr{S}(\mathbb{R}^n)_o$ with *Supp* $(\mathscr{F}(f)) \subset (1, \infty)$, we define

$$g := \left(\mathscr{A}^{-1} \circ \mathcal{M}^{-1} \right) f \,,$$

where $\mathscr{A}$ and $\mathcal{M}$ are as defined in the proof of Lemma 3.3.1. Then the interaction of g with the Fourier transform is best understood via the following commutative diagram, where each arrow is a bijection:

$$
\begin{tikzcd}
\mathscr{S}(\mathbb{R}^n)_o^\infty \arrow[r, "\mathcal{M}^{-1}"] \arrow[d, "\mathscr{F}"'] & \mathscr{S}(\mathbb{R}^n)_o^\infty \arrow[r, "\mathscr{A}^{-1}"] \arrow[d, "\mathscr{F}"'] & \mathscr{S}^2(S)_o^\infty \arrow[dl, "\wedge"] \\
\mathscr{S}(\mathbb{R})_e^\infty \arrow[r, "\mathfrak{m}^{-1}"] & \mathscr{S}(\mathbb{R})_e^\infty &
\end{tikzcd}
$$

Thus $g \in \mathscr{S}^2(S)_o^\infty$ with

$$\widehat{g}(\lambda) = \mathscr{F}\left(\mathcal{M}^{-1} f\right)(\lambda) = \frac{\lambda^{n-1}}{|\mathbf{c}(\lambda)|^{-2}} \mathscr{F} f(\lambda)\,. \tag{4.4.6}$$

So in particular, $Supp(\widehat{g}) \subset (1, \infty)$.

We also note that using the hypothesis, a computation similar to Lemma 4.3.2 implies that $\|\gamma_s(t)\| < \min\{\varepsilon, R_0\}$, for all $s \in [0, \delta_2)$ and $t \in [-T, T]$. Hence by (4.4.4), we see that

$$d\left(e, (\mathcal{E}_*\gamma)_s(t)\right) = \|\gamma_s(t)\| < \min\{\varepsilon, R_0\}\,. \tag{4.4.7}$$

Then for $s \in (\delta_1, \delta_2)$, using Lemmata 4.4.2 and 4.3.2, the local growth asymptotics of the density function of S (3.2.3), the relationship between $\widehat{g}$ and $\mathscr{F} f$ (4.4.6) and the isometric relation (4.4.7), we obtain

$$
\begin{aligned}
|T_{\mathcal{E}_*\gamma,4}\, g(s)| &\asymp \left| \int_1^\infty \mathscr{J}_{\frac{n-2}{2}}\left(\lambda\, d\left(e, (\mathcal{E}_*\gamma)_s(t(s))\right)\right) e^{it(s)\lambda^2}\, \widehat{g}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \right| \\
&= \left| \int_1^\infty \mathscr{J}_{\frac{n-2}{2}}\left(\lambda\, \|\gamma_s(t(s))\|\right) e^{it(s)\lambda^2} \mathscr{F} f(\lambda)\, \lambda^{n-1}\, d\lambda \right| \\
&= \left| \tilde{T}_\gamma f(s) \right|\,,
\end{aligned}
$$

where $\tilde{T}_\gamma$ is given by (4.4.1) . Thus again by the local growth asymptotics of the density function of S, it follows that

$$\left(\int_{\delta_1}^{\delta_2} |T_{\mathcal{E}_*\gamma,4}\, g(s)|^2 A(s)\, ds \right)^{1/2} \asymp \left(\int_{\delta_1}^{\delta_2} \left| \tilde{T}_\gamma f(s) \right|^2 s^{n-1}\, ds \right)^{1/2}. \tag{4.4.8}$$

Using (4.4.6) and the large frequency asymptotics of $|\mathbf{c}(\lambda)|^{-2}$ (1.2.7), we also compare the Homogeneous Sobolev norms of g and f:

$$\begin{aligned}
\|g\|_{\dot{H}^{1/4}(S)} &= \left(\int_1^\infty \lambda^{1/2}\,|\widehat{g}(\lambda)|^2\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\right)^{1/2} \\
&= \left(\int_1^\infty \lambda^{1/2}\,|\mathscr{F}f(\lambda)|^2\,\frac{\lambda^{2(n-1)}}{|\mathbf{c}(\lambda)|^{-4}}\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\right)^{1/2} \\
&\asymp \left(\int_1^\infty \lambda^{1/2}\,|\mathscr{F}f(\lambda)|^2\,\lambda^{n-1}\,d\lambda\right)^{1/2} \\
&= \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,.
\end{aligned} \tag{4.4.9}$$

Then by Lemma 4.4.2, Corollary 4.3.4, (4.4.8) and (4.4.9), the result follows. □

We now complete the proof of Theorem 4.1.5.

Proof: [Proof of Theorem 4.1.5] Let $f \in \mathscr{S}(\mathbb{R}^n)_o$. Then $\mathscr{F}f \in \mathscr{S}(\mathbb{R})_e$. Now let us choose an auxiliary non-negative even function $\psi \in C_c^\infty(\mathbb{R})$ such that $Supp(\psi) \subset \{\xi : \frac{1}{2} < |\xi| < 2\}$ and

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k}\xi) = 1\,,\ \xi \neq 0\,.$$

We set

$$\psi_1(\xi) := \sum_{k=-\infty}^{0} \psi(2^{-k}\xi)\,,\ \text{ and } \psi_2(\xi) := \sum_{k=1}^{\infty} \psi(2^{-k}\xi)\,.$$

Then both ψ_1 and ψ_2 are even non-negative C^∞ functions with $Supp(\psi_1) \subset (-2,2)$, $Supp(\psi_2) \subset \mathbb{R} \setminus (-1,1)$ and $\psi_1 + \psi_2 \equiv 1$. Then we consider,

$$\mathscr{F}f = (\mathscr{F}f)\,\psi_1 + (\mathscr{F}f)\,\psi_2\,.$$

As $(\mathscr{F}f)\,\psi_j$ for $j = 1,2$, are even Schwartz class functions on $\mathbb{R}$, by the Schwartz isomorphism theorem, there exist $f_1, f_2 \in \mathscr{S}(\mathbb{R}^n)_o$ such that $\mathscr{F}f_j = (\mathscr{F}f)\,\psi_j$ for $j = 1,2$. Then by Lemmata

4.4.1 and 4.4.3, we get the desired estimate on the linearized maximal function,

$$\begin{aligned}
&\left(\int_{\delta_1}^{\delta_2} \left|\tilde{T}_\gamma f(s)\right|^2 s^{n-1}\, ds\right)^{1/2} \\
\leq\ & \left(\int_{\delta_1}^{\delta_2} \left|\tilde{T}_\gamma f_1(s)\right|^2 s^{n-1}\, ds\right)^{1/2} + \left(\int_{\delta_1}^{\delta_2} \left|\tilde{T}_\gamma f_2(s)\right|^2 s^{n-1}\, ds\right)^{1/2} \\
\lesssim\ & \left(\int_0^2 \lambda^{1/2} \left|\mathscr{F} f_1(\lambda)\right|^2 \lambda^{n-1}\, d\lambda\right)^{1/2} + \left(\int_1^\infty \lambda^{1/2} \left|\mathscr{F} f_2(\lambda)\right|^2 \lambda^{n-1}\, d\lambda\right)^{1/2} \\
\leq\ & 2\, \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)} \,.
\end{aligned}$$

□

Finally, we prove Theorem 4.1.7:

Proof: [Proof of Theorem 4.1.7] By arguments mentioned in the Introduction, in order to prove the result on almost everywhere pointwise convergence, it suffices to prove that given any $0 < r_1 < r_2 < \infty$, there exists some fixed $T > 0$, such that one has for all $f \in \mathscr{S}(\mathbb{R}^n)_o$, the maximal estimate,

$$\left(\int_{r_1}^{r_2} \left(\sup_{t\in[-T,T]} \left|\tilde{S}_t f(\|\gamma_s(t)\|)\right|\right)^2 s^{n-1}\, ds\right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)} \,, \tag{4.4.10}$$

where the implicit constant can possibly depend only on r_1, r_2 and n. We first note that by compactness of $[0, r_2]$, there exists $0 < T < (r_1/2C_7)^2$, such that

$$\|\gamma_s(t)\| = s + C_7 t^{1/2} \,,$$

for all $s \in [0, r_2)$ and $t \in [-T, T]$. Then for all $s \in [0, r_2)$ and $t, t' \in [-T, T]$,

$$\left|\|\gamma_s(t)\| - \|\gamma_s(t')\|\right| = C_7 |t^{1/2} - t'^{1/2}| \leq C_7 |t - t'|^{1/2} \,.$$

Also for all $s, s' \in [0, r_2)$ and $t \in [-T, T]$,

$$|\|\gamma_s(t)\| - \|\gamma_{s'}(t)\|| = |s - s'| \,.$$

Therefore, γ satisfies $(\mathscr{H}_3)$ for $\alpha = 1/2$ and $(\mathscr{H}_4)$ for $s, s' \in [0, r_2)$ and $t, t' \in [-T, T]$, for some fixed $0 < T < (r_1/2C_7)^2$.

Let $\delta \in (0, r_2)$ be sufficiently small and set $\eta = r_2/\delta$ and $\delta' = r_1/\eta$, so that $\delta' < \delta$. For $f \in \mathscr{S}(\mathbb{R}^n)_o$, we consider

$$f_\eta(x) := f(\eta x)\,.$$

Now the solutions of the Schrödinger equation with initial data f and f_η are related as follows:

$$\begin{aligned}
\tilde{S}_t f(\|\gamma_s(t)\|) &= \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}\left(\lambda\|\gamma_s(t)\|\right) e^{it\lambda^2}\, \mathscr{F} f(\lambda)\, \lambda^{n-1}\, d\lambda \\
&= \eta^n \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}\left(\lambda\|\gamma_s(t)\|\right) e^{it\lambda^2}\, \mathscr{F} f_\eta(\eta\lambda)\, \lambda^{n-1}\, d\lambda \\
&= \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}\left(\lambda\frac{\|\gamma_s(t)\|}{\eta}\right) e^{i\left(\frac{t}{\eta^2}\right)\lambda^2}\, \mathscr{F} f_\eta(\lambda)\, \lambda^{n-1}\, d\lambda\,.
\end{aligned}$$

Now for $s \in [0, r_2)$ and $t \in [-T, T]$, we have

$$\frac{\|\gamma_s(t)\|}{\eta} = \frac{s}{\eta} + C_7\frac{t^{1/2}}{\eta} = \frac{s}{\eta} + C_7\left(\frac{t}{\eta^2}\right)^{1/2} = \|\gamma_{s/\eta}(t/\eta^2)\|\,.$$

Therefore, for $s \in [0, r_2)$ and $t \in [-T, T]$,

$$\begin{aligned}
\tilde{S}_t f(\|\gamma_s(t)\|) &= \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}\left(\lambda\|\gamma_{s/\eta}(t/\eta^2)\|\right) e^{i\left(\frac{t}{\eta^2}\right)\lambda^2}\, \mathscr{F} f_\eta(\lambda)\, \lambda^{n-1}\, d\lambda \\
&= \tilde{S}_{t/\eta^2} f_\eta(\|\gamma_{s/\eta}(t/\eta^2)\|)\,.
\end{aligned}$$

Then as $T < (r_1/2C_7)^2$, we have

$$\left(\frac{T}{\eta^2}\right)^{1/2} < \frac{\delta'}{2C_7}\,,$$

and hence by Theorem 4.1.5,

$$\begin{aligned}
&\left(\int_{r_1}^{r_2}\left(\sup_{t\in[-T,T]}\left|\tilde{S}_t f(\|\gamma_s(t)\|)\right|\right)^2 s^{n-1}\, ds\right)^{1/2} \\
=\;& \left(\int_{r_1}^{r_2}\left(\sup_{t\in[-T,T]}\left|\tilde{S}_{t/\eta^2} f_\eta(\|\gamma_{s/\eta}(t/\eta^2)\|)\right|\right)^2 s^{n-1}\, ds\right)^{1/2} \\
=\;& \left(\int_{r_1}^{r_2}\left(\sup_{t\in\left[-\frac{T}{\eta^2},\frac{T}{\eta^2}\right]}\left|\tilde{S}_t f_\eta(\|\gamma_{s/\eta}(t)\|)\right|\right)^2 s^{n-1}\, ds\right)^{1/2} \\
=\;& \eta^{n/2}\left(\int_{\delta'}^{\delta}\left(\sup_{t\in\left[-\frac{T}{\eta^2},\frac{T}{\eta^2}\right]}\left|\tilde{S}_t f_\eta(\|\gamma_s(t)\|)\right|\right)^2 s^{n-1}\, ds\right)^{1/2} \\
\lesssim\;& \eta^{n/2}\, \|f_\eta\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,.
\end{aligned}$$

We next note that

$$\begin{aligned}
\|f_\eta\|_{\dot{H}^{1/4}(\mathbb{R}^n)} &= \left(\int_0^\infty \lambda^{1/2}\, |\mathscr{F} f_\eta(\lambda)|^2 \lambda^{n-1}\, d\lambda\right)^{1/2} \\
&= \eta^{\left(\frac{1}{4}-\frac{n}{2}\right)} \left(\int_0^\infty \lambda^{1/2}\, |\mathscr{F} f(\lambda)|^2 \lambda^{n-1}\, d\lambda\right)^{1/2} \\
&= \eta^{\left(\frac{1}{4}-\frac{n}{2}\right)} \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,,
\end{aligned}$$

which we plug in the above to obtain

$$\left(\int_{r_1}^{r_2} \left(\sup_{t\in[-T,T]} \left|\tilde{S}_t f(\|\gamma_s(t)\|)\right|\right)^2 s^{n-1}\, ds\right)^{1/2} \lesssim \eta^{\frac{1}{4}} \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,.$$

Thus we obtain (4.4.10), which completes the proof. □

Chapter 5

Concluding remarks

In this chapter, we make some remarks and pose some new problems:

1. Functions satisfying the RMVP in a domain $D \subset M$ can be interpreted as functions which are harmonic with respect to a certain random walk $(X_n)_{n \geq 1}$ on D. Namely, the distribution of X_{n+1} given that $X_n = z \in D$ is the harmonic measure $\mu_{z,B}$ on the sphere ∂B, where B is the geodesic ball $B = B(z, \rho(z))$. Theorem 2.1.1 can then be interpreted as saying that continuous harmonic functions for the random walk with the appropriate boundary behaviour are also harmonic for the Laplace-Beltrami operator.

2. Theorem 2.1.3 assumes that the manifold M has strictly negative curvature. It would be interesting to study whether an analogue of this result holds for nonpositively curved manifolds, at least those which are Gromov hyperbolic. Such a result would cover the case of harmonic manifolds of purely exponential volume growth (which are nonpositively curved and Gromov hyperbolic), in particular it would cover all known examples of nonflat noncompact harmonic manifolds, namely the rank one symmetric spaces of noncompact type and the Damek-Ricci spaces.

3. For all the results obtained in chapter 2, it would be interesting to see if there are counterexamples if any of the hypotheses are relaxed. Constructing functions on a general Riemannian manifold which are not harmonic but satisfy the RMVP seems to be a difficult task however.

4. Keeping the problem of pointwise convergence of solutions in mind, one is naturally interested in obtaining maximal estimates analogous to Theorem 3.1.1 for the fractional

Schrödinger equations. We have investigated this problem, in fact, for more general dispersive equations on Damek-Ricci spaces in [De25.1].

5. On $\mathbb{H}^3 \cong SL(2,\mathbb{C})/SU(2)$, for $\beta > 3/2$, by Sobolev imbedding [CGM93, Theorem 4.7] and standard arguments, the non-tangential convergence of the solution of the Schrödinger equation (to the initial data) holds almost everywhere. Whereas by our counter-example (Theorem 4.1.1), the non-tangential convergence to the initial data fails on a set of positive ($SL(2,\mathbb{C})$-invariant) Riemannian measure if $\beta \leq 1/2$. This gives rise to the natural question of what happens when $\beta \in (1/2, 3/2]$. The analogue of Theorem 4.1.1 and the above question in the great generality of Damek-Ricci spaces turns out to be even more interesting.

6. In chapter 4, the two Euclidean results (Theorems 4.1.5 and 4.1.7) only give partial analogues of Theorem 4.1.2 for $\mathbb{R}^n$. Thus seeking a complete analogue of Theorem 4.1.2 for $\mathbb{R}^n$ is a natural question.

7. Finally, the ultimate goal would be to get a complete solution of the Carleson's problem (that is analogues of [DGL17, DZ19]) for Damek-Ricci spaces without any restriction on the symmetry of the initial data.

Bibliography

[AS85] Anderson M.T. and Schoen R. *Positive Harmonic Functions on Complete Manifolds of Negative Curvature*, Ann. Maths., Second Series, Vol. 121, No. 2 (Mar., 1985), pp 429-461.

[An91] Anker, J-P. *The spherical Fourier transform of rapidly decreasing functions. A simple proof of a characterization due to Harish-Chandra, Helgason, Trombi, and Varadarajan.* J. Funct. Anal. Volume 96, Issue 2, 1991, 331-349,

[ADY96] Anker, J-P., Damek, E. and Yacoub, C. *Spherical analysis on harmonic AN groups.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4, 23 (1996), no. 4, 643-679.

[AP14] Anker, J-P. and Pierfelice, V. *Wave and Klein-Gordon equations on hyperbolic spaces.* Anal. PDE, 2014, 7(4), pp.953-995.

[APV15] Anker, J-P., Pierfelice, V. and Vallarino, M. *The wave equation on Damek-Ricci spaces*, Ann. Mat. Pura Appl. (4) 194 (2015), no. 3, 731-758.

[As95] Astengo, F. *A class of L^p convolutors on harmonic extensions of H-type groups.* J. Lie Theory 5 (1995), no. 2, 147-164.

[ABCS] Astengo, F., Cowling, M., Di Blasio, B. and Sundari, M. *Hardy's uncertainty principle on certain Lie groups.* J. London Math. Soc. (2) 62 (2000), no. 2, 461–472.

[BH19] Benoist, Y. and Hulin, D. *Harmonic Measures on Negatively Curved Manifolds.* Ann. Inst. Fourier, Grenoble 69, 7 (2019), pp 2951-2971.

[BiD24] Biswas, K. and Dewan, U. *Restricted Mean Value Property on Riemannian Manifolds.* J. Geom. Anal. 34, 108(2024).

[BX98] Bloom, W.R. and Xu, Z. *Fourier transforms of Schwartz functions on Chébli-Trimèche Hypergroups.* Monatsh. Math. 125, 89-109 (1998).

[Bo16] Bourgain, J. *A note on the Schrödinger maximal function.* J. Anal. Math. 130 (2016), 393-396.

[BrH99] Bridson M. R. and Haefliger A. *Metric spaces of nonpositive curvature*, Grundlehren der mathematischen Wissenschaften, ISSN 0072-7830; 319, 1999.

[CM23] Cao, Z. and Miao, C. *Sharp pointwise convergence on the Schrödinger operator along one class of curves.* Bull. Sci. Math. 184 (2023) 103254.

[Ca80] Carleson, L. *Some analytic problems related to statistical mechanics, Euclidean harmonic analysis.* Lecture Notes in Math. 779, Springer, Berlin, 1980, 5-45.

[CLV12] Cho, C., Lee, S. and Vargas, A. *Problems on pointwise convergence of solutions to the Schrödinger equation.* J. Fourier Anal. Appl. 18 (2012), 972-994.

[Co83] Cowling, M. *Pointwise behavior of solutions to Schrödinger equations, harmonic analysis.* Lecture Notes in Math. 992. Springer, Berlin, 1983, 83-90.

[CDKR98] Cowling, M., Dooley, A., Korányi, A. and Ricci, F. *An approach to symmetric spaces of rank one via groups of Heisenberg type.* J. Geom. Anal. 8 (1998), no. 2, pp. 199–237.

[CGM93] Cowling, M., Giulini, S. and Meda, S. $L^p - L^q$ *estimates for functions of the Laplace-Beltrami operator on noncompact symmetric spaces I.* Duke. Math. J., vol. 72, no. 1, October 1993, pp. 109-150.

[DK82] Dahlberg, B.E.J. and Kenig, C.E. *A note on the almost everywhere behavior of solutions of the Schrödinger equation.* Lecture Notes in Math. 908. Springer-Verlag, Berlin, 1982, 205-208.

[De25] Dewan, U. *Regularity and pointwise convergence of solutions of the Schrödinger operator with radial initial data on Damek-Ricci spaces.*, Annali di Matematica Pura ed Applicata 204, 1161–1182 (2025).

[De25.1] Dewan, U. *Maximal estimates and pointwise convergence for solutions of certain dispersive equations with radial initial data on Damek-Ricci spaces.* arXiv:2501.08323

[DN17] Ding, Y. and Niu, Y. *Weighted maximal estimates along curve associated with dispersive equations.* Anal. Appl. vol. 15, no. 2, pp. 225-240 (2017).

[DGL17] Du, X., Guth, L. and Li, X. *A sharp Schrödinger maximal estimate in* $\mathbb{R}^2$. Ann. Math. 2017, 186, 607-640.

[DZ19] Du, X. and Zhang, R. *Sharp L^2 estimates of the Schrödinger maximal function in higher dimensions.* Ann. Math. 2019, 189, 837-861.

[Fe76] Fenton P.C. *Functions having the Restricted Mean Value Property*, J. Lond. Math. Soc., Volume s2-14, Issue 3, December 1976, pp 451-458.

[Fe79] Fenton P.C. *On Sufficient Conditions for Harmonicty*, Trans. Amer. Math. Soc., Volume 253, September 1979, pp 139-147.

[Fe87] Fenton P.C. *On the Restricted Mean Value Property*, Proc. Amer. Math. Soc.,Volume 100, Number 3, July 1987, pp 477-481.

[Gr09] Grafakos, L. *Modern Fourier Analysis.* Graduate Texts in Mathematics, 2nd ed, Springer Science+Business Media, LLC 2009.

[HN93] Hansen W. and Nadirashvili N. *A converse to the mean value theorem for harmonic functions*, Acta Math., 171 (1993), pp 139-163.

[HN94] Hansen, W. and Nadirashvili, N. *Littlewood's One Circle Problem*, J. Lond. Math. Soc., 50(2), (1994), 349–360.

[HRWW20] Heeren, B., Rumpf, M., Wardetzky, M. and Wirth, B. *Chapter 8 - Discrete Riemannian calculus on shell space, Editor(s): Andrea Bonito, Ricardo H. Nochetto, Handbook of Numerical Analysis*, Elsevier, vol. 21, 2020, pp. 621-679, ISSN 1570-8659, ISBN 9780444640031, https://doi.org/10.1016/bs.hna.2019.06.005.

[He00] Helgason, S. *Groups and geometric analysis, Integral geometry, invariant differential operators, and spherical functions.* Mathematical Surveys and Monographs, vol. 83. Providence, RI: American Mathematical Society, 2000.

[Ke34] Kellogg O.D. *Converses of Gauss' theorem on the arithmetic mean*, Trans. Amer. Math. Soc., 36, 1934, pp 227-242.

[Li68] Littlewood J. E. *Some problems in real and complex analysis*, Hath. Math. Monographs, Massachusetts, 1968.

[Mi24] Minguillón, J. *A note on almost everywhere convergence along tangential curves to the Schrödinger equation initial datum.* J. Geom. Anal. (2024) 34:333.

[Pr90] Prestini, E. *Radial functions and regularity of solutions to the Schrödinger equation.* Monatsh. Math. 109, 135-143 (1990).

[RS09] Ray, S.K. and Sarkar, R.P. *Fourier and Radon transform on harmonic NA groups.* Trans. Amer. Math. Soc. 361 (2009), no. 8, 4269–4297.

[Ro08] Rogers, K.M. *A local smoothing estimate for the Schrödinger equation.* Adv. Math. 219, pp. 2105-2122 (2008).

[SS89] Sjögren, P. and Sjölin, P. *Convergence properties for the time dependent Schrödinger equation.* Ann. Acad. Sci. Fenn. Math. 14 (1989), pp. 13-25.

[Sj87] Sjölin, P. *Regularity of solutions to the Schrödinger equation.* Duke Math J. 55(1987), 699-715.

[Sj97] Sjölin, P. *L^p maximal estimates for solutions to the Schrödinger equation.* Math. Scand., vol. 81, no. 1 (1997), pp. 35-68.

[ST78] Stanton, R. J. and Tomas, P. A. *Expansions for spherical functions on noncompact symmetric spaces.* Acta Math. 140 (1978), no. 3-4, pp. 251-276.

[St56] Stein, E.M. *Interpolation of Linear Operators.* Trans. Amer. Math. Soc. vol. 83, no. 2 (1956), 482–492.

[SW90] Stein, E.M. and Weiss, G. *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton University Press, Princeton, New Jersey, Sixth printing, 1990.

[Su83] Sullivan, D. *The Dirichlet problem at infinity for a negatively curved manifold*, J. Differential Geometry, 18, pp 723-732, 1983.

[Ve88] Vega, L. *Schrödinger equations: pointwise convergence to the initial data.* Proc. Amer. Math. Soc. 102(1988), 874-878.

[WZ19] Wang, X. and Zhang, C. *Pointwise Convergence of Solutions to the Schrödinger Equation on Manifolds.* Canad. J. Math. Vol. 71(4), 2019, 983-995.

[Wi50] Willmore T. J., *Mean Value Theorems in Harmonic Riemannian Spaces*, J. Lond. Math. Soc., Volume s1-25, Issue 1, January 1950, pp 54–57.

List of Publications

1. Kingshook Biswas and Utsav Dewan:

   Restricted Mean Value Property on Riemannian Manifolds.

   The Journal of Geometric Analysis (J. Geom. Anal. 34, 108(2024)).

2. Utsav Dewan:

   Regularity and pointwise convergence of solutions of the Schrödinger operator with radial initial data on Damek-Ricci spaces.

   Annali di Matematica Pura ed Applicata (1923-)

   (Annali di Matematica (2024) https://doi.org/10.1007/s10231-024-01523-2) .

3. Utsav Dewan:

   Pointwise convergence of solutions of the Schrödinger equation along general curves on Damek-Ricci spaces.

   Pre-print (arXiv:2411.14020) .

4. Utsav Dewan and Swagato K. Ray:

   Mapping properties of the local Schrödinger maximal function with radial initial data on Damek-Ricci spaces.

   Pre-print (arXiv:2411.04084) .