On Universal C^* Algebras associated to Operator Spaces and Generalized Crossed Products

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On Universal C^* Algebras associated to Operator Spaces and Generalized Crossed Products

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This thesis is dedicated to my mother and my wife, whose unconditional love, encouragement, and unwavering support have been my strength throughout this journey.

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Notation

Throughout this thesis, we adopt the following notation.

- $\mathcal{B}(H)$ denotes the C^*-algebra of bounded linear operators on a Hilbert space H.
- $\mathcal{K}(H)$ denotes the C^*-algebra of compact operators on H.
- $\mathcal{L}(X)$ denotes the collection of all adjointable operator on a C^*-correspondence X.
- $\mathcal{K}(X)$ denotes the collection of all adjointable compact operators on the C^*-correspondence X.
- $M(A)$ denotes the multiplier algebra of a C^*-algebra A.
- $\mathcal{M}(D)$ denotes the collection of positive Borel measures on a compact Hausdorff space D.
- $\mathcal{Z}(A)$ denotes the center of a C^*-algebra A.
- $\mathcal{B}_A := \{a \in A : \|a\| \leq 1\}$ denotes the closed unit ball of A.
- A^+ denotes the set of positive elements of a C^*-algebra A.
- $C^*(G)$ denotes the full group C^*-algebra of G.
- $\ell^2(G)$ denotes the Hilbert space of square-summable functions on G. For a discrete group G.
- $\lambda : G \to \mathcal{U}(\ell^2(G))$ denotes the left regular representation.

- Tr denotes the normalized trace on $M_n(\mathbb{C})$.
- $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit circle.
- $F \subset_{\text{fin}} G$ denotes a finite subset of G.
- E_H denotes the conditional expectation onto $C^*(H)$ for a subgroup $H \leq G$.
- τ denotes the canonical trace on $C^*(G)$ when G is amenable.

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Chapter 1

Introduction

The notion of *universal C^*-algebras* plays a fundamental role in the theory of operator algebras. It provides a systematic method for constructing C^*-algebras from abstract algebraic data described in terms of generators and relations. Given a set of generators together with a family of admissible relations, one can define a universal C^*-algebra as the completion of the free $*$-algebra generated by these elements with respect to the largest C^*-seminorm compatible with those relations. This universal property ensures that for any other C^*-algebra satisfying the same relations, there exists a unique $*$-homomorphism from the universal C^*-algebra into that algebra.

The study of universal C^*-algebras is of particular interest because it lies at the confluence of algebra, topology, and analysis. Universal constructions serve as a bridge between abstract algebraic structures and their concrete realisations as operator algebras on Hilbert spaces. Many important examples in operator algebra theory, including group C^*-algebras, Cuntz-Krieger algebras, Cuntz-Pimsner algebras, and the C^*-algebras associated with compact quantum groups, arise naturally as universal C^*-algebras determined by appropriate sets of relations. These constructions capture a wide range of dynamical and symmetry phenomena within a unified analytic framework.

This thesis is divided into two parts. The first part studies universal C^*-algebras arising from crossed product constructions of $C^*(G)$ associated with endomorphisms and completely positive maps. The second part investigates universal C^*-algebras arising from operator space structures determined by the generators of $\mathcal{U}_n^{nc}$ and related C^*-algebras. Taken together, these results demonstrate how universal constructions provide a natural framework for connecting algebraic relations, dynamical systems, and operator space theory within operator algebra.

In the first part of the thesis, universal C^*-algebras arising from crossed product constructions of a C^*-algebra by an endomorphism and a completely positive (CP) map are studied. The theory of C^*-algebras associated with dynamical systems has become a central theme in operator algebra theory, providing a unified analytic framework for understanding both reversible and irreversible dynamics. Among the earliest and most significant developments in this direction is the theory of *Cuntz-Pimsner algebras*, introduced by Pimsner as a generalization of both Cuntz-Krieger algebras and crossed products by automorphisms or endomorphisms. This construction associates a C^*-algebra to a given C^*-correspondence, thereby encoding the internal symmetries and dynamical behaviour of a system through operator-theoretic structures. The framework provides a powerful unifying language that connects several constructions in noncommutative geometry and operator theory.

Later, Exel [Exe03] introduced a framework for constructing crossed products by endomorphisms equipped with transfer operators. The resulting algebras, now known as *Exel crossed products*, extend the classical notion of crossed products to situations where the underlying dynamics may be irreversible. These algebras model a broader class of non-invertible or dissipative phenomena and reveal deep structural connections between the algebraic, analytic, and dynamical aspects of C^*-systems. Moreover, Exel's construction can be realised as a particular instance of a Cuntz-Pimsner algebra associated with a correspondence naturally determined by the endomorphism and its transfer operator, thereby establishing a connection between the two constructions.

In a more recent development, Kwaśniewski [Kwa17] further generalised Exel's approach by considering crossed products associated with completely positive (cp) maps. This generalization replaces the endomorphism in Exel's setting with a CP map, allowing for the study of quantum dynamical systems where the dynamics need not be multiplicative. The construction of crossed products by CP maps thus provides a natural extension of Exel's theory, capturing the behaviour of non-deterministic and irreversible processes. In this framework, the correspondence associated with a CP map plays a role analogous to that of the correspondence in Pimsner's theory, while at the same time introducing substantial differences in algebraic structure and representation theory.

In this part of the thesis, based on the work in [Ban25], several perspectives on generalised crossed products for $C^*(G)$ are explored and unified, including those arising from the Cuntz-Pimsner algebra, the Exel crossed product, and crossed products associated with completely positive maps. A central theme of this work is the comparative study of the crossed products of $C^*(G)$ by endomorphisms and by specific CP maps closely related to them. In particular, the relationships between these constructions are analyzed through the theory of C^*-correspondences, and the structural consequences of replacing endomorphisms with completely positive maps are investigated.

Beyond these results, we also examine the notion of the crossed product of $C^*(G)$ by a state. In addition, we analyze the crossed product of $C^*(G)$ by the conditional expectation E_H, where H is a subgroup of G. Furthermore, we undertake a classification of transfer operators on the commutative C^*-algebra $C(D)$, identifying how their structural properties influence the corresponding dynamical systems and their operator-algebraic realisations. These results shed new light on the interplay between CP maps, transfer operators, and C^*-correspondences, providing both concrete examples and a deeper theoretical understanding.

In Chapter §3, we begin by motivating the construction of the generalised crossed product of a C^*-algebra A, namely the crossed product of A with respect to an endomorphism or a completely positive (CP) map. Section §3.1 is devoted to a detailed study of transfer operators. After introducing the basic notions and examples, we state and prove a theorem characterizing transfer operators $\mathcal{L}_\alpha$ associated with an automorphism α, showing that every such transfer operator is of the form $\mathcal{L}_\alpha(a) = \alpha^{-1}(a)z$, where $z \in \mathcal{Z}(A) \cap A^+$. We also discuss several properties of the set of transfer operators corresponding to an endomorphism α.

In Section §3.1.1, we investigate the structure of transfer operators for $C(D)$. Given a continuous map $a : D \to D$, we obtain an endomorphism $\alpha_a : C(D) \to C(D)$ defined by $\alpha_a(f) = f \circ a$. In this setting, a series of structural results is established.
The first result provides an intrinsic characterization of transfer operators associated with α_a, describing their structure in terms of measures on D that reflect the dynamics of the map a. More precisely, a positive map $\rho : C(D) \to C(D)$ defined by

$$\rho(f)(x) = \int_D f \, d\mu_x$$

is a transfer operator for α_a if and only if the supports $\operatorname{supp}(\mu_x)$ are mutually disjoint and the map $a : D \to D$ is constant on each $\operatorname{supp}(\mu_x)$.
Examples of transfer operators are also provided in the case where the map $a : D \to D$ is surjective. A complete analysis is carried out when $a : D \to D$ is a continuous bijection (hence a homeomorphism), and the choice of transfer operator that yields the standard crossed product of $C(D)$ with respect to the automorphism α_a is identified.
Finally, the structure of transfer operators is analyzed in the case where $a : D \to D$ is a surjective local homeomorphism. In this situation, every transfer operator is shown to be of the form

$$\rho(f)(x) = \sum_{y \in a^{-1}(x)} c_y^x f(y),$$

where c_y^x are positive scalars. This result provides a complete characterization of transfer operators in this setting and highlights their connection with classical dynamical systems.

In Section §3.2, we define the crossed product of a C^*-algebra A with respect to a state $\psi : A \to \mathbb{C}$. The corresponding Cuntz-Pimsner algebra is constructed, and it is shown that the associated C^*-correspondence is $H_\psi \otimes A$, where $(\pi_\psi, H_\psi, \xi_\psi)$ denotes the GNS triple corresponding to (A, ψ). In Section §3.2.1, the special case $A = C^*(G)$ is considered, where G is a countably infinite discrete amenable group, and the crossed product is taken with respect to the canonical trace τ. It is first shown that the universal crossed product of $C^*(G)$ with respect to τ is isomorphic to the corresponding universal Toeplitz algebra. It is then proved that there does not exist a spatial representation of the system inside $B(\ell^2(G))$ that preserves the embedding of $C^*(G)$ into $B(\ell^2(G))$ given by the extension of the left regular representation. Finally, a representation $(\tilde{\pi}, T)$ of the system $(C^*(G), \tau)$ is constructed, and it is shown that the universal crossed product is isomorphic to the C^*-algebra $C^*(\tilde{\pi}, T)$.

In Section §3.3, we first discuss the universal C^*-algebra $\mathcal{E}_\alpha$ introduced by Hirshberg in [Hir02], where $\alpha : G \to G$ is an injective pure group homomorphism. The construction is motivated by its spatial realisation inside $B(\ell^2(G))$. We also prove that $\mathcal{E}_\alpha$ is isomorphic to the Exel crossed product of $C^*(G)$ with respect to the endomorphism $\gamma_\alpha : C^*(G) \to C^*(G)$, where γ_α denotes the extension of α to $C^*(G)$ via the universal property of $C^*(G)$, together with the transfer operator arising from the conditional expectation of $C^*(G)$ onto $C^*(\alpha(G))$.

In Section §3.3.2, we investigate crossed products of $C^*(G)$ with respect to the trace-preserving conditional expectation E_H, where $H < G$. We first show that when G is abelian there does not exist an operator $S \in B(\ell^2(G))$ such that (γ_λ, S) forms a representation of the system $(C^*(G), E_H)$, where λ denotes the left regular representation and γ_λ its extension to $C^*(G)$. This result is then extended to the case of virtually abelian groups.

We further explain how this phenomenon reflects a fundamental structural difference between crossed products of $C^*(G)$ arising from endomorphisms and those arising from completely positive maps. Indeed, taking a conditional expectation with respect to a subgroup $H < G$ is closely related to crossed products by endomorphisms. This relationship becomes clearer when one considers [Exe03, Proposition 2.6], which shows that

$$\mathcal{L}_\alpha = (\alpha|_{(\ker \alpha)^\perp})^{-1} \circ E_\alpha.$$

Moreover, [Kwa17, Theorem 4.7] establishes that the Exel crossed product associated with the Exel system $(A, \alpha, \mathcal{L}_\alpha)$ is isomorphic to the crossed product of A by the CP map $\mathcal{L}_\alpha$ in the sense of Kwaśniewski.
We then construct the associated Cuntz-Pimsner algebra and show that, in the case where $[G : H] = \infty$, every representation is automatically covariant. Finally, by constructing an appropriate representation of the C^*-dynamical system $(C^*(G), E_H)$, we prove that the universal crossed product is isomorphic to the C^*-algebra generated by this representation. This is achieved by defining a suitable gauge action of $\mathbb{T}$, thereby providing an explicit realisation of the crossed product.

The results discussed above focus on universal C^*-algebras arising from crossed product constructions associated with dynamical systems. In the second part of this thesis, the focus shifts to universal constructions arising from operator space structures.

In the second part of this thesis, we undertake a detailed study of operator spaces associated with Brown's universal noncommutative unitary C^*-algebra $\mathcal{U}_n^{nc}$, in particular, the operator space structure generated by its standard generators. We also study the universal C^*-algebra generated by this operator space and show that it is not isomorphic to $\mathcal{U}_n^{nc}$. Several operator algebraic properties of the resulting universal C^*-algebra are established, highlighting the relationship between the operator space structure and the associated universal C^*-algebra.
Recent developments in quantum information theory have provided strong motivation for the study of noncommutative structures that arise naturally in quantum systems. In particular, collections of operators and the theory of operator spaces and operator algebras have proved to be powerful tools for capturing the highly noncommutative nature of quantum phenomena. A notable example arises in the study of quantum correlations, that is, correlations between quantum subsystems, which require a noncommutative framework for their proper mathematical characterization, unlike their classical counterparts [St19, BKSt23, BHTT24]. This becomes particularly relevant under the no-signaling constraints that appear in quantum mechanics [BTT24].

While classical correlations can be described using stochastic or bistochastic matrices, their quantum analogues involve genuinely noncommutative objects that arise naturally within the C^*-algebraic framework. For instance, in [BHTT23, BHTT24], quantum no-signaling bicorrelations were formalized, providing a mathematical framework that extends classical correlation matrices to the quantum setting by employing techniques from operator space theory and tensor norms. Closely related to this perspective is the theory of non-local games, which provides a useful framework for demonstrating the advantages of quantum entanglement in achieving certain computational tasks [Pau16].
In particular, the study of Tsirelson's correlation sets and Bell's work on local hidden variable models is closely related to non-local games [Tsi93]. These developments have profound implications for operator algebras, since Tsirelson's correlation sets are closely connected to the Connes-Kirchberg problem [JNP+11, Fri12, Oza13]. This line of research culminated in the recent work of Ji, Natarajan, Vidick, Wright, and Yuen [JNV+22], who proposed a counterexample to the Connes-Kirchberg conjecture [Con76, Kir93], making essential use of techniques arising from the theory of non-local games.

It is in this context of generalizing quantum correlations to quantum spaces that quantum homomorphisms, which for a tuple of unital C^*-algebras $(\mathcal{A}, \mathbf{B})$ is any unital C^*-algebra $\mathcal{C}$ and $*$-homomorphism $\rho : \mathcal{A} \to \mathbf{B} \otimes \mathcal{C}$, arise naturally [BKSt23]. Naturally, the space of "all" quantum homomorphisms is a relevant object of study, and these correspond to universal quantum homomorphisms. Notably, such universal objects only exist (in the category of C^*-algebras) when $\mathbf{B}$ is finite dimensional [FMP24]. A key instance of such a universal quantum homomorphism is Brown's non commutative unitary C^*-algebra $\mathcal{U}_n^{nc}$, which is the universal quantum homomorphism when $\mathcal{A} = C^*(\mathbb{Z})$ and $\mathbf{B} = M_n(\mathbb{C})$, first defined and studied by Brown [Bro81], while attempting to generalize Atiyah's proof of Künneth theorem in K-theory to the case of C^*-algebras. Several of its interesting properties are known- for instance, McClanahan [McC92] showed that $\mathcal{U}_n^{nc}$ has no non-trivial projections and computed its K-theory. Residual finite dimensionality of $\mathcal{U}_n^{nc}$ was studied in [Har19, Man23b] while lifting property and primitivity of $\mathcal{U}_n^{nc}$ were studied in

[Man23b].

Here, in the second part of the thesis, which is based on [BMP25], we focus on operator spaces associated to $\mathcal{U}_n^{nc}$ and the structure of closely related universal C^*-algebras. The theory of operator spaces has been much studied with seminal contributions by several authors, including Ruan, Effros, Blecher, Paulsen and Pisier. The key result was obtained in Ruan's thesis [Rua88], which provided abstract characterizations for such spaces, in effect providing a "quantized" counterpart to the norm of Banach spaces, which thus signified the understanding that operator spaces are noncommutative analogues of Banach spaces. Operator spaces have been increasingly recognized as providing a robust mathematical language for analyzing quantum channels, entanglement and correlations- tools from operator space theory have been used to prove groundbreaking results in physics, for instance, to construct tripartite quantum states that can lead to arbitrarily large violations of Bell inequalities [PGWP$^+$08]. Beautiful work of Pisier [Pis96] also highlighted the usefulness of operator space techniques in relation to the Connes-Kirchberg conjecture.

In general, for a normed linear space $\mathcal{E}$, we obtained many norms on $M_n(\mathcal{E})$. Nevertheless, this family of norms admits minimal and maximal elements [BP91], with these structures by MIN($\mathcal{E}$) and MAX($\mathcal{E}$) respectively, which have been well studied [Ble92a, Ble92b, BP91, Pau96, Zha95]. These operator space structures satisfy the following relations: for any contractive map $\varphi : \mathcal{E} \to \mathbf{B}(\mathcal{H})$ and for all $(x_{ij}) \in M_n(\mathcal{E})$,

$$\|(x_{ij})\|_{\text{MIN}} \leq \|(\varphi(x_{ij}))\| \leq \|(x_{ij})\|_{\text{MAX}}\,, \text{ for all } n \in \mathbb{N} \text{ and } \mathcal{H} \text{ is a Hilbert space.}$$

We make a comprehensive study of operator spaces associated to Brown's noncommutative unitary C^*-algebra $\mathcal{U}_n^{nc}$, including maximal and minimal structures as well as those associated to the reduced C^*-algebras, $\mathcal{U}_{n,red}^{nc}$. Relatedly, we also study the universal C^*-algebra $C^*\langle M_n(\mathbb{C})^*\rangle$ of the operator space $M_n(\mathbb{C})^*$ (which is in fact the operator space structure associated to the generators of $\mathcal{U}_n^{nc}$) and show that it is isomorphic with the noncommutative C^*-algebra $\mathcal{C}_n^{nc}$, the universal unital C^*-algebra generated by elements u_{ij},

$1 \leq i, j \leq n$ where $[u_{ij}]$ is contractive. We exhibit several operator algebraic properties of $C^*\langle M_n(\mathbb{C})^*\rangle$- including, the Lifting property (LP), residual finite dimensionality and primitivity. Finally, motivated by natural quantum semigroup structures on $\mathcal{U}_n^{nc}$ [Man23b] and universal quantum homomorphisms of the form $\mathcal{U}(\mathcal{A}, \mathcal{A})$ for finite dimensional C^*-algebras $\mathcal{A}$ [BKSt23, FMP24], we discuss a natural compact quantum semigroup structure on $C^*\langle M_n(\mathbb{C})^*\rangle$, showing the separation by finite dimensional (co)-representations and characterizing invertible elements in its state space under convolution. However, we also show that this does not induce a compact quantum group structure on $C^*\langle M_n(\mathbb{C})^*\rangle$, suggesting that this quantum space might be associated to universal quantum homomorphisms [FMP24, St08], though not necessarily, invertible ones [SSt16].

In the section §4.1, we provide study C^*-algebraic structure of universal C^*-algebra of an operator space. First, we describe the operator space structure of the standard generators of $\mathcal{U}_n^{nc}$, which we denote by $\mathcal{E}_n^{nc}$, following Blecher [Ble92a] and Pisier [Pis20]. Then, we study the universal C^*-algebra $C^*\langle M_n(\mathbb{C})^*\rangle$ corresponding to this operator space structure of the generators of $\mathcal{U}_n^{nc}$. We deduce that $C^*\langle M_n(\mathbb{C})^*\rangle$ is not $*$-isomorphic to $\mathcal{U}_n^{nc}$, though $\mathcal{U}_n^{nc}$ is a quotient C^*-algebra of $C^*\langle M_n(\mathbb{C})^*\rangle$. Then, we identify this C^*-algebra $C^*\langle M_n(\mathbb{C})^*\rangle$ with the universal C^*-algebra $\mathcal{C}_n^{nc} = C^*\{u_{ij} : \ 1 \leq i, j \leq n, \ \|[u_{ij}]\| \leq 1\}$. Along with this identification, some C^*-algebraic properties are studied in this section, including LP, RFD, projections and primitivity.

The section §4.2 contains the maximal and minimal operator space structures of $M_n(\mathbb{C})^*$. In this context, we obtain the following: Let us consider ξ_{ij} as dual basis of $M_n(\mathbb{C})^*$, $1 \leq i, j \leq n$. Then, $\mathrm{MAX}(M_n(\mathbb{C})^*)$ embeds completely isometrically into $\mathcal{U}_n^{nc}$ via $\xi_{ij} \mapsto u_{ij}$ if and only if every contractive map $\varphi : M_n(\mathbb{C})^* \to \mathbf{B}(\mathcal{H})$ is completely contractive, where $\mathcal{H}$ is a Hilbert space, and u_{ij} are generators of $\mathcal{U}_n^{nc}$. Further, we prove that the $\mathrm{MIN}(M_n(\mathbb{C})^*)$ embeds completely isometrically into $C(\mathcal{U}(n))$, the space of all continuous functions on unitaries in $M_n(\mathbb{C})$.

In the section §4.3, we study the operator space structure of the standard generators of reduced C^*-algebra $\mathcal{U}_{n,\text{red}}^{nc}$, defined as the relative commutant of $\pi_{\varphi*\psi}(M_n(\mathbb{C}))$ in the reduced free product C^*-algebra $M_n(\mathbb{C}) *_{\text{red}} C(\mathbb{T})$, where φ and ψ are natural traces on $M_n(\mathbb{C})$ and $C(\mathbb{T})$ respectively. In our case, we have: The operator space structure of the standard generators of $\mathcal{U}_{n,\text{red}}^{nc}$ is completely isomorphic with $\mathcal{R}_{n^2} \cap \mathcal{C}_{n^2}$, where $\mathcal{R}_{n^2}$ and $\mathcal{C}_{n^2}$ are respectively row and column Hilbert subspaces of $\mathbf{B}(\ell^2) \oplus \mathbf{B}(\ell^2)$.

In last Chapter §5, we explore a natural compact quantum semigroup structure on $\mathcal{C}_n^{nc}$ and it representation theory. In fact, $\mathcal{U}_n^{nc}$ is a compact quantum subsemigroup of $\mathcal{C}_n^{nc}$ i.e., there exists a natural surjective $*$-homomorphism $\pi : \mathcal{C}_n^{nc} \to \mathcal{U}_n^{nc}$ which commutes with the co-multiplications. We explore the state space structure on $\mathcal{C}_n^{nc}$: in particular, we characterize the invertible elements of this compact semigroup of states- we show that the collection of all characters $\chi : \mathcal{C}_n^{nc} \to \mathbb{C}$ is semigroup isomorphic with $\mathbf{B}_{M_n(\mathbb{C})}$; consequently, every character need not be invertible but all invertible states on $\mathcal{C}_n^{nc}$ are characters. However, as we show, $\mathcal{C}_n^{nc}$ is not a compact quantum group with respect to this comultiplication.

Chapter 2

Preliminaries

In this section, we present some background about CP maps, universal C^*-algebras, Hilbert Bimodules and C^*-correspondences.

2.1 Operator Spaces and Operator Systems

The theory of operator spaces has been extensively studied following Ruan's work [Rua88], which provided an abstract characterization of such spaces. In the category, the objects are *Operator Spaces* and the morphisms are *Completely Bounded maps*. For a detailed discussion on the theory of operator spaces, we refer to Pisier [Pis03].

Definition 2.1.1. *An operator space is a closed subspace of* $\mathbf{B}(\mathcal{H})$ *for some Hilbert space* $\mathcal{H}$.

By the Gelfand-Naimark theorem, every C^* –algebra can be realised as a closed subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Consequently, we may view an operator space as a closed subspace of a C^*-algebra. A natural question that arises at this point is: *How does the notion of an operator space differ from that of a Banach space?* Addressing this distinction clarifies the additional structure carried by operator spaces. As a first step, we observe that every Banach space can be regarded as an operator space.

Lemma 2.1.2. *Let X be a Banach space. Then X can be isometrically embedded into a commutative C^*-algebra.*

Proof. Consider the dual Banach space X^* of X. Let B_{X^*} be the closed unit ball in X^*. Then if we consider the topological space $T = (B_{X^*}, wk^*)$, where wk^* is the weak* topology, we have an isometric embedding of X into $C(T)$. $\square$

The main distinction between the category of Banach spaces and the category of operator spaces lies in their morphisms. In the Banach space setting, morphisms are simply bounded linear maps. In contrast, for operator spaces, the morphisms are required to be *completely bounded* linear maps (which we will be defining next). This stronger notion arises naturally from the fact that, by virtue of being embedded in $B(\mathcal{H})$, an operator space $\mathcal{E}$ carries not only a norm structure on $\mathcal{E}$ itself, but also compatible matrix norm structures on each $M_n(\mathcal{E})$. For each $n \in \mathbb{N}$, the space $M_n(\mathcal{E})$ is identified with a subspace of $B(\mathcal{H}^{\oplus n})$ via the inclusion

$$[x_{ij}] \mapsto \left[(\xi_1, \dots, \xi_n) \mapsto \left(\sum_{j=1}^{n} x_{1j}\xi_j, \dots, \sum_{j=1}^{n} x_{nj}\xi_j \right) \right].$$

The *operator space norm* on $M_n(\mathcal{E})$ is then defined by

$$\|[x_{ij}]\|_{M_n(\mathcal{E})} := \left\|[x_{ij}]\right\|_{B(\mathcal{H}^{\oplus n})}.$$

The collection of norms $\{\| \cdot \|_{M_n(\mathcal{E})}\}_{n \geq 1}$ is called the *matrix norm structure* of $\mathcal{E}$.

Definition 2.1.3. *Let $\mathcal{E}$ and $\mathcal{F}$ be two operator spaces. For $n \geq 1$, consider the space*

$$M_n(\mathcal{E}) = \{(x_{ij}) : \ 1 \leq i, j \leq n, \ x_{ij} \in \mathcal{E}\}$$

be the space of all $n \times n$ matrices with entries in $\mathcal{E}$. Then for any linear map $\varphi : \mathcal{E} \to \mathcal{F}$, we define the linear maps as follows:

$$\varphi_n : M_n(\mathcal{E}) \to M_n(\mathcal{F}), \ \varphi_n\left((x_{ij})\right) = \left(\varphi(x_{ij})\right).$$

A linear map $\varphi : \mathcal{E} \to \mathcal{F}$ is called completely bounded if $\sup_{n\geq 1} \|\varphi_n\| < \infty$ and the completely bounded norm is denoted by $\|\varphi\|_{cb} = \sup_n \|\varphi_n\|$. The collection of all completely bounded maps from $\mathcal{E}$ to $\mathcal{F}$ is denoted by $CB(\mathcal{E}, \mathcal{F})$. If $\|\varphi\|_{cb} \leq 1$, φ is called completely contractive (c.c). Moreover, $\varphi : \mathcal{E} \to \mathcal{F}$ is called a complete isometry if each φ_n is an isometry for all n. Finally, two operator spaces $\mathcal{E}$ and $\mathcal{F}$ are called completely isomorphic if there exists a linear isomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ such that $\|\varphi\|_{cb}$ and $\|\varphi^{-1}\|_{cb}$ are finite.

2.1.1 Operator Systems and Completely Positive maps

Definition 2.1.4 (Operator System). *Let B be a unital C^*-algebra. A subspace $S \subseteq B$ is called an* operator system *if:*

1. $1_B \in S$,
2. *S is closed under the involution: if $x \in S$ then $x^* \in S$,*
3. *S is a complex linear space.*

An operator system need not be closed under multiplication, but it inherits an order structure from B via the positive cone

$$S^+ := \{x \in S : x \geq 0 \text{ in } B\}.$$

Remark 2.1.5. *Operator systems provide a natural framework for the study of positivity in operator algebra theory. From Definition 2.1.4, we observe that an operator system is equipped with a compatible family of positive cones on $M_n(\mathcal{S})$ for each $n \in \mathbb{N}$. This matrix-ordered structure captures positivity at all levels and allows one to distinguish between positive and completely positive maps, which we define next.*

Definition 2.1.6 (Positive and Completely Positive Maps). *Let $S \subseteq B(H)$ and $T \subseteq B(K)$ be operator systems. A linear map $\varphi : S \to T$ is called* positive *if $\varphi(S^+) \subseteq T^+$. For each $n \in \mathbb{N}$, we define the* amplification

$$\varphi_n : M_n(S) \to M_n(T), \quad \varphi_n([x_{ij}]) = [\varphi(x_{ij})],$$

where $M_n(S)$ is naturally identified with an operator system in $B(H^{\oplus n})$.

The map φ is called completely positive (CP) *if φ_n is positive for all $n \in \mathbb{N}$. If, in addition, $\varphi(1_S) = 1_T$, we call φ a* unital completely positive (UCP) map.

2.1.2 Ruan's Theorem

Let $\mathcal{E}$ be an operator space with $\mathcal{E} \subset \mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For each $n \in \mathbb{N}$, let $\|\cdot\|_n$ be the norm on $M_n(\mathcal{E})$. Let $x = (x_{ij})$ be an element in $M_n(\mathcal{E})$ and $\alpha, \beta \in M_n(\mathbb{C})$, then the following properties hold.

1. $\|\alpha x \beta\|_n \leq \|\alpha\|_{M_n(\mathbb{C})} \|x\|_n \|\beta\|_{M_n(\mathbb{C})}$.

2. For any element $y \in M_m(\mathcal{E})$, we have

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\| = \max\{\|x\|_n, \|y\|_m\}.$$

Indeed, (1) and (2) above are Ruan's axioms [Rua88]. Using Ruan's axioms, the following theorem characterizes an operator space.

Theorem 2.1.7. *Let $\mathcal{E}$ be a normed space. For each $n \in \mathbb{N}$, we have norm $\|\cdot\|_n$ on $M_n(\mathcal{E})$ which satisfies (1) and (2). Then, $\mathcal{E}$ is completely isometrically embedded into $\mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

2.1.3 Dual Operator Space Structure

Let $\mathcal{E}$ be an operator space, and denote by $\mathcal{E}^*$ its Banach space dual. The dual operator space structure on $\mathcal{E}^*$ can be described as follows.

Definition 2.1.8. *For each $n \geq 1$, we define a norm on $M_n(\mathcal{E}^*)$ as follows. For $[f_{ij}] \in M_n(\mathcal{E}^*)$, set*

$$\|[f_{ij}]\|_n = \sup\left\{\|[f_{ij}(a_{kl})]\| : m \geq 1,\ [a_{kl}] \in \mathbb{B}_{M_m(\mathcal{E})}\right\}.$$

By Theorem 2.1.7, the space $\mathcal{E}^*$ equipped with this sequence of norms is an operator space.

Remark 2.1.9. *For operator spaces $\mathcal{E}$ and $\mathcal{F}$, the space $CB(\mathcal{E}, \mathcal{F})$ of completely bounded maps carries a natural operator space structure. In particular, if we consider the dual operator space $\mathcal{E}^*$ as defined in Definition 2.1.8, we obtain a completely isometric isomorphism*

$$M_n(\mathcal{E}^*) \cong CB(\mathcal{E}, M_n), \quad n \geq 1.$$

2.1.4 Minimal and Maximal Operator Space Structure

For a normed linear space $\mathcal{E}$, each linear embedding into $\mathbf{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, induces an operator space structure on $\mathcal{E}$. However, there is a maximal and a minimal procedure of inducing operator space structures, as we describe below. We refer to [Pau92] for more details.

Definition 2.1.10. *Let $\mathcal{E}$ be a normed space. For each $n \geq 1$, we define the norm on $M_n(\mathcal{E})$ as follows.*

$$\|(x_{ij})\|_{\mathrm{MAX}} = \sup\{\|(\varphi(x_{ij}))\| : \varphi : \mathcal{E} \to \mathbf{B}(\mathcal{H}),\ \|\varphi\| \leq 1\}.$$

E together with the above family of norms is called the Maximal Operator Space*. We denote the maximal operator space structure on $\mathcal{E}$ by $\mathrm{MAX}(\mathcal{E})$.*

The *MAX norm* dominates every other matrix norm satisfying Ruan's Axioms. Next, we discuss the minimal operator space structure.

Definition 2.1.11. *Let $\mathcal{E}$ be a normed space. For each $n \geq 1$, we define the norm on $M_n(\mathcal{E})$ as follows.*

$$\|(x_{ij})\|_{\mathrm{MIN}} = \sup\{\|(f(x_{ij}))\| : f : \mathcal{E} \to \mathbb{C},\ \|f\| \leq 1\}.$$

We denote $\mathrm{MIN}(\mathcal{E})$ as the space $\mathcal{E}$ equipped with these matrix norms, which defines the minimal operator space structure on $\mathcal{E}$.

Remark 2.1.12. *With this definition, note that* $\mathrm{MIN}(\mathcal{E})$ *can be embedded completely isometrically into a commutative* C^**-algebra. In fact,* $\mathrm{MIN}(\mathcal{E})$ *can be embedded into* $C(\mathbf{B}_{\mathcal{E}^*})$*, where* $\mathbf{B}_{\mathcal{E}^*}$ *is the closed unit ball in dual of* $\mathcal{E}$*.*

The next lemma describes a relation between MIN and MAX operator space structures for more details we can look at [Pau92].

Lemma 2.1.13. *For a normed space* $\mathcal{E}$ *the dual operator space structure of* $\mathrm{MIN}(\mathcal{E})$ *and* $\mathrm{MAX}(\mathcal{E})$ *is related by* $\mathrm{MAX}(\mathcal{E})^* = \mathrm{MIN}(\mathcal{E}^*)$ *and* $\mathrm{MAX}(\mathcal{E}^*) = \mathrm{MIN}(\mathcal{E})^*$

2.2 Group C^*-algebras

Group C^*-algebras, particularly the full and reduced versions constitute a fundamental class within the theory of C^*-algebras, playing a central role in the study of operator algebras. Naturally, they will serve as an important set of examples in this work. In this section, we briefly recall the basic construction and properties of group C^*-algebras.
We consider our group G to be countable and discrete. Given such a countable discrete group G, the starting point for constructing group C^*-algebras is the *group ring* $\mathbb{C}[G]$. This is the complex vector space consisting of finite formal sums of the form

$$\sum_{g\in G} a_g\delta_g, \quad a_g \in \mathbb{C},$$

where addition and scalar multiplication are defined pointwise, and multiplication is given by convolution:

$$\left(\sum_{g\in G} a_g\delta_g\right) * \left(\sum_{h\in G} b_h\delta_h\right) = \sum_{k\in G}\left(\sum_{gh=k} a_g b_h\right)\delta_k.$$

This algebra is equipped with a $*$-operation defined by $\delta_g^* = \delta_{g^{-1}}$, extended linearly and conjugate-linearly.

$$(\sum_{g\in G} a_g\delta_g)^* = \sum_{g\in G} \overline{a_g}\delta_{g^{-1}}.$$

To obtain a C^*-algebra from $\mathbb{C}[G]$, one considers suitable completions under norms coming from representations.

2.2.1 Full Group C^*-Algebra $C^*(G)$

The *full group C^*-algebra*, denoted $C^*(G)$, is defined as the universal C^*-completion of $\mathbb{C}[G]$ with respect to all $*$-representations of $\mathbb{C}[G]$. We know that every unitary representation $u : G \to B(\mathcal{H})$ extends to a $*$-representation $\pi_u : \mathbb{C}[G] \to B(\mathcal{H})$ given by,

$$\pi_u \left(\sum_{g \in G} a_g \delta_g \right) = \sum_{g \in G} a_g u(g),$$

Considering the class of all such representations, we get the universal $*$-norm on $\mathbb{C}[G]$, for $x \in \mathbb{C}[G]$ we define,

$$\|x\|_{max} := sup \left\{ \|\pi_u(x)\| \;|\pi_u \text{ is a } * \text{-representation of } \mathbb{C}[G] \right\}$$

Now, completing $\mathbb{C}[G]$ with the above norm gives us the full group C^*-algebra $C^*(G)$. By virtue of its construction, $C^*(G)$ satisfies the following universal property:

Proposition 2.2.1. *For every unitary representation $u : G \to \mathcal{U}(\mathcal{H})$ of G on a Hilbert space $\mathcal{H}$, there exists a unique $*$-representation*

$$\pi_u : C^*(G) \to \mathcal{B}(\mathcal{H})$$

such that $\pi_u(\delta_g) = u(g)$ for all $g \in G$.

Remark 2.2.2. *The above remark holds for any C^*-algebra A, suppose if we have a group homomorphism $u : G \to \mathcal{U}(A) := \{a \in A \,|a^*a = aa^* = 1_A\}$ then there exists a unique $*$-homomorphism $\pi_u : C^*(G) \to A$, such that $\pi_u(\delta_g) = u(g)$ for all $g \in G$.*

We can observe that the universal property of $C^*(G)$ and the trivial representation of G, induces a non-trivial character on $\chi : C^*(G) \to \mathbb{C}$. Thus, $C^*(G)$ always admits a non-zero

$*$-homomorphism to $\mathbb{C}$.

2.2.2 Reduced Group C^*-Algebra $C_r^*(G)$

On the other hand, the *reduced group C^*-algebra*, denoted $C_r^*(G)$, is defined using the *left regular representation* $\lambda : G \to \mathcal{B}(\ell^2(G))$, where $\lambda(g)(\delta_h) = \delta_{gh}$. The representation λ extends to a faithful $*$-representation of $\mathbb{C}[G]$, and the reduced C^*-algebra is defined as the closure of $\lambda(\mathbb{C}[G])$ in the operator norm:

$$C_r^*(G) := \overline{\lambda(\mathbb{C}[G])} \subset \mathcal{B}(\ell^2(G)).$$

Proposition 2.2.3. *The vector state $\tau : C_r^*(G) \to \mathbb{C}$ defined by $x \mapsto \langle x\delta_e, \delta_e \rangle$ is a tracial state on $C_r^*(G)$.*

In general, there is a canonical surjective $*$-homomorphism $C^*(G) \twoheadrightarrow C_r^*(G)$, which is an isomorphism if and only if G is amenable, which we discuss next.

2.2.3 Amenable Groups

In this section, we present several characterizations of amenability, along with a few equivalent conditions that will serve as the working definitions of amenability for the remainder of this thesis.

Definition 2.2.4. *A countable discrete group G is said to be* amenable *if there exists a* state ϕ *on $\ell^\infty(G)$ that is* invariant *under the natural left translation action of G. That is, $\phi : \ell^\infty(G) \to \mathbb{C}$ is a positive linear functional of norm one such that*

$$\phi(f) = \phi(L_g f) \quad \text{for all } g \in G \text{ and } f \in \ell^\infty(G),$$

where the left translation $L_g f \in \ell^\infty(G)$ is defined by $(L_g f)(h) := f(g^{-1}h)$ for all $h \in G$.

The state ϕ is called an Invariant mean.

An equivalent and intuitive characterization of amenability for countable discrete groups is given by the *Følner condition*.

Definition 2.2.5. *A countable discrete group G is said to satisfy the* Følner condition *if for every finite subset $K \subset G$ and every $\varepsilon > 0$, there exists a finite, non-empty subset $F \subset G$ such that*

$$\frac{|kF \triangle F|}{|F|} < \varepsilon \quad \text{for all } k \in K,$$

where $\triangle$ denotes the symmetric difference of sets. This condition expresses the idea that the left translation action of G on itself becomes increasingly invariant on large finite subsets. A sequence of non-empty finite subsets $\{F_n\}_{n\in\mathbb{N}} \subset G$ such that for every $g \in G$,

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0,$$

is called a Følner sequence

Proposition 2.2.6. *Let G be a discrete group. The following are equivalent:*

1. *G is amenable.*
2. *G satisfies the Følner condition.*
3. *$C^*(G) = C^*_r(G)$, also the C^*-algebra is nuclear.*
4. *$C^*_r(G)$ has a character, i.e., one dimensional representation.*

Remark 2.2.7. *Examples of amenable groups include all* finite groups, abelian groups, *and more generally, all* solvable groups. *The class of amenable groups is closed under several important group-theoretic operations: taking* subgroups, quotients, extensions, *and* inductive limits. *Consequently, any group constructed from amenable groups using these operations remains amenable. In contrast,* free groups *on two or more generators provide a fundamental example of* non-amenable groups, *highlighting the boundary of this class.*

2.2.4 Operator space structure associated to generators of $C^*(\mathbb{F}_n)$ and $C^*_{red}(\mathbb{F}_n)$

We will briefly state some results about the operator space structure generated by the generators of the full and reduced group C^*-algebras with respect to $\mathbb{F}_n$. Let, $\{u_1, ..., u_{n-1}\}$ be the standard unitary generators of the full group C^*-algebra $C^*(\mathbb{F}_{n-1})$ and $u_0 = 1$. Now we define the operator space E_n spanned by u_0 and the generators $u_1, u_2, ..., u_{n-1}$, that is, $E_n = span\{u_0, u_1, ..., u_{n-1}\}$.

Lemma 2.2.8. *E_n is canonically isometric to ℓ^1_n, where ℓ^1_n is the n-dimensional ℓ^1 space.*

Proof. Let $(\alpha_k)_{k=0}^{n-1}$ be an element in $\mathbb{C}^n$. We already have $\|\sum_{k=0}^{n-1} \alpha_k u_k\| \leq \sum_{k=0}^{n-1} |\alpha_k|$. We intend to show the equality, so for the other direction we use the universal property of the group C^*-algebra $C^*(\mathbb{F}_n)$ and consider $\alpha_k/|\alpha_k|$ for $\alpha_k \neq 0$. □

Note that there is a one-to-one correspondence between elements $z = \sum u_k \otimes x_k \in E_n \otimes B(\ell^2)$ and $T_z : \ell^\infty \to B(\ell^2)$ such that

$$(\alpha_k)_{k=0}^{n-1} \mapsto \sum_{k=0}^{n-1} \alpha_k x_k \in B(\ell^2). \tag{2.1}$$

The next result describes the operator space structure generated by the generators of $C^*(\mathbb{F}_n)$.

Theorem 2.2.9. *The operator space E_n is completely isometrically isomorphic to the dual operator space $\ell^1_n = (\ell^\infty_n)^*$. In particular, $\|T_z\|_{cb} = \|z\|_{min}$ for all $z = \sum u_k \otimes x_k \in E_n \otimes B(\ell^2)$.*

For the proof of the above theorem, one can refer to [BO08, Lemma. 13.2.2]

The operator Space structure generated by the generators of $C^*_r(\mathbb{F}_n)$ was given by Uffe Haagerup and Gilles Pisier in [HP93]. For more details, we can refer to their paper; here, we will just state the statement of the theorem.

Theorem 2.2.10. *Let $E^{red}_n \subset M_n \oplus M_n$ be the operator space spanned by δ_i, where $\delta_i = e_{1i} \oplus e_{i1}$. Then there are linear mappings*

$$w : E_n^{red} \to C_r^*(\mathbb{F}_n) \text{ and } v : C_r^*(\mathbb{F}_n) \to E_n^{red}$$

such that

$$vw = I_{E_n^{red}} \text{ and } ||v||_{cb}||w||_{cb} \leq 2.$$

This shows that the operator space structure associated to the generators of $C_r^*(\mathbb{F}_n)$ is completely isomorphic to E_n^{red}.

2.3 The non-commutative unitary C^*–algebra

2.3.1 Universal C^*–algebra $\mathcal{U}_n^{nc}$

The algebra $\mathcal{U}_n^{nc}$ is an important object of study in the context of my work. In this section, we define $\mathcal{U}_n^{nc}$ and establish some of its basic structural properties.

Definition 2.3.1. *[Bro81] We define $\mathcal{U}_n^{nc}$ to be the universal unital C^*–algebra generated by elements $\{u_{ij} : 1 \leq i, j \leq n\}$, satisfying the relations which makes $U = [u_{ij}]$ an unitary matrix.*

This algebra is characterized by the following property: whenever $\mathcal{A}$ is a C^*–algebra equipped with elements $\{v_{ij}\}_{1\leq i,j\leq n}$ that form a unitary matrix $V = [v_{ij}] \in M_n(\mathcal{A})$, there exists a unique surjective $*$–homomorphism

$$\pi : \mathcal{U}_n^{nc} \longrightarrow \mathcal{A}$$

such that $\pi(u_{ij}) = v_{ij}$ for all $1 \leq i, j \leq n$.

Remark 2.3.2. *A relation can be drawn between $C^*(\mathbb{F}_n)$ and $\mathcal{U}_n^{nc}$, and this link is made clear using the universal property of $\mathcal{U}_n^{nc}$. Take the free group with n generators, namely $\mathbb{F}_n = \langle s_1, s_2, \ldots, s_n\rangle$. Then*

$$\begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n \end{bmatrix}$$

forms a unitary element in $M_n(C^*(\mathbb{F}_n))$. Therefore, due to the universal property of $\mathcal{U}_n^{nc}$, one obtains a uniquely determined surjective $*-$homomorphism

$$\pi : \mathcal{U}_n^{nc} \longrightarrow C^*(\mathbb{F}_n) \textit{ such that } \pi(u_{ij}) = \delta_{ij} s_i.$$

Moreover, the universal feature of $\mathbb{C}^*(\mathbb{F}_n)$ can also be seen as a consequence of the universality of $\mathcal{U}_n^{nc}$. In this sense, $\mathcal{U}_n^{nc}$ naturally extends $C^*(\mathbb{F}_n)$. Later, we will make analogous observations for their reduced versions as well.

Theorem 2.3.3. *[McC92][Man23b]The algebra $M_n(\mathbb{C}) * C(\mathbb{T})$ admits a $*$-isomorphism with $M_n(\mathbb{C})^c \otimes M_n(\mathbb{C})$, where $M_n(\mathbb{C})^c$ denotes the relative commutant of $M_n(\mathbb{C})$ inside $M_n(\mathbb{C}) * C(\mathbb{T})$.*

Proof. We include the proof here for continuity. Let $x \in M_n(\mathbb{C})^c$ and $y \in M_n(\mathbb{C})$. Since $xy = yx$, the universal property of the tensor product yields a $*$-homomorphism

$$\alpha : M_n(\mathbb{C})^c \otimes M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C}) * C(\mathbb{T}), \qquad \alpha(x \otimes y) = xy.$$

Define maps

$$\beta_1 : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})^c \otimes M_n(\mathbb{C}), \quad x \mapsto 1 \otimes x,$$

and

$$\beta_2 : C(\mathbb{T}) = C^*(\mathbb{Z}) \longrightarrow M_n(\mathbb{C})^c \otimes M_n(\mathbb{C}), \qquad 1 \mapsto \sum_{i,j} \Big(\sum_k e_{ki} u e_{jk} \Big) \otimes e_{ij}.$$

where u is the canonical unitary defined by $u(z) = z$ for $z \in \mathbb{T}$, which generates the algebra $C(\mathbb{T})$. Universal property of free product of C^*-algebras gives us, that there exists a $*$-homomorphism

$$\beta = \beta_1 * \beta_2 : M_n(\mathbb{C}) * C(\mathbb{T}) \longrightarrow M_n(\mathbb{C})^c \otimes M_n(\mathbb{C}).$$

Now, some simple calculations give us $\alpha \circ \beta = \beta \circ \alpha = \mathrm{id}$.

$\square$

The next theorem in [McC92]provides us with a very important identification of $\mathcal{U}_n^{nc}$.

Theorem 2.3.4. *[McC92] The algebra $\mathcal{U}_n^{nc}$ can be identified with the relative commutant $M_n(\mathbb{C})^c$ inside $M_n(\mathbb{C}) * C(\mathbb{T})$. This identification is realised through the mapping*

$$u_{ij} \mapsto \sum_k e_{ki} u e_{jk},$$

where e_{ij} denotes the standard matrix units and u is the canonical unitary defined by $u(z) = z$ for $z \in \mathbb{T}$, which generates the algebra $C(\mathbb{T})$.

2.3.2 Reduced Free products of C^*-algebras

We first recall the free product of C^*-algebras with respect to given states. A detailed discussion can be found in [Avi82].

Let $\mathcal{A}$ and $\mathbf{B}$ be two C^*-algebras with states $\varphi : \mathcal{A} \to \mathbb{C}$ and $\psi : \mathbf{B} \to \mathbb{C}$. Denote the GNS representation associated to these two states $\varphi : \mathcal{A} \to \mathbb{C}$ and $\psi : \mathbf{B} \to \mathbb{C}$ by $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ and $(\pi_\psi, \mathcal{H}_\psi, \xi_\psi)$, respectively. Consider $\mathcal{H}_\varphi^\circ = \xi_\varphi^\perp$, $\mathcal{H}_\psi^\circ = \xi_\psi^\perp$. First, we describe the construction of the free product of Hilbert spaces for these states. Consider a Hilbert space

$$\mathcal{H}_{\varphi*\psi} = \mathbb{C}\xi_{\varphi*\psi} \oplus \mathcal{H}_\varphi^\circ \oplus \mathcal{H}_\psi^\circ \oplus (\mathcal{H}_\varphi^\circ \otimes \mathcal{H}_\psi^\circ) \oplus (\mathcal{H}_\psi^\circ \otimes \mathcal{H}_\varphi^\circ) \oplus (\mathcal{H}_\varphi^\circ \otimes \mathcal{H}_\psi^\circ \otimes \mathcal{H}_\varphi^\circ) \oplus \cdots,$$

with distinguish vector $\xi_{\varphi*\psi}$. We frequently identify $\mathcal{H}_\varphi$ with $\mathbb{C}\xi_{\varphi*\psi} \oplus \mathcal{H}_\varphi^\circ$ and $\mathcal{H}_\psi$ with $\mathbb{C}\xi_{\varphi*\psi} \oplus \mathcal{H}_\psi^\circ$. Now, we construct a representation $\pi_{\varphi*\psi}$ of the C^*-algebra $\mathcal{A} * \mathbf{B}$ on the Hilbert space $\mathcal{H}_{\varphi*\psi}$ as follows. For $a \in \mathcal{A}$ and $h_\varphi^i \in \mathcal{H}_\varphi^\circ$, $h_\psi^i \in \mathcal{H}_\psi^\circ$,

$$\pi_\varphi(a) h_\varphi^i = (\pi_\varphi(a) h_\varphi^i)_\circ + \lambda \xi_\varphi,$$

$$\pi_\varphi(a) \xi_\varphi = (\pi_\varphi(a) \xi_\varphi)_\circ + \mu \xi_\varphi,$$

where $\mathcal{H}_\varphi = \mathcal{H}_\varphi^\circ \oplus \mathbb{C}\xi_\varphi$ and $(\pi_\varphi(a) h_\varphi^i)_\circ, (\pi_\varphi(a)\xi_\varphi)_\circ \in \mathcal{H}_\varphi^\circ$.

Thus, define

$$\pi_{\varphi*\psi}(a)\xi_{\varphi*\psi} = (\pi_\varphi(a)\xi_\varphi)_\circ + \mu\xi_{\varphi*\psi}$$

$$\pi_{\varphi*\psi}(a)(h^1_\varphi \otimes h^1_\psi \otimes \cdots) = (\pi_\varphi(a)h^1_\varphi)_\circ \otimes h^1_\psi \otimes \cdots + \lambda(h^1_\psi \otimes \cdots).$$

$$\pi_{\varphi*\psi}(a)(h^1_\psi \otimes h^2_\psi \otimes \cdots) = (\pi_\varphi(a)\xi_\varphi)_\circ \otimes h^1_\psi \otimes h^2_\psi \otimes \cdots + \lambda(h^1_\psi \otimes h^2_\psi \otimes \cdots).$$

Similarly, we define $\pi_{\varphi*\psi}$ for $b \in \mathbf{B}$ on the Hilbert space $\mathcal{H}_{\varphi*\psi}$. Now, by the universal property of full free product of C^*-algebras, we can define $\pi_{\varphi*\psi}$ on $\mathcal{A} *_{\mathbb{C}} \mathbf{B}$ determined by $(\pi_{\varphi*\psi}|_{\mathcal{A}},\ \pi_{\varphi*\psi}|_{\mathbf{B}})$.

Definition 2.3.5. *The Reduced Free product of C^*-algebras A and B with respect to states ϕ and ψ is denoted by $A *_{red} B$ and is defined by $A *_{red} B := \pi_{\phi*\psi}(A *_{\mathbb{C}} B)$.*

Lemma 2.3.6. *[Avi82] If the states in consideration ϕ and ψ are faithful, then $\xi_{\phi*\psi}$ is a separating vector for $A *_{\text{alg}} B$. Consequently, $\xi_{\phi*\psi}$ is a separating vector for $A *_{red} B$ if ϕ and ψ are traces. In particular, $\pi_{\phi*\psi}$ is faithful on $A *_{red} B$.*

Remark 2.3.7. *We can define a state $\phi * \psi(T) = \langle T\xi_{\phi*\psi}, \xi_{\phi*\psi}\rangle$. Given that ϕ and ψ are faithful, we can conclude that $\phi * \psi$ is also faithful as $\xi_{\phi*\psi}$ is separating on $A *_{\mathbb{C}} B$. Also, if ϕ and ψ are traces, then $\phi * \psi$ is a faithful trace.*

Next, we mention a result proved by Avitzour in [Avi82] which gives a classification of simplicity for reduced free products.

Proposition 2.3.8. *Let A and B be C^*-algebras with faithful traces ϕ and ψ, respectively. If there are unitaries $k \in A$ and $l, m \in B$ such that $\phi(k) = 0$ and $\psi(l) = \psi(m) = \psi(l^*m) = 0$. Then $A *_{red} B$ is simple and $\phi * \psi$ is the unique trace on it.*

2.3.3 Reduced C^*-algebra $\mathcal{U}^{nc}_{n,red}$

We use the theory of reduced free products to construct $\mathcal{U}^{nc}_{n,red}$. Recall that Theorem 2.3.4 states $\mathcal{U}^{nc}_n$ is isomorphic to the relative commutant $M_n(\mathbb{C})^c$ in $M_n(\mathbb{C}) * C(\mathbb{T})$. Therefore, constructing the reduced free product $M_n(\mathbb{C})*_{red}C(\mathbb{T})$ plays a crucial role in the construction of $\mathcal{U}^{nc}_{n,red}$.

Definition 2.3.9. *Consider the natural traces* $\mathrm{Tr} : M_n \to \mathbb{C}$ *and* $h_{\mathbb{T}} : C(\mathbb{T}) \to \mathbb{C}$*, defined by,*

$$\mathrm{Tr}([a_{ij}]) = \frac{1}{n}\sum a_{ii},\ [a_{ij}] \in M_n \text{ and } h_{\mathbb{T}}(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta,\ f \in C(\mathbb{T})$$

It can be deduced that the GNS-Space for $\mathrm{Tr} : M_n \to \mathbb{C}$ *and* $h_{\mathbb{T}} : C(\mathbb{T}) \to \mathbb{C}$ *are* $(\mathcal{H}_{\mathrm{Tr}}, \pi_{\mathrm{Tr}}, \xi_{Tr})$ *and* $(\mathcal{H}_{h_{\mathbb{T}}}, \pi_{h_{\mathbb{T}}}, \xi_{h_{\mathbb{T}}})$ *respectively, where,*

$$H_{\mathrm{Tr}} = \mathbb{C}^{n^2},\ \pi_{\mathrm{Tr}}(x) = x \oplus x \oplus ... \oplus x,\ \xi_{(Tr)} = e_1 \oplus e_1 \oplus ... \oplus e_1,\ \text{and}$$
$$H_{h_{\mathbb{T}}} = L^2(\mathbb{T}),\ \pi_{h_{\mathbb{T}}}(f) = L_f,\ \xi_{h_{\mathbb{T}}} = 1,$$

where, $x \in \mathbb{C}^{n^2}$ *and* $L_f(g) = f \cdot g,\ g \in L^2(\mathbb{T})$*.*
$\mathcal{U}^{nc}_{n,red}$ *is the relative commutant* $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(M_n)^c$ *of* $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(M_n)$ *in* $M_n *_{red} C(\mathbb{T})$*.*

The following result further highlights the similarity in behaviour between $\mathcal{U}^{nc}_n$ and $\mathcal{U}^{nc}_{n,red}$, analogous to the relationship between the full and reduced group C^*-algebras $C^*(\mathbb{F}_n)$ and $C^*_{\mathrm{red}}(\mathbb{F}_n)$. In particular, it generalizes the classical fact that the reduced group C^*-algebra $C^*_{\mathrm{red}}(\mathbb{F}_n)$ is a homomorphic image of the full group C^*-algebra $C^*(\mathbb{F}_n)$ via the canonical homomorphism.

Lemma 2.3.10. *There is a surjective $*$-homomorphism* $\pi_{\mathrm{Tr}*h_{\mathbb{T}}} : \mathcal{U}^{nc}_n \to \mathcal{U}^{nc}_{n,red}$

Proof. First we observe that, $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(M^c_n) \subseteq \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(M_n)^c$ and $M^c_n \cong \mathcal{U}^{nc}_n$. Thus, $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}|_{M^c_n}$ gives a homomorphism from $\mathcal{U}^{nc}_n$ to $\mathcal{U}^{nc}_{n,red}$, and we show that it is surjective.
Let $x \in \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(M_n)^c$. Then $x = \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(x_\circ)$, for $x_\circ \in M_n *_{\mathbb{C}} C(\mathbb{T})$. Now, observe that $\sum e_{k1}x_\circ e_{1k}$ is in M^c_n and what we intend to show that is $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(\sum e_{k1}x_\circ e_{1k}) = x$ i.e. $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}|_{\mathcal{U}^{nc}_n}$

is surjective.

$$
\begin{aligned}
x &= x\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(\sum_{k=1}^{n} e_{k1}e_{1k}) \\
&= \sum_{k=1}^{n} \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(e_{k1})x\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(e_{1k}), \{\text{ Since, } x \in \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(M_n)^c\} \\
&= \sum_{k=1}^{n} \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(e_{k1})\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(x_\circ)\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(e_{1k}) \\
&= \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(\sum_{k} e_{k1}x_\circ e_{1k})
\end{aligned}
$$

Thus, $\mathcal{U}^{nc}_{n,red} = \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(\mathcal{U}^{nc}_n)$ and $\mathcal{U}^{nc}_{n,red}$ is generated by $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})$. We will identify $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})$ with u_{ij} and moreover, any $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(x)$ with x, for any $x \in M_n *^{\mathrm{alg}}_{\mathbb{C}} C(\mathbb{T})$ as $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}$ is an isomorphism restricted to the algebraic free product.

$\square$

Remark 2.3.11. *We can also conclude that*

$$M_n \otimes \mathcal{U}^{nc}_{n,red} \cong M_n *_{\mathrm{red}} C(\mathbb{T}).$$

Using Proposition 2.3.8, we can show that $M_n *_{\mathrm{red}} C(\mathbb{T})$ is simple. We choose an element $a \in M_n$ to be any trace-zero matrix. For $b, c \in C(\mathbb{T})$, let $b = z$ and $c = \bar{z}$, where z denotes the canonical generating unitary in $C(\mathbb{T})$ and the rest follows.
Now since, $M_n *_{\mathrm{red}} C(\mathbb{T})$ is simple, it implies that $M_n \otimes \mathcal{U}^{nc}_{n,red}$ is also simple, which in turn also implies that $\mathcal{U}^{nc}_{n,red}$ is simple.

2.4 Universal C^*-algebra

In [Kwa17], Kwaśniewski defined an equivalent definition for Universal C^*-algebra which is much more compatible with the current topic of generalised crossed products. Kwaśniewski defined a universal C^*-algebra for a set $\mathcal{G}$ and the set of relations $\mathcal{R}$, where $\mathcal{R}$ is a collection of maps from $\mathcal{G}$ to C^*-algebras.

$$\mathcal{R} = \{\beta_A |\ \beta_A : \mathcal{G} \to A\}$$

Now, we introduce a preorder order on the set $\mathcal{R}$. First note that for every $\beta_A \in \mathcal{R}$ we get a C^*-subalgebra $C^*\langle \beta_A(\mathcal{G})\rangle$ generated by $\mathcal{G}$ inside A. Now, we say $\beta_B \preceq \beta_A$ if the map $\beta_A(a) \mapsto \beta_B(a),\ \ a \in \mathcal{G}$, extends to a surjective homomorphism from $C^*\langle \beta_A(\mathcal{G})\rangle$ onto $C^*\langle \beta_B(\mathcal{G})\rangle$. We say the above-mentioned relations are equivalent and denote it by $\beta_A \approx \beta_B$ if $\beta_A \preceq \beta_B$ and $\beta_B \preceq \beta_A$.

Definition 2.4.1. *Assuming, the class of Relations $\mathcal{R}$ admits a upper bound $i \in \mathcal{R}$ such that $\beta_A \preceq i,\ \ \forall \beta_A \in \mathcal{R}$, then we say the universal algebra generated by $\mathcal{G}$ is $C^*(\mathcal{G}, \mathcal{R}) := C^*\langle i(G)\rangle$ and i is the universal representation of $\mathcal{G}$ for $\mathcal{R}$.*

By preorder relations, we can comment about the uniqueness of the universal C^*-algebra $C^*(\mathcal{G}, \mathcal{R})$. Since if there exist two universal representations, say i_1 and i_2, then we can easily conclude that $i_1 \approx i_2$ which leads to a $*$-isomorphism between the resulting Universal C^*-algebra.

Definition 2.4.2. *[Kwa17]We say that the class of relations $\mathcal{R}$ associated with a set $\mathcal{G}$ is* "closed under product" *if, for every family of maps $\beta_i : \mathcal{G} \to A_i$, $i \in I$, belonging to $\mathcal{R}$, the following conditions hold:*

1. *$\prod_{i\in I} \beta_i(a)$ lies in $\prod_{i\in I} A_i$ for all $a \in \mathcal{G}$.*

2. *There exists an injective homomorphism $g : \prod_{i\in I} A_i \to A$ such that the map $\beta_A : \mathcal{G} \to A$ is defined by*

$$\beta_A(a) := g\left(\prod_{i\in I} \beta_i(a)\right), \qquad a \in \mathcal{G},$$

also belongs to $\mathcal{R}$.

The next proposition in [Kwa17, Proposition 2.8] gives us a condition guaranteeing the existence of the universal representation for a set $\mathcal{G}$ with respect to the set of relations $\mathcal{R}$.

Proposition 2.4.3. *[Kwa17] Suppose the family of relations $\mathcal{R}$ on a generating set $\mathcal{G}$ is "closed under products" then $\mathcal{R}$ possesses an upper bound, and hence the universal C^*-algebra determined by $\mathcal{G}$ with relations from $\mathcal{R}$ exists.*

2.5 Hilbert C^*-Modules and C^*-Correspondences

We provide a brief introduction to *Hilbert C^*-modules* and *C^*-correspondences*. For a more detailed discussion, [Lan95] serves as a standard reference.

Given a C^*-algebra A. A (right) *Hilbert C^*-module* over a C^*-algebra A is a complex vector space X, endowed with the following structure:

- a right A-module structure, and
- an A-valued inner product $\langle \cdot, \cdot \rangle_A : X \times X \to A$, we consider linearity on the second variable and conjugate linearity on the first.

satisfying the following properties for all $x, y, z \in X$, $a \in A$, and $\lambda \in \mathbb{C}$:

1. $\langle x, x \rangle_A \geq 0$, and $\langle x, x \rangle_A = 0$ implies $x = 0$,
2. $\langle x, y + \lambda z \rangle_A = \langle x, y \rangle_A + \lambda \langle x, z \rangle_A$,
3. $\langle x, y.a \rangle_A = \langle x, y \rangle_A a$,
4. $\langle x, y \rangle_A = \langle y, x \rangle_A^*$.

We require X to be complete in the norm $\|x\| = \|\langle x, x \rangle_A\|^{1/2}$, making it a Banach space.

Definition 2.5.1. *Let A be a C^*-algebra. A C^*-*correspondence *over A is a (right) Hilbert C^*-module X over A together with a$*$-homomorphism $\phi : A \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ denotes the C^*-algebra of adjointable operators on X. The map ϕ gives X a left A-module structure via $a \cdot x := \phi(a)(x)$, satisfying*

$$\phi(a)(x \cdot b) = (\phi(a)x) \cdot b \quad \textit{for all } a, b \in A,\ x \in X.$$

These structures play a central role in the study of operator algebras, particularly in the construction of Cuntz–Pimsner algebras.

Definition 2.5.2. *Let X be a C^*-correspondence over a C^*-algebra A, with left action $\phi : A \to \mathcal{L}(X)$. We say that X is an* essential C^*-correspondence *if*

$$\overline{\text{span}}\{\phi(a)x : a \in A,\, x \in X\} = X.$$

Equivalently, the left action of A on X is said to be nondegenerate.

Example 2.5.3. *Let A be a C^*-algebra. Then A itself becomes a C^*-correspondence over A in the following natural way:*

- *The Hilbert A-module structure is given by the right multiplication on A and the A-valued inner product defined as*
$$\langle a, b\rangle_A := a^*b, \quad \textit{for all } a, b \in A.$$
- *The left action of A is given by left multiplication, i.e., $\phi(a)(b) := ab$. This defines a $*$-homomorphism $\phi : A \to \mathcal{L}(A)$, since left multiplication is adjointable.*

This makes A a C^*-correspondence over itself.
More generally, any $*$-homomorphism $\phi : A \to \mathcal{L}(X)$ gives rise to a C^*-correspondence structure on a Hilbert A-module X, and such correspondences play a fundamental role in the theory of Cuntz–Pimsner algebras.

2.6 Relative Cuntz-Pimsner Algebra

Let A be a C^*-algebra and X be a C^*-correspondence with the left action of A given by the $*$-homomorphism $\phi : A \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the collection of all adjointable operators on X (one can show that it is a C^*-algebra), thus the left module action of A on X is given by $a.x = \phi(a)(x)$.

Definition 2.6.1. *We say the pair* (π, π_X) *is a representation for the pair* (A, X) *consisting of a* $*$*-representation* $\pi : A \to B(H)$ *and a* A*-linear map* $\pi_X : X \to B(H)$ *such that* $\pi_X(x)^*\pi_X(y) = \pi(\langle x, y\rangle_A)$*, for all* $x, y \in X$ *and* $\pi(a)\pi_X(x) = \pi_X(\phi(a)x)$ *for all* $a \in A$ *and* $x \in X$.

Remark 2.6.2. *If we observe the relation* $\pi_X(x)^*\pi_X(y) = \pi(\langle x, y\rangle_A)$ *what it really says is that the maps* π *and* π_X *interact in such a way that they preserve the inner product structure of* X *over* A *when mapped to the canonical* C^**-correspondence structure of* $B(H)$ *over itself, that is,*

$$\pi(\langle x, y\rangle_A) = \langle \pi_X(x), \pi_X(y)\rangle_{B(H)}.$$

Given a representation (π, π_X) we denote the C^*-algebra generated by the union of $\pi(A)$ and $\pi_X(X)$ by $C^*(\pi, \pi_X)$.

Remark 2.6.3. *For a representation* (π, π_X)*,* π_X *is always contractive,*

$$\begin{aligned}
\|\pi_X(x)\|^2 &= \|\pi_X(x)^*\pi_X(x)\| \\
&= \|\pi(\langle x, x\rangle_A)\| \\
&\leq \|\langle x, x\rangle_A\| \\
&= \|x\|^2
\end{aligned}$$

also, if π *is faithful,* π_X *is an isometry.*

In this case, we try to create a universal C^*-algebra with respect to A and X as we defined in Definition 2.4.1. We choose the generating set $\mathcal{G} := A \cup X$ and the set of relations $\mathcal{R}$ consists of representations we define for (A, X) in Definition 2.6.1. One can verify that $\mathcal{R}$ satisfies the conditions in Definition 2.4.2, which makes it closed under product. Now, from Proposition 2.4.3 we can confirm the existence of the universal representation (i_A, i_X) and we call the universal C^*-algebra corresponding to it, the Toeplitz Algebra $\mathcal{T}(A, X) := C^*(i_A, i_X) = C^*(i_A(A) \cup i_X(X))$.

Definition 2.6.4. *For the universal representation* (i_A, i_X) *define the universal* C^**-algebra generated by* $i_A(A)$ *and* $i_X(X)$ *i.e.* $\mathcal{T}(A, X) := C^*(i_A, i_X)$ *we call it the Toeplitz Algebra.*

Let us denote the collection of all adjointable compact operators by $\mathcal{K}(X) = \overline{span\{\theta_{x,y} | \ x, y \in X\}}$ where $\theta_{x,y}(w) = x\langle y, w\rangle_A$ for $x, y, w \in X$. For any representation (π, π_X) one can define a homomorphism $(\pi, \pi_X)^{(1)} : \mathcal{K}(X) \to B(H)$ which satisfy,

- $(\pi, \pi_X)^{(1)}(\theta_{x,y}) = \pi_X(x)\pi_X(y)^*$
- $(\pi, \pi_X)^{(1)}(T)\pi_X(x) = \pi_X(Tx)$

for $x, y \in X$ and $T \in \mathcal{K}(X)$.

We can check the second condition by choosing $T = \theta_{l,m}, \ l, m \in X$

$$\begin{aligned}
(\pi, \pi_X)^{(1)}(\theta_{l,m})\pi_X(x) &= \pi_X(l)\pi_X(m)^*\pi_X(x) \\
&= \pi_X(l)\pi(\langle m, x\rangle_A) \\
&= \pi_X(l\langle m, x\rangle_A) \\
&= \pi_X(\theta_{l,m}(x))
\end{aligned}$$

Let $J(X) := \phi^{-1}(\mathcal{K}(X))$, also note that $\mathcal{K}(X)$ is an ideal in $\mathcal{L}(X)$ which implies $J(X)$ is an ideal in A. Now, given any representation (π, π_X) the restriction $(\pi, \pi_X)^{(1)} \circ \phi|_{J(X)}$ and $\pi|_{J(X)}$ are representations on $J(X)$.

Definition 2.6.5. *We say that the representation (π, π_X) of (A, X) is J-*covariant *if the condition*

$$(\pi, \pi_X)^{(1)}(\phi(a)) = \pi(a)$$

holds for all $a \in J$, where J is an ideal in $J(X)$.

Now, consider the same set $\mathcal{G} = A \cup X$ and $\mathcal{R}'$ the new set of relations for $\mathcal{G}$ be the collection of all J-*Covariant representation for the pair* (A, X). We first observe that $\mathcal{R}' \subseteq \mathcal{R}$ and similarly to $\mathcal{R}$, we observe that $\mathcal{R}'$ also satisfies the properties of being *closed under product* according to the Definition 2.4.2. Thus, using Proposition 2.4.3 we can conclude there exists a universal representation for $\mathcal{G} = A \cup X$ with respect to $\mathcal{R}'$.

Definition 2.6.6. *Let (j_A, j_X) be the "universal J-covariant representation" of (A, X). Then the relative Cuntz-Pimsner Algebra is defined as $\mathcal{O}(J, X) := C^*(j_A, j_X)$*

Remark 2.6.7. *It can be shown that,*

$$\mathcal{O}(J, X) = \mathcal{T}(A, X)/\mathcal{I}$$

where, $\mathcal{I}$ is the ideal generated by $\{i_A(a) - (i_A, i_X)^{(1)}(\phi(a)) : a \in J\}$. Muhly and Solel in their paper [MS98] actually defines $\mathcal{O}(J, X)$ as Quotients of $\mathcal{T}(A, X)$

From [Kat03b, Propostion 2.21] and [MS98, Proposition 3.3] the next proposition follows,

Proposition 2.6.8. *For the pair (A, X) and considering the ideal J in $J(X)$. The universal representation $j_A : A \to \mathcal{O}(J, X)$ is injective* if and only if $J \subseteq (\ker\phi)^\perp$.

Katsura in his paper [Kat03a] and [Kat03b] observed that the most natural ideal to consider is $J_X := (\ker\phi)^\perp \cap J(X)$, it is the largest ideal in $J(X)$ where we get the universal representation is injective due to Proposition 2.6.8. It is also the ideal to which the restriction of ϕ is injective into $\mathcal{K}(X)$.
The next theorem gives more insight into the connection between Katsura's ideal and the ideal of redundancies introduced in the next section. This also serves as a motivation for *Covariant representation*

Proposition 2.6.9. *For the pair (A, X), where A is a C^*-algebra and X is C^*-correspondence over A, along with a injective representation (π, π_X), if $a \in A$ satisfies $\pi(a) \in (\pi, \pi_X)^{(1)}(\mathcal{K}(X))$, then we have $a \in J_X$ and $\pi(a) = (\pi, \pi_X)^{(1)}(\phi(a))$.*

Proof. For $a \in A$ with $\pi(a) \in (\pi, \pi_X)^{(1)}(\mathcal{K}(X))$. Let $k \in \mathcal{K}(X)$ be an element with $\pi(a) = (\pi, \pi_X)^{(1)}(k)$. For each $x \in X$, we have

$$\pi_X(\phi(a)x) = \pi(a)\pi_X(x) = (\pi, \pi_X)^{(1)}(k)\pi_X(x) = \pi_X(kx).$$

Now, π is injective $\Rightarrow \pi_X$ is injective$\Rightarrow \phi(a)x = kx$ for every $x \in X \Rightarrow \phi(a) = k \in \mathcal{K}(X)$. Thus we get $\pi(a) = (\pi, \pi_X)^{(1)}(\phi(a))$. Take $b \in \ker\phi$ and we will show that $ab = 0$. We get

$$\pi(ab) = \pi(a)\pi(b) = (\pi, \pi_X)^{(1)}(\phi(a))\pi(b) = (\pi, \pi_X)^{(1)}(\phi(a)\phi(b)) = 0$$

Since π is injective, we get $ab = 0 \Rightarrow a \in (\ker\phi)^{\perp}$, and as we have already shown that, $\phi(a) = k \in \mathcal{K}(X) \Rightarrow a \in J(X) \Rightarrow a \in J(X) \cap (\ker\phi)^{\perp} = J_X$ $\square$

Definition 2.6.10. *The (Unrelative) Cuntz-Pimsner Algebra associated with the C^*-algebra A and the C^*-correspondence X is the one we get with respect to the Katsura's Ideal J_X, i.e. $\mathcal{O}(J_X, X)$.*

Remark 2.6.11. *If a C^*-correspondence X is essential in the sense of Definition 2.5.2, it suffices to only consider representation (π, π_X) on $B(H)$ where π is non-degenerate, due to [BRV10, Lemma 3.4]. Consequently, since throughout this thesis we restrict our attention to essential C^*-correspondences, it is enough to consider representation (π, π_X) for which π is non-degenerate.*

2.7 Crossed Products of C^*-Algebras

Crossed products are fundamental constructions in the theory of C^*-algebras, used to study and encode the action of a group on a C^*-algebra. Given a C^*-algebra A and a group G acting on A via $*$-automorphisms, the crossed product $A \rtimes_\alpha G$ is a new C^*-algebra that encodes both the algebraic structure of A and the dynamics of the group action.

Formally, let $\alpha : G \to \mathrm{Aut}(A)$ be a group homomorphism from a countable discrete group G into the group of $*$-automorphisms of A, giving a dynamical system (A, G, α). The crossed product $A \rtimes_\alpha G$ is constructed by completing the $*$-algebra of compactly supported functions $f : G \to A$, under convolution and involution twisted by α.

2.7.1 Universal Crossed Product by a Discrete Countable Group

Let A be a C^*-algebra, and let G be a discrete countable group. Suppose we are given an action $\alpha : G \to \mathrm{Aut}(A)$, which turns (A, G, α) into a C^*-dynamical system.

We now describe the construction of the *universal (or full) crossed product* $A \rtimes_\alpha G$.

Algebraic Preliminaries

Consider the $*$-algebra $C_c(G, A)$ of finitely supported functions $f : G \to A$, equipped with pointwise linear operations and the following convolution product and involution:

$$(f * g)(t) := \sum_{s \in G} f(s)\alpha_s\Big(g(s^{-1}t)\Big), \quad f^*(t) := \alpha_t\left(f(t^{-1})^*\right).$$

This algebra $C_c(G, A)$ plays the role of a dense $*$-subalgebra of the crossed product.

Covariant Representations

A *covariant representation* of the system (A, G, α) on a Hilbert space $\mathcal{H}$ consists of a pair (π, U), where

- $\pi : A \to B(\mathcal{H})$ is a$*$-representation,
- $U : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation,

satisfying the covariance condition:

$$U_g \pi(a) U_g^* = \pi(\alpha_g(a)), \quad \forall g \in G,\ a \in A.$$

Each such covariant pair defines a$*$-representation $\pi \times U$ of $C_c(G, A)$ on $\mathcal{H}$ via:

$$(\pi \times U)(f) := \sum_{g \in G} \pi(f(g))U_g.$$

Definition of the Universal Crossed Product

The *universal norm* on $C_c(G, A)$ is defined by

$$\|f\| := \sup_{(\pi,U)} \|(\pi \times U)(f)\|,$$

where the supremum is taken over all covariant representations (π, U) of the system (A, G, α). The *universal crossed product* $A \rtimes_\alpha G$ is then defined as the completion of $C_c(G, A)$ with respect to this norm. That is,

$$A \rtimes_\alpha G := \overline{C_c(G, A)}^{\|\cdot\|}.$$

This crossed product satisfies the following *universal property*: for every covariant representation (π, U), there exists a unique$*$-homomorphism

$$\pi \times U : A \rtimes_\alpha G \to B(\mathcal{H})$$

extending the integrated form $f \mapsto \sum_{g \in G} \pi(f(g))U_g$.

2.7.2 Reduced Crossed Product by a Discrete Countable Group

We now turn to the construction of the *reduced crossed product* associated to a C^*-dynamical system (A, G, α), where G is a discrete countable group and A is a C^*-algebra.

The Regular Covariant Representation

Let $\mathcal{H}$ be a Hilbert space on which A acts faithfully via a $*$-representation $\pi : A \to B(\mathcal{H})$ and for the group G we can define the *left regular representation* to $\mathcal{U}(\ell^2(G))$. Define the Hilbert space

$$\mathcal{H}_G := \ell^2(G) \otimes \mathcal{H},$$

and define the *regular covariant representation* $(\tilde{\pi}, \tilde{\lambda})$ on $\mathcal{H}_G$ by extending π and λ as follows:

- For each $a \in A$, define $\tilde{\pi} : A \to B(\ell^2(G) \otimes \mathcal{H})$ as

$$\tilde{\pi}(a)(\delta_g \otimes \xi) := \delta_g \otimes \pi\Big(\alpha_{g^{-1}}(a)\Big)\xi.$$

- For each $h \in G$, define $\tilde{\lambda} : G \to B(\ell^2(G) \otimes \mathcal{H})$ as

$$\tilde{\lambda}_h(\delta_g \otimes \xi) := \delta_{hg} \otimes \xi.$$

Here, δ_g denotes the standard orthonormal basis of $\ell^2(G)$.

It is straightforward to verify that $(\tilde{\pi}, \tilde{\lambda})$ is a covariant representation of (A, G, α).

Definition 2.7.1. *As before, we consider the $*$-algebra $C_c(G, A)$ equipped with the convolution product and involution:*

$$(f * g)(t) := \sum_{s \in G} f(s)\alpha_s\Big(g(s^{-1}t)\Big), \quad f^*(t) := \alpha_t\left(f(t^{-1})^*\right).$$

The reduced crossed product norm is defined via the regular covariant representation:

$$\|f\|_r := \left\|(\tilde{\pi} \times \tilde{\lambda})(f)\right\|.$$

Then the reduced crossed product $A \rtimes_{\alpha,r} G$ *is defined as the completion of $C_c(G, A)$ with respect to $\|\cdot\|_r$:*

$$A \rtimes_{\alpha,r} G := \overline{C_c(G, A)}^{\|\cdot\|_r}.$$

Remark 2.7.2. *We state some basic facts for crossed products*

- *The reduced crossed product is obtained by integrating the regular covariant representation and hence is a quotient of the universal crossed product.*

$$A \rtimes_{\alpha,r} G \cong (A \rtimes_\alpha G)/\mathcal{J},$$

  where $\mathcal{J}$ is the kernel of the regular representation.

- *When G is amenable, the full and reduced crossed products coincide:*

$$A \rtimes_\alpha G \cong A \rtimes_{\alpha,r} G.$$

The fact can be generalised to amenable actions of G on A.

- *The reduced crossed product $A \rtimes_{\alpha,r} G$ doesn't depend on the choice of faithful representation of $A \subseteq B(\mathcal{H})$. (proof of the fact can be found in [BO08, proposition 4.1.5])*
- *The map $E : C_C(G, A) \to A$ defined by $E(f) = f(e)$, extends to a faithful conditional expectation from $A \rtimes_{\alpha,r} G$ onto A.*

In this discussion, we restrict attention to crossed products of a C^*-algebra A by automorphisms, that is, crossed products by $\mathbb{Z}$. Since $\mathbb{Z}$ is abelian and therefore amenable, the full and reduced crossed products agree in this setting.

2.8 Exel's Construction of Crossed Products by an endomorphism

Exel in his paper [Exe03] gave a new definition of the crossed product of unital a C^* Algebra A with respect to $*$ -endomorphism $\alpha : A \to A$ which depends not only on the pair (A, α) but also on the choice of a *transfer operator* associated to the endomorphism α.
We restrict ourselves here to only unital C^*-algebras unless we say otherwise. Let A be a unital C^*-algebra and let $\alpha : A \to A$ be a $*$-endomorphism (here we are not assuming α to be injective) of it.

Definition 2.8.1. *A continuous linear map $\mathcal{L}_\alpha : A \to A$ is said to be a transfer operator for the pair (A, α) if $\mathcal{L}_\alpha$ satisfies the following,*

1. *$\mathcal{L}_\alpha$ is a positive map.*
2. *$\mathcal{L}_\alpha(\alpha(a)b) = a\mathcal{L}_\alpha(b) \;\; \forall a, b \in A$.*

We call the tuple $(A, \alpha, \mathcal{L}_\alpha)$ as *Exel-System*, the goal is to construct the *Crossed Product* C^*-algebra given an Exel system.

Definition 2.8.2. *Let $(A, \alpha, \mathcal{L}_\alpha)$ be an Exel system. A pair (π, S) is called a representation of the system if $\pi\colon A \to B(\mathcal{H})$ is a non-degenerate $*$-representation of A on a Hilbert space $\mathcal{H}$ and $S \in B(\mathcal{H})$ satisfies, for all $a \in A$,*

- $S^*\pi(a)S = \pi(\mathcal{L}_\alpha(a))$,
- $S\pi(a) = \pi(\alpha(a))S$.

For every representation (π, S) the C^*-algebra $C^*(\pi, S) := C^*(\pi(A) \cup S\pi(A))$ is the C^*-algebra associated to (π, S).

Given an Exel system $(A, \alpha, \mathcal{L}_\alpha)$, let us recall Definition 2.4.1, we wish to construct the universal algebra for the generating set $\mathcal{G} = A \cup \{s\}$, where s is an abstract element, and the collection of relations $\mathcal{R}$ consists of maps of the form $\gamma_{(\pi,S)} : \mathcal{G} \to B(\mathcal{H})$ defined by, $\gamma_{(\pi,S)}|_A = \pi$ and $\gamma(s) = S$, where $\pi : A \to B(\mathcal{H})$ and $S \in B(\mathcal{H})$ and (π, S) is a representation for $(A, \alpha, \mathcal{L}_\alpha)$.

Remark 2.8.3. *We will mostly use the notation (π, S) instead of $\gamma_{(\pi,S)}$ in our discussion from now on and instead of relations, we will refer to them as representations of the Exel-system $(A, \alpha, \mathcal{L}_\alpha)$.*

We can check our set of relations $\mathcal{R}$ in this case satisfies the property of being closed under product (Definition 2.4.2). Hence Proposition 2.4.3 confirms the existence of the universal representation, (i_A, t).

Definition 2.8.4. *The universal C^*-algebra with respect to the universal representation (i_A, t) is the C^*-algebra associated to (i_A, t) defined by $\mathcal{T}(A, \alpha, \mathcal{L}_\alpha) := C^*(i_A, t) = C^*(i_A(A) \cup i_A(A)S)$ and called the* Toeplitz algebra *of $(A, \alpha, \mathcal{L}_\alpha)$.*

Definition 2.8.5. *For a representation (π, S) of $(A, \alpha, \mathcal{L}_\alpha)$, a* redundancy *is defined as a pair $(\pi(a), k)$, where $a \in A$ and $k \in \overline{\pi(A)SS^*\pi(A)}$, satisfying*

$$\pi(a)\pi(b)S = k\pi(b)S, \quad \forall\, b \in A.$$

Definition 2.8.6. *The* Exel crossed product $A \rtimes_{\alpha,\mathcal{L}_\alpha} \mathbb{N}$ *associated with the system* $(A, \alpha, \mathcal{L}_\alpha)$ *is defined as the quotient* C^**-algebra of* $\mathcal{T}(A, \alpha, \mathcal{L}_\alpha)$ *by the ideal generated by elements of the form*

$$\{\, i_A(a) - k : \; a \in \overline{A\alpha(A)A}, \; (i_A(a), k) \text{ is a redundancy for } (i_A, t) \,\}.$$

2.8.1 Connection to Cuntz-Pimsner Algebra

For the Exel-system $(A, \alpha, \mathcal{L}_\alpha)$ let $A_{\mathcal{L}_\alpha}$ be the underlying vector space of A, we can define a *right* A*-Module* structure on $A_{\mathcal{L}_\alpha}$ by defining the right action as,

$$m.a = m\alpha(a), \; \forall \, m \in A_{\mathcal{L}_\alpha}, \; a \in A.$$

and we can define a A-valued semi-inner product $\langle ., . \rangle$ on $A_{\mathcal{L}_\alpha}$ by $\langle m, n \rangle = \mathcal{L}_\alpha(m^* n), \; \forall m, n \in A_{\mathcal{L}_\alpha}$ and $\mathcal{N} = \{m \in A_{\mathcal{L}_\alpha} : \langle m, m \rangle = 0\}$ is the null space. The semi-inner product induces an inner product on the space $A_{\mathcal{L}_\alpha}/\mathcal{N}$ and its completion yields a Hilbert A-module denoted by $M_{\mathcal{L}_\alpha} := \overline{A_{\mathcal{L}_\alpha}/\mathcal{N}}$.

Denoting by $q : \mathcal{A}_{\mathcal{L}_\alpha} \to A_{\mathcal{L}_\alpha}$ the quotient map we can get that $a.q(m) := q(am), \; \forall a \in A_{\mathcal{L}_\alpha}, a \in A$, which gives the left action on $M_{\mathcal{L}_\alpha}$, thus making $M_{\mathcal{L}_\alpha}$ into a C^*-correspondence over A, we can see that the left multiplication is defined on $M_{\mathcal{L}_\alpha}$ by the $*$-homomorphism $\phi : A \to \mathcal{L}(M_{\mathcal{L}_\alpha})$ defined by $\phi(a)(q(m)) := a.m = q(am)$.

In [BRV10] one can find the proof of the following proposition, which actually establishes the connection between the two universal Toeplitz-algebras, one coming from the Exel system $(A, \alpha, \mathcal{L}_\alpha)$ and the other coming from the Cuntz-Pimsner dynamical system of $(A, M_{\mathcal{L}_\alpha})$.

Proposition 2.8.7. *We have a one-to-one correspondence between representation* (π, S) *of* $(A, \alpha, \mathcal{L}_\alpha)$ *and representation* $(\pi, \pi_{M_{\mathcal{L}_\alpha}})$ *of* $(A, M_{\mathcal{L}_\alpha})$*. In particular* $\mathcal{T}(A, \alpha, \mathcal{L}_\alpha) \cong \mathcal{T}(A, M_{\mathcal{L}_\alpha})$*.*

Definition 2.8.8. *The* Exel–Royer crossed product $\mathcal{O}(A, \alpha, \mathcal{L}_\alpha)$*, corresponding to an Exel system* $(A, \alpha, \mathcal{L}_\alpha)$*, is defined as the quotient of* $\mathcal{T}(A, \alpha, \mathcal{L}_\alpha)$ *by the ideal generated by the following* *set:*

$$\{i_A(a) - k : \ a \in J_{M_{\mathcal{L}_\alpha}} \text{ and } (i_A(a), k) \text{ is a redundancy for the universal representation } (i_A, t)\}$$

where $J_{M_{\mathcal{L}_\alpha}}$ *is the Katsura's ideal for* $(A, M_{\mathcal{L}_\alpha})$

It can be shown quite easily that $\mathcal{O}(A, \alpha, \mathcal{L}_\alpha) = \mathcal{O}(J_{M_{\mathcal{L}_\alpha}}, M_{\mathcal{L}_\alpha})$.

2.9 Crossed Product by CP map

Given a unital C^*-algebra A together with a completely positive map $\rho : A \to A$, the pair (A, ρ) is referred to as a C^*-*dynamical system*. In his paper [Kwa17], Kwaśniewski introduces the *relative crossed product* for (A, ρ) similar to what Exel did for $(A, \alpha, \mathcal{L}_\alpha)$.

Definition 2.9.1. *A representation of the* C^**-dynamical system* (A, ρ) *consists of a pair* (π, S)*, where* $\pi : A \to B(\mathcal{H})$ *is a* $*$*-homomorphism and* $S \in B(\mathcal{H})$*, satisfying the relation*

$$S^*\pi(a)S = \pi(\rho(a)) \quad \text{for all } a \in A.$$

If π is faithful we call the representation (π, S) *faithful*. The algebra associated to a representation (π, S) is $C^*(\pi, S) := C^*(\pi(A) \cup \pi(A)S)$. Again, choosing the generating set to be $A \cup \{s\}$ and setting the relations to be the collection of representations as defined above for (A, ρ). We can show the existence of the universal representation (i_A, t) for (A, ρ) and we call the universal C^*-algebra as $\mathcal{T}(A, \rho) := C^*(i_A, t) = C^*(i_A(A) \cup i_A(A)t)$. Next, we define *redundancy* in the present context and proceed similarly to Exel's Construction.

Definition 2.9.2. *For a representation* (π, S) *of the* C^**-dynamical system* (A, ρ)*, a* redundancy *is a pair* $(\pi(a), k)$ *where* $a \in A$ *and* $k \in \overline{\pi(A)S\pi(A)S^*\pi(A)}$*, satisfying*

$$\pi(a)\pi(b)S = k\pi(b)S \quad \text{for all } b \in A.$$

For a representation (π, S), consider the set

$$\mathcal{J}_{(\pi,S)} = \{a \in A : (\pi(a), k) \text{ is a redundancy of } (\pi, S) \text{ with the condition } \pi(a) = k\}.$$

It turns out that $\mathcal{J}_{(\pi,S)}$ forms an ideal in A. Moreover, one can verify that

$$\mathcal{J}_{(\pi,S)} = \{a \in A : \pi(a) \in \overline{\pi(A)S\pi(A)S^*\pi(A)}\}.$$

This ideal is referred to as the *ideal of covariance* corresponding to the representation (π, S). For an Ideal J in A. We say (π, S) is J-covariant if $J \cap J(X_\rho) \subseteq J_{(\pi,S)}$.

We now introduce another very important ideal for our discussion.

Definition 2.9.3. *For a positive map $\varrho : A \to B$ between any two C^*-algebras A and B, its GNS-kernel N_ϱ is defined by,*

$$N_\varrho := \{a \in A : \ \varrho((ab)^*ab) = 0, \textit{ for all } b \in A\} \tag{2.2}$$

Proposition 2.9.4. *The set N_ϱ coincides with the largest ideal of A that is contained in the kernel of the mapping $\varrho : A \to B$.*

Proof. First, observe that N_ϱ is a closed right ideal of A. Indeed, for any $a, b \in A$, the inequality

$$b^*a^*ab \leq \|a^*a\| \, b^*b$$

holds, and hence applying ϱ gives

$$\varrho((ab)^*ab) \leq \|a^*a\| \, \varrho(b^*b).$$

This shows that N_ϱ is also closed under multiplication from the left, and thus forms a two-sided ideal.

Now, if $a \in N_\varrho$ is positive, then $a^{1/4} \in N_\varrho$. Consequently,

$$\varrho\Big((a^{1/4}a^{1/4})^*\, a^{1/4}a^{1/4}\Big) = 0,$$

which implies $N_\varrho \subseteq \ker\varrho$.

Finally, let I be any ideal in A such that $I \subseteq \ker\varrho$. Then, by definition of N_ϱ, one has $I \subseteq N_\varrho$. Thus N_ϱ is indeed the largest ideal of A contained in $\ker\varrho$. $\square$

Definition 2.9.5. *We say that the map ϱ is* almost faithful *on an ideal $I \subseteq A$ provided the following holds: whenever $a \in I$ satisfies $\varrho((ab)^*ab) = 0$, for every $b \in A$, then necessarily $a = 0$. Equivalently, ϱ is almost faithful on I if and only if*

$$I \subseteq N_\varrho^\perp.$$

In the specific case of our assumed completely positive map $\rho : A \to A$, we may analogously introduce the set N_ρ, which we shall refer to as the GNS-kernel of ρ.

Remark 2.9.6. *We consider our C^*-dynamical system (A, ρ), where $\rho : A \to A$ is a CP map. Let (π, S) be a representation of our C^*-dynamical system. Then for every $a \in N_\rho$ the pair $(\pi(a), 0)$ becomes a redundancy for the representation (π, S) because,*

$$\begin{aligned} \|\pi(a)\pi(b)S\|^2 &\leq \|S^*\pi(ab)^*\pi(ab)S\| \\ &\leq \|\rho((ab)^*ab)\| \\ &= 0 \end{aligned} \tag{2.3}$$

for all $b \in A$. Thus $a \in \mathcal{J}_{(\pi,S)} \cap N_\rho$ which implies that $\pi(a) = 0$. Accordingly if (π, S) is faithful then $J_{(\pi,S)} \subseteq N_\rho^\perp$. An argument of this sort actually motivates the definition of Katsura's Ideal.

Now we define the Crossed product of A with respect to ρ.

Definition 2.9.7. *The* crossed product *$C^*(A, \rho)$ of A by ρ is constructed as the quotient of the*

Toeplitz C^*-algebra $\mathcal{T}(A, \rho)$ by the ideal generated by the set

$$\{i_A(a) - k : \ a \in N_\rho^\perp \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\} \tag{2.4}$$

where (i_A, t) is the universal representation of (A, ρ). For any ideal J in A we define the relative crossed product *$C^*(A, \rho; J)$ to be the quotient of the Toeplitz C^*-algebra $\mathcal{T}(A, \rho)$ by the set*

$$\{i_A(a) - k : \ a \in J \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\} \tag{2.5}$$

We denote by (j_A, s) the representation of (A, ρ) that generates $C^*(A, \rho, J)$.

2.9.1 Crossed Product and relative Cuntz-Pimsner algebras

Paschke already considered a C^*-correspondence associated with a completely positive map [Pas73]. We define X_ρ to be a Hausdorff completion of the algebraic tensor product $A \odot A$ with respect to the seminorm associated to the A-valued sesquilinear form given by,

$$\langle a \odot b, c \odot d\rangle := b^* \rho(a^* c) d, \qquad \text{where } \ a, b, c, d \in A. \tag{2.6}$$

We denote the image of the simple tensor product $a \odot b$ in X_ρ by $a \otimes b$. It can be verified that the space X_ρ acquires the structure of a C^*-correspondence over A. The left and right module actions are given, respectively, by

$$a \cdot (b \otimes c) := (ab) \otimes c, \qquad (b \otimes c) \cdot a := b \otimes (ca),$$

for all $a, b, c \in A$. One can verify here that X_ρ is essential.

We now briefly introduce the KSGNS-construction to draw a parallel with the above construction.

Let A be a C^*-algebra and let X be a C^*-correspondence over A with left action $\phi_X : A \to \mathcal{L}(X)$. Let $\rho : A \to \mathcal{L}(X)$ be a completely positive map.

We construct a C^*-correspondence $X \otimes_\rho X$. Consider the algebraic tensor product $X \odot X$ with right A-module structure given by

$$(x \odot y) \cdot a := x \odot (y \cdot a), \qquad x, y \in X, \ a \in A.$$

Define an A-valued sesquilinear form on $X \odot X$ by

$$\langle x_1 \odot y_1, \ x_2 \odot y_2 \rangle := \langle y_1, \ \rho(\langle x_1, x_2 \rangle_X)\, y_2 \rangle_X, \qquad x_i, y_i \in X.$$

Since ρ is completely positive, this form is positive semidefinite. Let

$$\mathcal{N}_\rho := \{z \in X \odot X \mid \langle z, z \rangle = 0\}.$$

Then $(X \odot X)/\mathcal{N}_\rho$ is a pre-Hilbert A-module, and its completion is denoted by

$$X \otimes_\rho X.$$

The right A-action descends from the algebraic tensor product, and the left action of A on $X \otimes_\rho X$ is given by

$$a \cdot (x \otimes y) := (\phi_X(a)x) \otimes y, \qquad a \in A, \ x, y \in X.$$

With these operations, $X \otimes_\rho X$ is a C^*-correspondence over A. When $X = A$ and $\rho : A \to A \subseteq \mathcal{L}(A)$, this construction coincides with the classical KSGNS correspondence associated to ρ [Lan95].

It is now straightforward to observe that the above construction is nothing but the KSGNS construction for $A \otimes_\rho A$ considering A as C^*-correspondence over itself.

Now we state several important theorems about the construction that establish the connection between the crossed product for the system (A, ρ) and the Cuntz-Pimsner algebra for (A, X_ρ). For more details, one can refer to [Kwa17]. In our discussion, as mentioned above, we assume A to be unital.

Proposition 2.9.8. *[Kwa17]Let X_ρ denote the C^*-correspondence associated with (A, ρ). There exists a bijective correspondence between representations (π, S) of the system (A, ρ) (as defined in Definition 2.9.1) and representations (π, π_{X_ρ}) of the pair (A, X_ρ) (as per Definition 2.6.1). The correspondence is described as follows:*

$$\pi_{X_\rho}(a \otimes b) = \pi(a)S\pi(b), \quad \text{for } a \otimes b \in X_\rho,$$

and

$$S^*\pi_{X_\rho}(a \otimes b)S = \pi(\rho(a)b), \quad a, b \in A,$$

with the condition that

$$S^*|_{(\pi_{X_\rho}(X_\rho)H)^\perp} \equiv 0.$$

Furthermore, for the associated representations, one has

$$C^*(\pi, S) := C^*(\pi(A) \cup S\pi(A)) \cong C^*(\pi, \pi_{X_\rho}) := C^*\Big(\pi(A) \cup \pi_{X_\rho}(X_\rho)\Big).$$

The following result is fundamental, as it establishes a precise relationship between the redundancy ideal and the quotient ideal in the construction of the Cuntz-Pimsner algebra.

Proposition 2.9.9. *[Kwa17]For a faithful representation (π, S) of (A, ρ) and let the corresponding representation of (A, X_ρ) given by (2.9) be (π, π_{X_ρ}). $(\pi(a), k)$ is a redundancy of (π, S) if and only if $a \in J_{X_\rho}$ and $k = (\pi, \pi_{X_\rho})^{(1)}(\phi(a))$, where $\phi : A \to \mathcal{L}(X_\rho)$ gives the left action.*

Proof. The proof revolves around the fact of understanding that, $(\pi, \pi_{X_\rho})^{(1)}$ maps $\mathcal{K}(X_\rho)$ to the subalgebra $\overline{\pi(A)S\pi(A)S^*\pi(A)}$, using Proposition 2.9.8.

$$\begin{aligned} (\pi, \pi_{X_\rho})^{(1)}(\theta_{a\otimes b, c\otimes d}) &= \pi_{X_\rho}(a \otimes b)\pi_{X_\rho}(c \otimes d)^* \\ &= \pi(a)S\pi(b)\pi(d)S^*\pi(c) \in \overline{\pi(A)S\pi(A)S^*\pi(A)} \end{aligned} \tag{2.7}$$

Thus if $a \in J(X_\rho)$ and $k = (\pi, \pi_{X_\rho})^{(1)}(\phi(a)) \in \overline{\pi(A)S\pi(A)S^*\pi(A)}$. Then for $a, a_1, a_2 \in A$.

$$\begin{aligned}
\pi(a)\pi(a_1)S\pi(a_2) &= \pi(a)\pi_{X_\rho}(a_1 \otimes a_2) \\
&= \pi_{X_\rho}(\phi(a)(a_1 \otimes a_2)) \\
&= (\pi, \pi_{X_\rho})^{(1)}(\phi(a))\pi_{X_\rho}(a_1 \otimes a_2) \\
&= (\pi, \pi_{X_\rho})^{(1)}(\phi(a))\pi(a_1)S\pi(a_2).
\end{aligned} \tag{2.8}$$

Now, using non-degeneracy of π we can see that

$$\pi(a)\pi(a_1)S = (\pi, \pi_{X_\rho})^{(1)}(\phi(a))\pi(a_1)S = k\pi(a_1)S \quad \text{for all } a_1 \in A$$

Hence $(\pi(a), k)$ is a redundancy.

For the other direction let $(\pi(a), k)$ be a redundancy for the representation (π, S). Then $k = (\pi, \pi_{X_\rho})^{(1)}(p)$ for some $t \in K(X_\rho)$ as $k \in \overline{\pi(A)S\pi(A)S^*\pi(A)} = (\pi, \pi_{X_\rho})^{(1)}(\mathcal{K}(X_\rho))$. Now, for $a_1, a_2 \in A$ we get,

$$\begin{aligned}
\pi_{X_\rho}(\phi(a)a_1 \otimes a_2) &= \pi(a)\pi_{X_\rho}(a_1 \otimes a_2) \\
&= \pi(a)\pi(a_1)S\pi(a_2) \\
&= k\pi(a_1)S\pi(a_2) = (\pi, \pi_{X_\rho})^{(1)}(p)\pi_{X_\rho}(a_1 \otimes a_2) \\
&= \pi_{X_\rho}(p(a_1 \otimes a_2)).
\end{aligned} \tag{2.9}$$

π is injective$\Rightarrow$ π_{X_ρ} is injective $\Rightarrow \phi(a) = p$ $\square$

The following remark is the key step in showing that the relative crossed-product construction with respect to an ideal J in A for (A, ρ) agrees with the relative Cuntz–Pimsner algebra associated to (A, X_ρ) with respect to the ideal $J \cap J(X_\rho)$.

Remark 2.9.10. *Using Proposition 2.9.9, we observe that the sets generating the ideals we quotient* *the Toeplitz C^*-algebra for the respective construction are the same.*

$$\begin{aligned}
&\{i_A(a)-k:\ a\in J \text{ and } (i_A(a),k) \text{ is a redundancy of } (i_A,t)\}\\
=&\{i_A(a)-k:\ a\in J\cap J(X_\rho) \text{ and } (i_A(a),k) \text{ is a redundancy of } (i_A,t)\}\\
=&\{i_A(a)-(\pi,\pi_{X_\rho})^{(1)}(\phi(a)):\ a\in J\cap J(X_\rho)\}
\end{aligned}$$

Thus, we have the following theorem.

Theorem 2.9.11. *Let X_ρ denote the C^*-correspondence associated with the system (A,ρ). For an ideal $J\subseteq A$, one has*

$$C^*(A,\rho;J)=C^*(A,\rho;J\cap J(X_\rho))\ \cong\ \mathcal{O}(J\cap J(X_\rho),X_\rho).$$

The universal homomorphism $j_A: A\to C^*(A,\rho;J)$ is injective iff when the inclusion

$$J\cap J(X_\rho)\subseteq (N_\rho)^\perp$$

holds. Consequently, this condition ensures that

$$C^*(A,\rho)\ \cong\ \mathcal{O}_{X_\rho},$$

and, in particular, the canonical map $j_A: A\to C^*(A,\rho)$ is injective.

[Kwa17, Proposition 3.13] will serve as an important tool in our study of $C^*(A,\rho)$, and we state it here for the convenience of the reader. It follows from Katsura's gauge-invariant uniqueness theorem for Cuntz-Pimsner algebras [Kat03b, Theorem 6.4].

Proposition 2.9.12. *Let (π,S) be a covariant representation of (A,ϱ). Then $C^*(\pi,S)\cong C^*(A,\varrho)$ iff (π,S) is faithful and there exists a strongly continuous action $\beta:\mathbb{T}\to Aut(C^*(\pi,S))$ such that $\beta_z(\pi(a))=\pi(a)$ and $\beta_z(\pi(a)S)=z\pi(a)S$ for all $a\in A$ and $z\in\mathbb{T}$.*

In this thesis, we will dive into the specifics of some important constructions of this sort.

Chapter 3

Crossed product of a C^*-algebra with respect to an endomorphism

This chapter is based on my work [Ban25], where the basic framework of the construction was introduced in Section 2.8. In the present section, we provide a detailed analysis of these constructions and examine concrete examples in the context of $C^*(G)$.

Motivation from the crossed product of a C^*-algebra by an automorphism

Suppose A is a C^*-algebra and $\alpha : A \to A$ is an automorphism. The crossed product by an automorphism is equivalent to the crossed product with respect to the group $\mathbb{Z}$, and is denoted by $A \rtimes_\alpha \mathbb{Z}$. The action of $n \in \mathbb{Z}$ on $a \in A$ is given by $\alpha^n(a)$. Since $\mathbb{Z}$ is cyclic, the homomorphism from $\mathbb{Z}$ to $\mathrm{Aut}(A)$ is completely determined by the image of 1.
A covariant representation for this system is a pair (π, u), where $\pi : A \to B(H)$ is a $*$-representation of A and $u : \mathbb{Z} \to U(H)$ is a unitary representation of $\mathbb{Z}$ (again completely determined by $u(1)$), such that

$$u(n)\pi(x)u(n)^* \;=\; \pi(\alpha^n(x)), \quad \forall x \in A,\, n \in \mathbb{Z}.$$

Equivalently, we may describe the covariant representation as a pair (π, U), where $U := u(1)$

is a unitary on H, satisfying

$$U\pi(x)U^* \;=\; \pi(\alpha(x)), \quad \forall x \in A.$$

This perspective naturally motivates the study of crossed products by an endomorphism or, more generally, by a completely positive (CP) map, which may be viewed as crossed products by the semigroup $\mathbb{N}$.

3.1 On Transfer Operators

We restrict ourselves here to unital C^*-algebras unless stated otherwise. Let A be a unital C^*-algebra and let $\alpha : A \to A$ be a $*$-endomorphism (of A). Note that we do not assume α to be injective here.
Recall from Definition 2.8.1 that a map $\mathcal{L}_\alpha : A \to A$ is called a transfer operator for the pair (A, α) if it satisfies the following conditions:

1. $\mathcal{L}_\alpha$ is a positive map.
2. $\mathcal{L}_\alpha(\alpha(a)b) = a\mathcal{L}_\alpha(b) \ \ \forall a, b \in A$.

Throughout this section, whenever we refer to $\mathcal{L}_\alpha$, we mean a transfer operator for the pair (A, α). The triple $(A, \alpha, \mathcal{L}_\alpha)$ will be referred to as an Exel system. Since $\mathcal{L}_\alpha$ is positive, it is self-adjoint. Consequently, we also obtain, $\mathcal{L}_\alpha(a\alpha(b)) = \mathcal{L}_\alpha(a)b \ \ \forall a, b \in A$. Thus, $\mathrm{Range}(\mathcal{L}_\alpha)$ is always a two-sided ideal.

Next, we characterize all transfer operators corresponding to an automorphism.

Proposition 3.1.1. *If α is an automorphism of A, then any transfer operator corresponding to α is of the form $\mathcal{L}_\alpha(a) = \alpha^{-1}(a)z$, where $z \in \mathcal{Z}(A) \cap A^+$.*

Proof. For the C^*-algebra A and the automorphism $\alpha : A \to A$ let $\mathcal{L}_\alpha$ be a transfer operator associated to α then,

$$\mathcal{L}_\alpha(a) = \mathcal{L}_\alpha(\alpha(\alpha^{-1}(a))1) = \alpha^{-1}(a)\mathcal{L}_\alpha(1) \tag{3.1}$$

From the Definition 2.8.1 of transfer operators, it can be observed that $\mathcal{L}_\alpha(1) \in \mathcal{Z}(A)^+$ as we can see that,

$$\mathcal{L}_\alpha(\alpha(a)) = a\mathcal{L}_\alpha(1) = \mathcal{L}_\alpha(1)a. \tag{3.2}$$

This proves one direction. Conversely we verify that the map $\mathcal{L}_\alpha : A \longrightarrow A, \quad \mathcal{L}_\alpha(a) = \alpha^{-1}(a)z$, with $z \in \mathcal{Z}(A) \cap A^+$, satisfies the desired conditions. Clearly, $\mathcal{L}_\alpha$ is positive. Moreover, for the bimodule condition, we compute:

$$\mathcal{L}_\alpha(\alpha(a)b) = \alpha^{-1}(\alpha(a)b)z = a\,\alpha^{-1}(b)z = a\,\mathcal{L}_\alpha(b).$$

Thus, $\mathcal{L}_\alpha$ is indeed a transfer operator.

$\square$

Remark 3.1.2. *If we apply Exel's construction to a C^*-algebra A with respect to an automorphism α, and choose the transfer operator to be α^{-1}, the resulting algebra is precisely the crossed product $A \rtimes_\alpha \mathbb{Z}$. Thus, Exel's construction can be seen as a generalization of crossed products by $\mathbb{Z}$.*

Note that if $\mathcal{L}_\alpha(1) = 1$, we can conclude that $\mathcal{L}_\alpha$ is surjective as for any $a \in A$, we have $\mathcal{L}_\alpha(\alpha(a)) = a\mathcal{L}_\alpha(1) = a$.

Remark 3.1.3. *For (A, α), we give some easy ways to generate new transfer operators.*

1. *If we have a finite collection of transfer operators, say $\{\mathcal{L}_i| \ i \in I\}$, then $\mathcal{L} = \sum_{i\in I} \mathcal{L}_i$ is also*

a transfer operator.

$$\begin{aligned}\mathcal{L}(\alpha(a)b) &= \sum_{i\in I}\mathcal{L}_i(\alpha(a)b)\\ &= \sum_{i\in I} a\mathcal{L}_i(b)\\ &= a\sum_{i\in I}\mathcal{L}_i(b)\\ &= a\mathcal{L}(b)\end{aligned}$$

2. *Similarly, suppose $\mathcal{L}$ is a transfer operator for (A,α). Then for any $z \in \mathcal{Z}(A)\cap A^+$, the map*

$$\mathcal{L}'(a) = \mathcal{L}(a)z$$

is also a transfer operator. Hence, given a family of transfer operators $\{\mathcal{L}_i \mid i \in I\}$ and corresponding elements $z_i \in \mathcal{Z}(A)\cap A^+$, the map

$$\mathcal{L}(a) = \sum_{i\in I}\mathcal{L}_i(a)z_i$$

is again a transfer operator.

3. *If $\mathcal{L}$ is a transfer operator for (A,α). Then, $\mathcal{L}^n$ is the transfer operator for (A,α^n).*

$$\begin{aligned}\mathcal{L}^n(\alpha^n(a)b) &= \mathcal{L}^{n-1}(\mathcal{L}(\alpha^n(a)b))\\ &= \mathcal{L}^{n-1}(\alpha^{n-1}(a)\mathcal{L}(b))\\ &= a\mathcal{L}^n(b)\end{aligned}$$

4. *We consider a finite family of endomorphisms $\{\alpha_i\}_{i\in I}$ of the C^*-algebra A, where $|I| < \infty$, defined by*

$$\alpha_i(a) = v_i a v_i^*, \qquad a \in A,$$

with

$$v_i v_i^* \neq 1, \qquad v_i^* v_i = 1, \qquad v_i^* v_j = 0 \text{ for } i \neq j.$$

Now define an endomorphism $\alpha : A \to A$ by

$$\alpha(a) = \sum_{i \in I} \alpha_i(a).$$

Then, for every subset $J \subseteq I$, one can define a transfer operator for the pair (A, α).

More precisely, for each $J \subseteq I$, define

$$\mathcal{L}_J : A \to A, \qquad \mathcal{L}_J(a) = \sum_{j \in J} v_j^* a v_j.$$

Now we show that this map is a transfer operator for (A, α).

$$\begin{aligned}
\mathcal{L}_J(\alpha(a)b) &= \sum_{j \in J} v_j^* (\sum_{i \in I} v_i a v_i^*) b v_j \\
&= \sum_{i \in I, j \in J \subseteq I} v_j^* v_i a v_i^* b v_j \\
&= \sum_{j \in J} a v_j^* b v_j, \\
&= a \sum_{j \in J} v_j^* b v_j \\
&= a \mathcal{L}_J(b)
\end{aligned}$$

The key question here is: under what conditions and by what methods can a transfer operator be constructed from a given endomorphism? In his paper [Exe03], Exel provides a method to construct a transfer operator from a conditional expectation from A onto $\mathrm{Range}(\alpha)$. Consider the map $E_\alpha : A \to A$ defined by $E_\alpha(a) = \alpha \circ \mathcal{L}_\alpha$ where (A, α) is the given system.

We observe

$$E_\alpha(a\alpha(b)) = \alpha(\mathcal{L}(a\alpha(b))) = \alpha(\mathcal{L}(a)b) = E_\alpha(a)\alpha(b)$$

Similarly,

$$E_\alpha(\alpha(a)b) = \alpha(a)E_\alpha(b)$$

Hence, E_α is $\mathrm{Range}(\alpha)$ -linear with respect to the bimodule structure of A over $\mathrm{Range}(\alpha)$.

Proposition 3.1.4. *[Exe03](Prop. 2.3) For the system (A, α) we consider the transfer operator $\mathcal{L}_\alpha$, then the following are equivalent:*

1. *$E_\alpha(= \alpha \circ \mathcal{L}_\alpha) : A \to A$ is a conditional expectation from A onto the subalgebra $\mathrm{Range}(\alpha)$.*
2. *$\alpha \circ \mathcal{L}_\alpha \circ \alpha = \alpha$,*
3. *$\alpha(\mathcal{L}_\alpha(1)) = \alpha(1)$.*

Proof. (1) $\Rightarrow$ (2) We know that E_α is a projection onto $\mathrm{Range}(\alpha)$ if it is a conditional expectation. Hence, we get

$$\alpha \circ \mathcal{L}_\alpha \circ \alpha(a) = E_\alpha(\alpha(a)) = \alpha(a).$$

(2) $\Rightarrow$ (3) From the definition of $\mathcal{L}_\alpha$ we can conclude that $\mathcal{L}_\alpha(\alpha(1)) = \mathcal{L}_\alpha(1)$. Hence, using (2) we get,

$$\begin{aligned} \alpha(\mathcal{L}_\alpha(1)) &= \alpha(\mathcal{L}_\alpha(\alpha(1))) \\ &= \alpha \circ \mathcal{L}_\alpha \circ \alpha(1) \\ &= \alpha(1) \end{aligned}$$

(3) $\Rightarrow$ (1) Here, we actually prove (2) from (3), which in turn proves our map E_α is a projection onto $\mathrm{Range}(\alpha)$. Now, from (3) we can conclude that E_α is contractive and if we

can prove that (2) follows from (3) using Tomiyama's theorem [BO08, Theorem 1.5.10], we can conclude that E_α is a conditional expectation. So, we prove (3) $\Rightarrow$ (2)

$$\begin{aligned}\alpha(\mathcal{L}_\alpha(\alpha(a))) &= \alpha(\mathcal{L}_\alpha(\alpha(a)1)\\ &= \alpha(a\mathcal{L}_\alpha(1))\\ &= \alpha(a)\alpha(\mathcal{L}_\alpha(1))\\ &= \alpha(a)\alpha(1) = \alpha(a)\end{aligned}$$

Thus, as stated above, we can conclude that E_α is a conditional expectation. □

Remark 3.1.5. *For a system (A, α), if the transfer operator $\mathcal{L}_\alpha$ satisfies the above equivalent conditions it is called nondegenerate. In this case we can write the C^* -Algebra A as a direct sum of* $\ker(\alpha)$ *and* $\mathrm{Range}(\mathcal{L}_\alpha)$

$$A = \ker(\alpha) \oplus \mathrm{Range}(\mathcal{L}_\alpha)$$

This is because for any $a \in A$ we can write $a = a - \mathcal{L}_\alpha(\alpha(a)) + \mathcal{L}_\alpha(\alpha(a))$, where $a - \mathcal{L}_\alpha(\alpha(a)) \in \ker(\alpha)$, since $\mathcal{L}_\alpha$ is non-degenerate and using condition (2) in Proposition 3.1.4. Also, if $a \in \ker(\alpha)$, then $a\mathcal{L}_\alpha(b) = \mathcal{L}_\alpha(\alpha(a)b) = 0$. Similarly, $\mathcal{L}_\alpha(b)a = \mathcal{L}_\alpha(b\alpha(a)) = 0$, which implies that $\ker(\alpha)$ and $\mathrm{Range}(\mathcal{L}_\alpha)$ are mutually orthogonal.

The following proposition, due to Exel [Exe03], provides a method for constructing a transfer operator for the C^*-algebra A associated with an endomorphism $\alpha : A \to A$, assuming the existence of a conditional expectation onto $\mathrm{Range}(\alpha)$.

Proposition 3.1.6. *Suppose there exists a conditional expectation E_α from A onto $\mathrm{Range}(\alpha)$ for the system (A, α). Let $\mathcal{I}$ be an ideal in A such that*

$$A = \mathcal{I} \oplus \ker(\alpha).$$

Then the map $\mathcal{Q} := (\alpha|_{\mathcal{I}})^{-1} \circ E_\alpha$ is a non-degenerate transfer operator for (A, α). Moreover, we *have*

$$E_\alpha = \alpha \circ \mathcal{Q}.$$

Proof. It follows directly that $\mathcal{L}_\alpha$ is positive. Let us denote $(\alpha|_\mathcal{I})^{-1}$ by $\hat{\alpha}$. Now for $a, b \in A$,

$$\begin{aligned}\mathcal{L}_\alpha(\alpha(a)b) &= \hat{\alpha}(E_\alpha(\alpha(a)b)) \\ &= \hat{\alpha}(\alpha(a))\hat{\alpha}(E_\alpha(b))\end{aligned}$$

Now α is injective when restricted to $\mathcal{I}$. Hence it suffices to prove that $\alpha(\hat{\alpha}(\alpha(a))\hat{\alpha}(E_\alpha(b))) = \alpha(a\mathcal{L}_\alpha(b))$, now from injectivity of α we can conclude that $\hat{\alpha}(\alpha(a))\hat{\alpha}(E_\alpha(b)) = a\mathcal{L}_\alpha(b)$

$$\begin{aligned}\alpha(\hat{\alpha}(\alpha(a))\hat{\alpha}(E_\alpha(b))) &= \alpha(\hat{\alpha}(\alpha(a)))\alpha(\hat{\alpha}(E_\alpha(b))) \\ &= \alpha(a)E_\alpha(b) \\ &= \alpha(a\mathcal{L}_\alpha(b))\end{aligned}$$

Hence, we have shown that $\mathcal{L}_\alpha$ is a transfer operator. To prove it is nondegenerate, we observe that,

$$\begin{aligned}\alpha \circ \mathcal{L}_\alpha(a) &= \alpha(\hat{\alpha}(E_\alpha(a))) \\ &= E_\alpha(a)\end{aligned}$$

Hence we obtain that, $\mathcal{L}_\alpha$ is non-degenerate transfer operator. $\square$

3.1.1 Transfer Operator for $C(D)$

Let D be a compact Hausdorff space. We choose $A = C(D)$, the C^*-algebra of complex valued continuous functions on D. Our goal is to find properties of a transfer operator $\mathcal{L}_\alpha$ for the system $(C(D), \alpha)$, where α is an $*$-endomorphism. Any $*$-endomorphism of $C(D)$ corresponds to a continuous map from D to itself. When we talk about endomorphism of a

C^*-algebra, we mean $*$-endomorphism.

Now, given a continuous map $a : D \to D$ we obtain an endomorphism $\alpha_a : C(D) \to C(D)$ given by $\alpha_a(f) = f \circ a$ for $f \in C(D)$ and, as mentioned above, any endomorphism of C(D) is of this form. We intend to find a suitable transfer operator for such a α_a. Recall that any positive map $\rho : C(D) \to C(D)$ induces a correspondence $x \mapsto \mu_x$ between points $x \in D$ and positive Borel measures on D $\mu_x \in \mathcal{M}(D)$, such that,

$$\rho(f)(x) = \int_D f d\mu_x$$

Proposition 3.1.7. *Suppose $\rho : C(D) \to C(D)$ is a positive map from $C(D)$ to $C(D)$ defined by $\rho(f)(x) = \int_D f d\mu_x$. If ρ is a transfer operator with respect to the endomorphism $\alpha_a : C(D) \to C(D)$ defined by $\alpha_a(f) = f \circ a$, where $a : D \to D$ is a continuous map then supports of μ_x, $\mathrm{supp}(\mu_x)$ are mutually disjoint, where $\mathrm{supp}(\mu_x)$ denotes the smallest closed subset $F \subseteq D$ such that $\mu_x(D \backslash F) = 0$.*

Proof. If ρ is a transfer operator then ρ must satisfy the following condition for any $f, g \in C(D)$,

$$
\begin{aligned}
&\rho((\alpha_a(f))g)(x) = f(x)\rho(g)(x)\\
\Rightarrow &\rho((f \circ a)g)(x) = f(x)\rho(g)(x)\\
\Rightarrow &\int_D (f \circ a)(y)g(y)d\mu_x(y) = f(x)\int_D g(y)d\mu_x(y)\\
\Rightarrow &\int_D \{(f \circ a)(y) - f(x)\}g(y)d\mu_x(y) = 0
\end{aligned}
$$

Now, considering, $g = (f \circ a)(y) - f(x)$ we get,

$$
\begin{aligned}
&\int_D \{(f \circ a)(y) - f(x)\}^2 d\mu_x(y) = 0,\\
\Rightarrow &(f \circ a)(y) - f(x) = 0, \ \forall\ y \in \operatorname{supp}(\mu_x)
\end{aligned}
$$

From the above calculation, we now try to conclude that all the $\operatorname{supp}(\mu_x)$ are mutually disjoint and $a(y) = x \ \forall y \in \operatorname{supp}(\mu_x)$. Suppose for $y_1 \in \operatorname{supp}(\mu_x)$ we get $a(y_1) \neq x$ then there exist $f \in C(D)$ because $C(D)$ separates points in D such that $f(a(y_1)) \neq f(x)$ so we arrive at a contradiction. Hence $a(y) = x, \forall y \in \operatorname{supp}(\mu_x)$, so if for $x, x^{'} \in D$ we have $y' \in \operatorname{supp}(\mu_x) \cap \operatorname{supp}(\mu'_x)$ we have $a(y') = x = x'$ thus proving that $\operatorname{supp}(\mu_x)$ are all mutually disjoint.

□

Remark 3.1.8. *We can observe that the converse of the above proposition also holds. Thus we can state the following: a positive map $\rho : C(D) \to C(D)$ defined by $\rho(f)(x) = \int_D f d\mu_x$, is a transfer operator for $C(D)$ with respect to the endomorphism α_a defined by $\alpha_a(f) = f \circ a$, where $a : D \to D$ is a continuous map if and only if the supports* $\operatorname{supp}(\mu_x)$ *are disjoint and $a : D \to D$*

takes the value x for every $y \in supp(\mu_x)$. It follows from the fact that,

$$
\begin{aligned}
(f \circ a)(y) - f(x) = 0,\ \forall y \in \operatorname{supp}(\mu_x) \\
\Rightarrow \int_D \{(f \circ a)(y) - f(x)\} g(y) d\mu_x(y) = 0
\end{aligned}
$$

We actually now look at specific cases for $C(D)$ and observe some examples of transfer operators arriving in those situations.

1. **Endomorphism coming from a surjective continuous map:**

   Let us assume a is surjective and admits a continuous right inverse h. Thus, our endomorphism is $\alpha_a(f) = f \circ a$. Now, we define $\mathcal{L}_a : C(D) \to C(D)$ by,

   $$
   \begin{aligned}
   \mathcal{L}_a(f)(x) &= \int_D f d\delta_{h(x)} \\
   &= f(h(x))
   \end{aligned}
   $$

   where $\delta_{h(x)}$ is the Dirac measure on $h(x)$. We now check whether it satisfies the properties for being a transfer operator,

   $$
   \begin{aligned}
   \mathcal{L}_a(\alpha_a(f)g)(x) &= \alpha_a(f)(h(x))g(h(x)) \\
   &= f(a(h(x)))\mathcal{L}_a(g)(x) \\
   &= f(x)\mathcal{L}_a(g)(x)
   \end{aligned}
   $$

   Since $\mathcal{L}_a$ is a positive map, the above computation shows that it satisfies the transfer operator identity.

   Hence, for D if such a surjective continuous map $a : D \to D$ exists that has a continuous right inverse, we get an Exel system $(C(D), \alpha_a, \mathcal{L}_a)$. Also, we note that a being surjective means that our endomorphism α_a is injective, and in this case, even our transfer operator $\mathcal{L}_a$ ends up being an endomorphism.

Remark 3.1.9. *Suppose that the map $a : D \to D$ is continuous and bijective. From the fact that D is compact and Hausdorff, we can conclude that a is a homeomorphism. Thus, α_a is an automorphism. As previously discussed for an automorphism, any transfer operator is of the form $\beta_a(f) = g\{f \circ a^{-1}\}$, where $g \in \mathcal{Z}(C(D)) \cap C(D)^+ = C(D)^+$. In this case, for any $x \in D$, the corresponding element $\mu_x \in \mathcal{M}(D)$ is given by $\mu_x := g(x)\delta_{a^{-1}(x)}$. In the special case where we choose $g = 1$, we recover the standard crossed product construction, following Exel's framework, which we will discuss in detail in the next section.*

2. **Endomorphism coming from a surjective local homeomorphism:**

Building on the above remark, it is natural to ask whether we can describe a transfer operator for a system where the map $a : D \to D$ is not a homeomorphism, but the next best thing, a surjective local homeomorphism.

Since $a : D \to D$ is a local homeomorphism and D is compact, it implies that the set of preimages of a given $x \in D$,

$$a^{-1}(x) := \{y \in D |\ a(y) = x\}$$

is a finite set. Moreover, from Proposition 3.1.7 we understand that for a positive map defined by,

$$\rho(f)(x) := \int_D f(y) d\mu_x(y)$$

we have $\text{supp}(\mu_x)$ is disjoint and more importantly we have that $a(y) = x,\ \forall y \in supp(\mu_x)$. Hence we obtain that $\text{supp}(\mu_x) \subseteq a^{-1}(x)$ and $a^{-1}(x)$ is finite, so we can conclude here that μ_x is some positive linear combination of Dirac measures

$\{\delta_y,\ y \in a^{-1}(x)\}$.

Hence the transfer operator in this case will be of the form,

$$\rho(f)(x) = \sum_{y \in a^{-1}(x)} c_y^x f(y)$$

Thus, in this case, we obtain a complete classification of all possible transfer operators.

Remark 3.1.10. *From the above discussion we conclude that for an endomorphism $\alpha_a : C(D) \to C(D)$ corresponding to a local surjective map $a : D \to D$ the possible transfer operators $\rho : C(D) \to C(D)$ will be of the form,*

$$\rho(f)(x) = \sum_{y \in a^{-1}(x)} c_y^x f(y)$$

where c_y^x are positive scalars.

Remark 3.1.11. *In the context of classical dynamical systems, the pair (D, γ), where D is a compact topological space and $\gamma : D \to D$ is a surjective local homeomorphism, the map,*

$$\mathcal{L}(f)(x) = \frac{1}{|\gamma^{-1}(x)|} \sum_{y \in \gamma^{-1}(x)} f(y),\ \forall\ f \in C(D),\ x \in D$$

is called a transfer operator for the pair (D, γ) and has important uses in classical dynamics [Rue89].

3.2 Crossed product of a C^*-algebra with respect to a state

Let A be a unital C^*-algebra and let $\psi : A \to \mathbb{C}$ be a state on A. Consider the map $\varrho_\psi : A \to A$ defined by

$$\varrho_\psi(a) = \psi(a)1_A, \quad a \in A$$

It is evident that ϱ_ψ is a UCP map. Hence, we can apply Kwaśniewski's construction. Hence we say the pair (π, S) where $\pi : A \to B(\mathcal{H})$ and $S \in B(\mathcal{H})$ is a representation for our system

(A, ψ) if

$$S^*\pi(a)S = \pi(\varrho_\psi(a)) = \psi(a)\pi(1_A) = \psi(a)1_{B(\mathcal{H})}, \quad a \in A$$

The associated C^*-correspondence X_ψ, as in Kwaśniewski's construction, is obtained by completing the algebraic tensor product $A \odot A$ with respect to the A-valued inner product induced by the semi-inner product given by,

$$\begin{aligned}\langle a \otimes b, c \otimes d\rangle_\psi &= b^*\rho_\psi(a^*c)d \\ &= b^*\psi(a^*c)\, 1_A\, d \\ &= \psi(a^*c)\, b^*d.\end{aligned} \tag{3.3}$$

To obtain an inner product, we quotient $A \odot A$ by the null space

$$\mathcal{N} = \left\{ \sum_i a_i \otimes b_i \;\middle|\; \left\langle \sum_i a_i \otimes b_i,\ \sum_j a_j \otimes b_j \right\rangle_\psi = 0 \right\}.$$

Expanding the inner product, we obtain

$$\mathcal{N} = \left\{ \sum_i a_i \otimes b_i \;\middle|\; \sum_{i,j} \psi(a_i^* a_j)\, b_i^* b_j = 0 \right\}.$$

Thus, the associated Hilbert C^*-module is given by

$$X_\psi = (A \odot A)/\mathcal{N}.$$

Remark 3.2.1. *For any Hilbert space H and a C^*-algebra A, the algebraic tensor product $H \otimes_{\mathrm{alg}} A$, is a right A-module with an A-valued inner product given by,*

$$\langle \xi \otimes a, \eta \otimes b\rangle = \langle \xi, \eta\rangle a^*b, \ \textit{where } \xi, \eta \in H \textit{ and } a, b \in A \tag{3.4}$$

Thus $H \otimes_{\mathrm{alg}} A$ is an inner product right A-module and we denote $H \otimes A$ to be its completion.

Now let $(\pi_\psi, H_\psi, \xi_\psi)$ be the GNS-representation of A with respect to ψ. Now we can consider the right Hilbert A-module $H_\psi \otimes A$, with the inner product,

$$\begin{aligned} \langle \eta_a \otimes b, \eta_c \otimes d \rangle &= \langle \eta_a, \eta_c \rangle b^* d \\ &= \psi(c^* a) b^* d \end{aligned} \tag{3.5}$$

where $\eta_a, \eta_c \in H_\psi$ and $b, d \in A$.

Now,

$$\begin{aligned} \langle \xi_\psi \otimes 1, \pi_\psi \otimes 1(a) \xi_\psi \otimes 1 \rangle &= \langle \eta_1 \otimes 1, \eta_a \otimes 1 \rangle \\ &= \psi(a) 1_A \\ &= \varrho_\psi(a) \end{aligned} \tag{3.6}$$

since ξ_ψ is cyclic, we get,

$$\overline{(\pi_\psi \otimes 1)(A)(\xi_\psi \otimes 1)A} = H_\psi \otimes A \tag{3.7}$$

Hence, the space $\overline{A \odot A/\mathcal{N}}$ is isomorphic to $H_\psi \otimes A$, where H_ψ denotes the GNS Hilbert space of (A, ψ). In particular, $H_\psi \otimes A$ carries the structure of a C^*-correspondence in our setting, with the left action given by $\pi_\psi \otimes 1 : A \longrightarrow \mathcal{L}(H_\psi \otimes A)$, defined as follows:

$$(\pi_\psi \otimes 1)(a)(\eta_b \otimes c) = \eta_{ab} \otimes c$$

for $a, b, c \in A$ and $\eta_b \in H_\psi$. Also, the left multiplication defined here is consistent with the notion of left multiplication used in Kwaśniewski's construction.

Definition 3.2.2. *Let A be a unital C^*-algebra and ψ be a state on A. We define $A \rtimes_\psi \mathbb{N}$ the* crossed product of A by the state ψ *as the Cuntz-Pimsner algebra associated with the C^*-correspondence $(H_\psi \otimes A)$, where H_ψ is the GNS Hilbert space of ψ which coincides with Kwaśniewski's crossed-product construction for A with respect to ϱ_ψ. The corresponding crossed* *product is given by*

$$A \rtimes_{\psi} \mathbb{N} := C^*(A, \varrho_{\psi}) \cong \mathcal{O}(J_{H_{\psi}\otimes A},\, H_{\psi} \otimes A),$$

3.2.1 Special case for $A = C^*(G)$ and the canonical trace on it

Let G be a countable discrete amenable group, with the identity element $e \in G$. If we consider our C^*-algebra to be $C^*(G)$ and let τ be the canonical trace defined by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ and the corresponding CP map for Kwaśniewski's construction is $\rho_{\tau} : C^*(G) \to C^*(G)$ defined by $\rho_{\tau}(x) = \tau(x)1_{C^*(G)}$ for $x \in C^*(G)$. In this case we know the GNS-tuple for $C^*(G)$ with respect to τ is $(\gamma_{\lambda}, \ell^2(G), \delta_e)$, where γ_{λ} is the extension of the left regular representation λ to $C^*(G)$ using the universality of $C^*(G)$. Thus as we already discussed our required crossed product of $C^*(G)$ with respect to ρ_{τ} is given by,

$$C^*(G) \rtimes_{\tau} \mathbb{N} := C^*(C^*(G), \rho_{\tau}) \cong \mathcal{O}(J_{\ell^2(G)\otimes C^*(G)}, \ell^2(G) \otimes C^*(G)) \tag{3.8}$$

where, $\mathcal{O}(J_{\ell^2(G)\otimes C^*(G)}, \ell^2(G) \otimes C^*(G))$ denotes the Cuntz-Pimsner algebra associated with the C^*-correspondence $\ell^2(G)\otimes C^*(G)$, where the left action is given by the $*$-homomorphism

$$\gamma_{\lambda} \otimes 1 : C^*(G) \longrightarrow \mathcal{L}(\ell^2(G) \otimes C^*(G)),$$

defined as follows:

$$\gamma_{\lambda} \otimes 1(u_g)(\xi_h \otimes u_k) = \xi_{gh} \otimes u_k \tag{3.9}$$

where $g, h, k \in G$, $\{\xi_g : g \in G\}$ denotes the orthonormal basis of $l^2(G)$ and u_g is the unitary element in $\mathbb{C}[G]$ corresponding to $g \in G$.

Proposition 3.2.3. *For an infinite countable discrete group G, the universal crossed product of $C^*(G)$ with respect to τ is isomorphic to the universal Toeplitz algebra we get for $C^*(G)$ with respect* *to ρ_τ, that is,*

$$C^*(G) \rtimes_\tau \mathbb{N} := C^*(C^*(G), \rho_\tau) \cong \mathcal{O}_{\ell^2(G)\otimes C^*(G)} \cong \mathcal{T}(C^*(G), \rho_\tau) \cong \mathcal{T}(C^*(G), \ell^2(G) \otimes C^*(G)) \tag{3.10}$$

Proof. From Theorem 2.9.11 and Proposition 2.9.8 it is evident that all we need to show is,

$$\mathcal{O}(J_{\ell^2(G)\otimes C^*(G)}, \ell^2(G) \otimes C^*(G)) \cong \mathcal{T}(C^*(G), \ell^2(G) \otimes C^*(G))$$

Now if we recall the Definition 2.6.5, and choose a representation $(\pi, \pi_{\ell^2(G)\otimes C^*(G)})$, to show that our representation is covariant we need to check if the condition $(\pi, \pi_{\ell^2(G)\otimes C^*(G)})^{(1)}(\lambda \otimes 1(a)) = \pi(a)$ holds for all $a \in J_{\ell^2(G)\otimes C^*(G)} \subseteq J(\ell^2(G) \otimes C^*(G)) \Rightarrow \gamma_\lambda \otimes 1(a) \in \mathcal{K}(\ell^2(G) \otimes C^*(G))$. Now, $\mathcal{K}(\ell^2(G) \otimes C^*(G)) = \mathcal{K}(\ell^2(G)) \otimes C^*(G)$. Hence, for $\sum_{s\in F \subset_{Fin} G} a_s u_s \in \mathbb{C}[G]$,

$$(\gamma_\lambda \otimes 1)(\sum_{s\in F} a_s u_s) = \sum_{s\in F} a_s \lambda_s \otimes 1 \tag{3.11}$$

Applying the map $(\mathrm{id} \otimes \tau)$ on (3.12), we get,

$$(\mathrm{id} \otimes \tau)\left((\lambda \otimes 1)(\sum_{s\in F} a_s u_s)\right) = a_e \lambda_e \tag{3.12}$$

Now, observe that $(\mathrm{id} \otimes \tau)(\mathcal{K}(\ell^2(G)) \otimes C^*(G)) = \mathcal{K}(\ell^2(G))$. But since G is countably infinite, λ_e cannot be compact. Hence $\gamma_\lambda \otimes 1(\sum_{s\in F\subset_{Fin} G} a_s u_s) \notin \mathcal{K}(\ell^2(G)) \otimes C^*(G))$. Now following the same line of proof from Proposition 3.3.10 we can prove that for all $a \in C^*(G)$, $\gamma_\lambda \otimes 1(a) \notin \mathcal{K}(\ell^2(G)) \otimes C^*(G))$. Therefore, in our case every representation automatically satisfies the covariance condition. Hence, the claim is proved.

$\square$

Since we are working with $C^*(G)$ and τ one of the direct question one can ask is, "can we get a spatial structure of the crossed product inside $B(\ell^2(G))$, or can we find a representation of $(C^*(G), \rho_\tau)$ given by (γ_λ, S), where $S \in B(\ell^2(G))$ such that it satisfies the conditions in Definition 2.9.1,

$$\begin{aligned} S^*\gamma_\lambda(u_g)S &= \gamma_\lambda(\rho_\tau(u_g)) \\ \Rightarrow S^*\lambda_g S &= \tau(u_g)1_{C^*(G)} \\ &= \begin{cases} 1_{C^*(G)} & \text{if } g = e \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3.13}$$

Now, from the above expression we can conclude that $S \in B(\ell^2(G)$ is an isometry.

Theorem 3.2.4. *There does not exist an isometry $S \in B(\ell^2(G))$ such that,*

$$S^*\lambda_g S = \begin{cases} 1_{C^*(G)} & \textit{if } g = e \\ 0 & \textit{otherwise} \end{cases} \tag{3.14}$$

Proof. Follows as a special case of Proposition 3.3.9. □

Hence we obtain that there does not exist a $S \in B(\ell^2(G))$ such that (γ_λ, S) becomes a representation.

We now define a representation for the system $(C^*(G), \rho_\tau)$. We will see later on in this section that this turns out to be a very important representation. Let $\mathcal{H} = \bigoplus_n \ell^2(G)^{\otimes n}$ we define $T \in B(\bigoplus_n \ell^2(G)^{\otimes n})$ such that $T\xi = \delta_e \otimes \xi$ and we define $\tilde{\pi} : C^*(G) \to B(\bigoplus_n \ell^2(G)^{\otimes n})$ by,

$$\tilde{\pi}(u_g) = \lambda(g) \otimes \mathrm{id} \tag{3.15}$$

for $g \in G$

To prove $(\tilde{\pi}, T)$ is a representation we need to show,

$$T^*\tilde{\pi}(u_g)T = \tilde{\pi}(\rho_\tau(u_g))$$

Now,

$$\begin{aligned}\langle T^*\tilde{\pi}(u_g)T\xi,\eta\rangle &= \langle \tilde{\pi}(u_g)T\xi, T\eta\rangle \\ &= \langle \tilde{\pi}(u_g)\delta_e\otimes\xi, \delta_e\otimes\eta\rangle \\ &= \langle \delta_g\otimes\xi, \delta_e\otimes\eta\rangle \\ &= \langle \tilde{\pi}(\rho_\tau(u_g))\xi,\eta\rangle\end{aligned} \tag{3.16}$$

Thus, by Definition 2.9.1, $(\tilde{\pi}, T)$ is a representation of $(C^*(G), \rho_\tau)$, and the associated C^*-algebra is given by

$$C^*(\tilde{\pi}, T) = C^*\Big(\tilde{\pi}(C^*(G)) \cup T\tilde{\pi}(C^*(G))\Big).$$

Moreover, by Proposition 3.2.3, $(\tilde{\pi}, T)$ is a covariant representation.
Now we define a gauge action of $\mathbb{T}$ on $C^*(\tilde{\pi}, T)$. For $\xi \in \ell^2(G)^{\otimes n}$ such that $\xi = \bigoplus_n \xi_n$ and for $z \in \mathbb{T}$ we define $U_z(\xi) = \bigoplus_n z^n\xi_n$.

$$\begin{aligned}U_z\tilde{\pi}(u_g)U_{\overline{z}}(\xi) &= U_z\tilde{\pi}(u_g)(\bigoplus_n \overline{z}^n\xi_n) \\ &= U_z(\overline{z}\lambda(g)(\xi_1) \oplus \overline{z}^2\xi_2 \oplus \ldots) \\ &= z\overline{z}\lambda(g)(\xi_1) \oplus z^2\overline{z}^2\xi_2 \oplus \ldots \\ &= \tilde{\pi}(u_g)(\xi)\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}U_zTU_{\overline{z}}(\xi) &= U_zT(\bigoplus_n \overline{z}^n\xi_n) \\ &= U_z(\bigoplus_n \overline{z}^n\delta_e\otimes\xi_n) \\ &= \bigoplus_n z^{n+1}\overline{z}^n\delta_e\otimes\xi_n \\ &= zT(\xi)\end{aligned} \tag{3.18}$$

Proposition 3.2.5. *Let G be an infinite discrete amenable group. The crossed product of $C^*(G)$ by its canonical trace τ is isomorphic to the spatial C^*-algebra $C^*(\tilde{\pi}, T)$, where*

$$\tilde{\pi} : C^*(G) \to B\left(\bigoplus_{n=0}^{\infty} \ell^2(G)^{\otimes n}\right)$$

is the representation defined by

$$\tilde{\pi}(u_g) = \lambda(g) \otimes \mathrm{id},$$

with λ denoting the left regular representation of G, and $T \in B(\bigoplus_{n=0}^{\infty} \ell^2(G)^{\otimes n})$ is defined by

$$T\xi = \delta_e \otimes \xi.$$

Consequently,

$$C^*(G) \rtimes_\tau \mathbb{N} := C^*(C^*(G), \rho_\tau) \cong C^*(\tilde{\pi}, T).$$

Proof. Using Proposition 3.2.3, we know that $(\tilde{\pi}, T)$ is a covariant representation. We define a gauge action $\beta : \mathbb{T} \to \mathrm{Aut}(C^*(\tilde{\pi}, T))$ defined by,

$$\beta_z(x) = U_z x U_{\overline{z}} \tag{3.19}$$

for $z \in \mathbb{T}$ and $x \in C^*(\tilde{\pi}, T)$.
Now using (3.18), (3.19) and Proposition 2.9.12 we get $\mathbb{C}^*(C^*(G), \rho_\tau) \cong C^*(\tilde{\pi}, T)$. $\square$

3.3 Generalised Crossed Product Structures on $C^*(G)$

In this section, we discuss various generalised crossed-product structures on $C^*(G)$, primarily in connection with endomorphisms and certain completely positive maps, $C^*(G)$ along with $C(G)$ play an important role in connecting group theoretic structures with C^*-algebraic structures, particularly in the context of *Quantum Groups*.

3.3.1 Crossed Product by an endomorphism

Let G be a discrete group, and let $\alpha : G \hookrightarrow G$ be an injective homomorphism. We call α *pure* if $\bigcap_{n=0}^{\infty} \alpha^n(G) = \{e\}$, where e is the identity element for G. Let u_g be the image of g in $\mathbb{C}[G]$. We assume $[G : \alpha(G)] < \infty$.
In [Hir02], Hirshberg defined a universal C^*-algebra $\mathcal{E}_\alpha$ generated by $\mathbb{C}[G]$ and an isometry S.

Definition 3.3.1. *([Hir02, Definition 1.1]) The universal C^*-algebra $\mathcal{E}_\alpha$ is generated by $\mathbb{C}[G]$ and an isometry S, satisfying the relations:*

1. $Su_x = u_{\alpha(x)}S$ *for* $x \in G$
2. $S^*u_xS = 0$ *if* $x \notin \alpha(G)$
3. *For any complete list of right coset representatives* $x_1, x_2, ...x_n$ *of* $\alpha(G)$

$$\sum_{k=1}^{n} u_{x_k^{-1}} SS^* u_{x_k} = 1 \tag{3.20}$$

We first discuss the motivation for this definition and then define the corresponding *Exel system*. After defining the system we show that $\mathcal{E}_\alpha$ is *isomorphic* to the Exel crossed product arising from this Exel system. Unless stated otherwise, we assume that G is amenable.

Motivation coming from a spatial construction in $B(\ell^2(G))$

Since G is amenable, the algebra $C^*(G)$ is faithfully represented inside $B(\ell^2(G))$ via the $*$-representation obtained as the extension of the left regular representation $\lambda : G \to \mathcal{U}(\ell^2(G))$. Now, given an endomorphism $\alpha : G \to G$, we obtain an isometry $S \in B(\ell^2(G))$ defined by $S(\xi_g) = \xi_{\alpha(g)}$ and hence, $S^*(\xi_g) = \begin{cases} \xi_{\alpha^{-1}(g)} & \text{if } g \in \alpha(G) \\ 0 & \text{otherwise} \end{cases}$. For $g \in G$, we define the corresponding elements inside $\lambda_g \in B(\ell^2(G))$ such that $\lambda_g(\xi_h) = \xi_{gh}$. Then,

$$S\lambda_g(\xi_h) = S\xi_{gh} = \xi_{\alpha(gh)} = \lambda_{\alpha(g)}S(\xi_h) \tag{3.21}$$

Now for $g \notin \alpha(G)$

$$\begin{aligned}\langle S^*\lambda_g S(\xi_h), \xi_k\rangle &= \langle \lambda_g \xi_{\alpha(h)}, S\xi_k\rangle \\ &= \langle \xi_{g\alpha(h)}, \xi_{\alpha(k)}\rangle \\ &= 0 \ \{\text{Since, } g \notin \alpha(G)\}\end{aligned} \tag{3.22}$$

Thus, (3.22) and (3.23) lay the foundation for the first two conditions in Definition 3.3.1. We considered $x_1, x_2, ...x_n$ to be the complete list of right coset representatives. Now for the third condition, we observe,

$$\begin{aligned}\lambda_{x_k^{-1}} SS^*\lambda_{x_k}(\xi_r) &= \lambda_{x_k^{-1}} SS^*(\xi_{x_k r}) \\ &= \begin{cases}\lambda_{x_k^{-1}} S\xi_{\alpha^{-1}(x_k r)} & \text{if } x_k r \in \alpha(G) \\ 0 & \text{otherwise}\end{cases} \\ &= \begin{cases}\xi_r & \text{if } r \in x_k^{-1}\alpha(G) \\ 0 & \text{otherwise}\end{cases}\end{aligned} \tag{3.23}$$

Since $\{x_1, x_2, ...x_n\}$ gives us the complete list of right coset representatives of $\alpha(G)$, it follows that $\{x_1^{-1}, x_2^{-1}, ...x_n^{-1}\}$ gives us the complete list of left coset representative. From (3.24) we can see that $\lambda_{x_k^{-1}} SS^*\lambda_{x_k}(\xi_r) = 1_{x_k^{-1}\alpha(G)}(r)\xi_r$. Thus, $\lambda_{x_k^{-1}} SS^*\lambda_{x_k}$ is the projection onto the subspace $\ell^2(x_k^{-1}\alpha(G))$. Since, $G = \bigsqcup_{k=1}^n x_k^{-1}\alpha(G)$ we obtain the important relation,

$$\sum_{k=1}^{n} \lambda_{x_k^{-1}} SS^*\lambda_{x_k}(\xi_r) = 1 \tag{3.24}$$

which gives us the motivation for defining the third relation in Definition 3.3.1.

Comparison with Exel Crossed Product

We know that $\alpha : G \to G$ can be extended to an endomorphism $\gamma_\alpha : C^*(G) \to C^*(G)$. We now consider the conditional expectation $E_\alpha : C^*(G) \to C^*(\alpha(G))$ defined by,

$$E_\alpha(\sum_{s\in G} a_s u_s) = \sum_{s\in G} 1_{\alpha(G)}(s) a_s u_s, \text{ where } 1_{\alpha(G)}(s) = \begin{cases} 1 & \text{if } s \in \alpha(G) \\ 0 & \text{otherwise} \end{cases} \tag{3.25}$$

Now, using Proposition 3.1.6([?, Proposition 2.6]) we can find our transfer operator corresponding to the conditional expectation E_α.

Using [BO08, Proposition 2.5.8] we observe that, α is injective $\Rightarrow \gamma_\alpha$ is also injective $\Rightarrow$ $\ker(\gamma_\alpha) = 0$. We can define $\beta_\alpha : C^*(\alpha(G)) \to C^*(G)$ where, $\beta_\alpha = (\gamma_\alpha)^{-1}$. Thus, the transfer operator $\mathcal{L}_\alpha$ is given by $\beta_\alpha \circ E_\alpha$ which can be elaborated as,

$$\mathcal{L}_\alpha(\sum a_s u_s) = \beta_\alpha \circ E_\alpha(\sum a_s u_s) = \sum 1_{\alpha(G)}(s) a_s u_{\alpha^{-1}(s)} \tag{3.26}$$

where $1_{\alpha(G)}$ is the indicator function.

For the universal representation (i, S) we obtain our Toeplitz Algebra $\mathcal{T}(C^*(G), \gamma_\alpha, \mathcal{L}_\alpha)$ generated by $i(C^*(G))$ and S such that the following holds,

1. $S^* i(x) S = i(\mathcal{L}_\alpha(x))$, for $x \in C^*(G)$
2. $S i(x) = i(\gamma_\alpha(x)) S$, for $x \in C^*(G)$

Since i is injective, we may identify $i(x)$ with x. Also,

$$S^* u_s S = \begin{cases} u_{\alpha^{-1}(s)} & \text{if } s \in \alpha(G) \\ 0 & \text{otherwise} \end{cases}$$

this actually gives us the first two condition of the universal algebra in [Hir02].

Analysis of Redundancies:

Recalling Definition 2.8.5, for an Exel system $(A, \alpha, \mathcal{L})$ and a representation (π, S) of the system, we say that a pair $(\pi(a), k)$ is a *redundancy* if for every $b \in A$ one has

$$\pi(a)\pi(b)S \;=\; k\pi(b)S,$$

where $a \in A$ and $k \in \overline{\pi(A)SS^*\pi(A)}$.

Note that for any representation (π, S) for the system $(C^*(G), \gamma_\alpha, \mathcal{L}_\alpha)$ S is always an isometry which is due to the fact $C^*(G)$ is unital and as per our definition $\mathcal{L}_\alpha(1) = 1$. Hence, we have $S^*S = S^*\pi(1)S = \pi(\mathcal{L}_\alpha(1)) = 1$

We now prove that the pair $(1, \sum_k u_{x_k^{-1}} SS^* u_{x_k})$ coming from the definition of $\mathcal{E}_\alpha$ (Definition 3.3.1) is a redundancy for the universal representation (i_A, S) of the Exel system $(C^*(G), \lambda_\alpha, \mathcal{L}_\alpha)$. Since the universal representation is injective, we identify $i_A(a)$ with a.

To prove that the above pair is a redundancy, we show that

$$\sum_s a_s u_s S = (\sum_k u_{x_k^{-1}} SS^* u_{x_k})(\sum_s a_s u_s)S, \text{ for all } \sum_s a_s u_s \in \mathbb{C}G$$

Hence we try to simplify the right hand side,

$$\begin{aligned}
(\sum_k u_{x_k^{-1}} SS^* u_{x_k})(\sum_s a_s u_s)S &= \sum_{k,s} u_{x_k^{-1}} SS^* u_{x_k} a_s u_s S \\
&= \sum_{k,s} a_s u_{x_k^{-1}} SS^* u_{x_k s} S
\end{aligned}$$

Now,

$$a_s u_{x_k^{-1}} S S^* u_{x_k s} S = \begin{cases} a_s u_{x_k^{-1}} S u_{\alpha^{-1}(x_k s)}, & \text{if } x_k s \in \alpha(G) \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} a_s u_{x_k^{-1}} u_{x_k s} S, & \text{if } x_k s \in \alpha(G) \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} a_s u_s S, & \text{if } s \in x_k^{-1}\alpha(G) \\ 0 & \text{otherwise} \end{cases}$$

Since x_k^{-1} are left coset representatives we have,

$$\sum_s a_s u_s S = (\sum_k u_{x_k^{-1}} S S^* u_{x_k})(\sum_s a_s u_s) S$$

proving $(1, \sum_k u_{x_k^{-1}} S S^* u_{x_k})$ is a redundancy for the universal representation (i_A, S) which means when we quotient the Toeplitz Algebra $C^*(i_A, S)$ with the ideal generated by the set $\{i_A(a) - k : \ (i_A(a), k) \text{ is a redundancy for } (i_A, S)\}$ to get our Exel crossed product we get $\sum_k u_{x_k^{-1}} S S^* u_{x_k} = 1$, as we get in the definition of $\mathcal{E}_\alpha$ in [Hir02].
We now show that Condition (3) in Definition 3.3.1 for $\mathcal{E}_\alpha$ is sufficient to eliminate all elements of the redundancy ideal.
Now, suppose a pair (a, k) be a redundancy pair. Then,

$$\begin{aligned} &a u_{x_l^{-1}} S = k u_{x_l^{-1}} S \\ \Rightarrow &a u_{x_l^{-1}} S S^* u_{x_l} = k u_{x_l^{-1}} S S^* u_{x_l} \\ \Rightarrow &a \sum_l u_{x_l^{-1}} S S^* u_{x_l} = k \sum_l u_{x_l^{-1}} S S^* u_{x_l} \end{aligned}$$

Thus, we obtain $a = k$ in the final crossed product construction for the Exel construction, which actually shows that $\mathcal{E}_\alpha$ is actually equivalent to the construction of Exel crossed products.

Remark 3.3.2. *In [CV13, Corollary 2.7] they proved that, for abelian groups, the universal C^*-algebra with respect to a pure injective endomorphism isomorphic to its spatial structure embedded inside $B(\ell^2(G))$. In the next section, we will see how different it is to consider the case of CP maps instead of an endomorphism.*

3.3.2 Crossed Product by a CP map

In the above setting, it is natural to consider the crossed product of $C^*(G)$ with respect to a completely positive map (CP map), as defined in [Kwa17]. In this section, we not only examine the construction of such C^*-algebras for particular CP maps, but also provide a comparison with our earlier discussion of crossed products by endomorphisms.
It is most natural to first consider a conditional expectation as a candidate for our investigation. In our case, we focus on the conditional expectation with respect to a proper subgroup $H < G$, defined by

$$E_H(\sum_{g\in G} a_g u_g) = \sum_{g\in G} 1_H(g) a_g u_g, \text{ for } g \in G. \tag{3.27}$$

where 1_H is the indicator function. At present we are not assuming any condition on $[G : H]$. We consider the C^*-dynamical system $(C^*(G), E_H)$. Recalling Definition 2.9.1 a pair (π, S) is a representation if,

$$S^*\pi(\sum_{s\in G} a_s u_s)S = \pi(E_H(\sum_{s\in G} a_s u_s)) = \sum_{s\in G} 1_H(s) a_s \pi(u_s) \tag{3.28}$$

Remark 3.3.3. *Note that taking a conditional expectation with respect to a subgroup $H < G$ is not a significant deviation from the above construction with an endomorphism, which is more evident if we consider [Kwa17, Theorem 4.7] which implies that the Exel crossed product with respect to the Exel system $(A, \alpha, \mathcal{L}_\alpha)$ is isomorphic to the crossed product of A with respect to the CP map $\mathcal{L}_\alpha$ following Kwaśniewski's Construction. Now, recall Proposition 3.1.6, we define $\mathcal{L}_\alpha = (\alpha|_{(\ker\alpha)^\perp})^{-1} \circ E_\alpha$. So at first glance it may seem that the treatment of $C^*(C^*(G), \mathcal{L}_\alpha)$ and $C^*(C^*(G), E_\alpha)$ should be quite* *similar but it turns out to be quite different which is evident from the following construction. Rather than considering the conditional expectation E_α onto $C^*(\alpha(G))$, we instead work with E_H for an arbitrary proper subgroup $H < G$, thereby providing a more general framework for our discussion.*

Since we are concerned with $C^*(G)$ it is quite natural to expect finding a representation inside $B(\ell^2(G))$ with respect to the representation $\pi_\lambda : C^*(G) \to B(\ell^2(G))$ coming as an extension of the left regular representation $\lambda : G \to \mathcal{U}(\ell^2(G))$. Such a representation is particularly natural when G is abelian in the context of Remark 3.3.2. So first we try to find if there exist $S \in B(\ell^2(G))$ such that (γ_λ, S) becomes a representation for $(C^*(G), E_H)$. So essentially we want to find $S \in B(\ell^2(G))$ such that,

$$S^* u_g S = \begin{cases} u_g & \text{if } g \in H \\ 0 & \text{otherwise} \end{cases} \tag{3.29}$$

First note that here S becomes an isometry.

Proposition 3.3.4. *For an abelian discrete group G and a proper subgroup $H < G$, there does not exist $S \in B(\ell^2(G))$ such that S is an isometry and,*

$$S^* u_g S = \begin{cases} u_g & \textit{if } g \in H \\ 0 & \textit{otherwise} \end{cases}$$

where $u_g \in \mathbb{C}[G]$ is the unitary element corresponding to $g \in G$.

We first show the following lemma,

Lemma 3.3.5. *For $g \in H$, we have $u_g S = S u_g$*

Proof. First note that,

$$\begin{aligned}
&S^*u_gS = u_g,\ g \in H \\
\Rightarrow &(SS^*)u_gS = Su_g \\
\Rightarrow &\|(SS^*)(u_gSx)\| = \|Su_gx\| = \|x\| = \|u_gSx\|, \text{for } x \in \ell^2(G)
\end{aligned}$$

Thus SS^* is a projection onto $\text{Range}(S)$, we observe from above that SS^* preserves the norm of u_gS for $g \in H$. Thus we can claim that $u_gS \in \text{Range}(S)$

Let,

$$\begin{aligned}
&u_gSx = Sy,\ x, y \in \ell^2(G) \\
\Rightarrow &S^*u_gSx = S^*Sy \\
\Rightarrow &u_gx = y
\end{aligned}$$

Thus, we have $u_gSx = Sy = Su_gx \Rightarrow u_gS = Su_g$. □

Note that the above lemma does not assume G to be abelian. Let, K be an abelian compact group with Haar probability measure μ such that $\hat{K} = G$. We recall that $\ell^2(G)$ is isometrically isomorphic to $L^2(K)$ via the Fourier transform. Henceforth, we identify elements of G with characters $\chi : K \to S^1$. Now, we begin with the proof of Proposition 3.3.4.

Proof. Note that $\{\chi : \chi \in \hat{K}\}$ form an orthonormal basis of $L^2(K)$. The corresponding version of Proposition 3.3.4 in $L^2(K)$ would be stating there does not exist an isometry $S \in B(L^2(K))$ such that

$$S^*U_\chi S = \begin{cases} U_\chi & \text{if } \chi \in H \\ 0 & \text{otherwise} \end{cases} \tag{3.30}$$

where $U_\chi(h) = \chi h$ for all characters χ and $h \in L^2(K)$.

Let $S(1) = f$. From Lemma 3.3.5,

$$SU_\chi = U_\chi S$$

for all $\chi \in H$, which implies

$$S(\chi) = \chi f$$

for all $\chi \in H$. Since S is an isometry, $\{\chi f : \chi \in H\}$ is orthonormal. Hence,

$$\widehat{|f|^2}(\chi) = \int_K |f|^2 \bar{\chi} = \langle f, \chi f \rangle = 0 \tag{3.31}$$

for all $\chi \in H \setminus \{1\}$.

If $\chi \in G \setminus H$, (3.31) implies $\chi f \in (\operatorname{im} S)^\perp$. Thus,

$$\widehat{|f|^2}(\chi) = \langle \chi f, f \rangle = 0 \tag{3.32}$$

for all $\chi \in G \setminus H$, since $f \in \operatorname{im} S$. From (3.31) and (3.32), $\widehat{|f|^2}(\chi) = 0$ for all $\chi \neq 1$ and hence, $|f| = c$ a.e. on K. Moreover, since $S(1) = f$ and S is an isometry it implies that $\int_K |f|^2 = 1$, hence $|f| = 1$ a.e on K.

Now let $\chi_0 \in G \setminus H$ and let $S(\chi_0) = f_0$. Since $\chi_0 \perp H$ and S is an isometry,

$$\widehat{f_0 \bar{f}}(\chi) = \int_K f_0 \bar{f} \bar{\chi} = \langle f_0, \chi f \rangle = \langle S\chi_\circ, S\chi \rangle = 0 \tag{3.33}$$

for all $\chi \in H$.

Also from (3.31), $\chi f \in (\operatorname{im} S)^\perp$ for all $\chi \in G \setminus H$. This implies

$$\begin{aligned} &\langle f_0, \chi f \rangle = 0 \\ \Rightarrow &\widehat{f_0 \bar{f}}(\chi) = 0 \end{aligned} \tag{3.34}$$

for all $\chi \in G \setminus H$. From (3.34) and (3.35), $\widehat{f_o \bar{f}} \equiv 0$, implying $f_0 \bar{f} \equiv 0$. Since $|f| = 1$ a.e., this implies $f_0 = 0$ a.e., contradicting the fact that $||f_0||_2 = 1$.

$\square$

Remark 3.3.6. *Hence, for an abelian group G, there is no spatial construction of the system $(C^*(G), E_H)$ inside $B(\ell^2(G))$ which actually shows how different the case becomes for crossed product by a CP map compared to an endomorphism.*

We now discuss the above result for virtually-abelian discrete countable groups. The result remains true, though the proof is a bit more complicated. We need to discuss the Plancherel theorem for virtually-abelian groups first.

Definition 3.3.7. *A group G is called virtually abelian if G has a finite index abelian subgroup.*

Let G be a discrete virtually abelian group. For an irreducible unitary representation $\pi : G \to B(\mathcal{H}_\pi)$ and $f \in L^1(G)$, define

$$\widehat{f}(\pi) = \int_G f(g)\pi(g)^* dg.$$

This reduces to the Fourier transform if G is abelian. Let $[\pi]$ denote the unitary equivalence class of π and let $\widehat{G}$ be the set of all such equivalence classes. This is called the *unitary dual* of G. Note that for two unitarily equivalent irreducible representations π and π', $\widehat{f}(\pi)$ and $\widehat{f}(\pi')$ are also unitarily equivalent. The Plancherel theorem can now be stated as :

Theorem 3.3.8. *[Fol16] There exists a measure μ on $\widehat{G}$ such that for all $f \in L^2(G) \cap L^1(G)$, for μ - almost every $[\pi] \in \widehat{G}$, $\widehat{f}(\pi)$ is a Hilbert-Schmidt operator and*

$$||f||_2^2 = \int_{\widehat{G}} ||\widehat{f}(\pi)||_{HS}^2 \, d\mu(\pi).$$

Subsequently, we have

$$L^2(G) \cong \int_{\widehat{G}}^{\oplus} L^2(\mathcal{H}_\pi) \, d\mu(\pi)$$

where $L^2(\mathcal{H}_\pi)$ denotes the Hilbert space of all Hilbert-Schmidt operators on $\mathcal{H}_\pi$ and $||.||_{HS}$ denotes the corresponding norm.

[Fol16] states the above theorem for type I groups, so the above result thus holds for virtually abelian groups since for countable groups both notions are equivalent by a famous result of Thoma [Tho64].

Thus, for each $[\pi] \in \widehat{G}$ we choose a representative π of the equivalence class, which we fix. The map $\mathcal{F} : L^1(G) \cap L^2(G) \to \int_{\widehat{G}}^{\oplus} L^2(\mathcal{H}_\pi)\, d\mu(\pi)$ given by

$$\mathcal{F}(f)(\pi) = \widehat{f}(\pi)$$

extends to a unitary between the two spaces, which we also denote by $\mathcal{F}$. This is the non-abelian version of the Fourier transform. We are now in a position to state and prove the next result.

Proposition 3.3.9. *For a virtually abelian discrete group G and a proper subgroup $H < G$, there does not exist $S \in B(\ell^2(G))$ such that*

$$S^* u_g S = \begin{cases} u_g & \textit{if } g \in H \\ 0 & \textit{otherwise} \end{cases}$$

where $u_g \in \mathbb{C}[G]$ is the unitary element corresponding to $g \in G$.

Proof. Assume such an S exists. For $g \in G$ let δ_g denote the indicator function on G. Note that $\{\delta_g : g \in G\}$ is an orthonormal basis of $l^2(G)$. By Lemma 3.3.5,

$$S u_g = u_g S$$

for all $g \in H$. Let $S\delta_e = f$. Denote by f^* the map

$$f^*(g) = \overline{f(g^{-1})}.$$

Then $S\delta_g = u_g f = \delta_g * f$ for $g \in H$. By orthogonality, $\langle u_g f, f \rangle = 0$ for all $g \in H \setminus \{e\}$. Now if $g \in G \setminus H$, $u_g f \perp \mathrm{im}(S)$ and therefore,

$$\langle u_g f, f \rangle = 0$$

for all $g \neq e$. Note that

$$\begin{aligned} \langle f, u_g f \rangle &= \sum_{x \in G} \overline{f(g^{-1}x)} f(x) \\ &= f * f^*(g) \end{aligned}$$

for all $g \in G$. Hence,

$$f * f^* = \delta_e. \tag{3.35}$$

Let $g_0 \in G \setminus H$ and let $S\delta_{g_0} = \phi$. Since $\langle S\delta_{g_0}, \delta_g \rangle = 0$ for all $g \in H$, $\langle u_g f, \phi \rangle$ = 0 for all $g \in H$. If $g \in G \setminus H$, $\langle u_g f, \phi \rangle = 0$ since $\phi \in \mathrm{im}(S)$. Hence, $\langle \phi, u_g f \rangle = 0$ for all $g \in G$, implying, like the previous argument,

$$\phi * f^* \equiv 0. \tag{3.36}$$

Let $f_n \in l^1(G) \cap l^2(G)$ such that $f_n \to f$ in $l^2(G)$. Note that for any $h \in l^2(G)$, $h * f^*(g) = \langle h, u_g f \rangle$ and $\sum_{g \in G} |\langle h, u_g f \rangle|^2 \leq \|h\|_2^2$, since $\{u_g f : g \in G\}$ is an orthonormal set. Thus we have,

$$
\begin{aligned}
||(f_n - f) * f^*||^2 &= \sum_{g \in G} |\langle f_n - f, u_g f \rangle|^2 \\
&\leq ||f_n - f||^2 \\
&\to 0.
\end{aligned} \tag{3.37}
$$

This implies that $\mathcal{F}(f_n * f^*) \to \mathcal{F}(f * f^*)$ in $\int_{\widehat{G}}^{\oplus} L^2(\mathcal{H}_\pi)\, d\mu(\pi)$. Thus we get

$$
\begin{aligned}
&\int_{\widehat{G}} ||\mathcal{F}(f_n * f^*)(\pi) - \mathcal{F}(f * f^*)(\pi)||^2 \, d\mu(\pi) \\
&= \int_{\widehat{G}} ||\widehat{f_n}(\pi)\mathcal{F}(f^*)(\pi) - \mathcal{F}(f * f^*)(\pi)||^2 \, d\mu(\pi) \\
&\to 0
\end{aligned} \tag{3.38}
$$

Passing on to a subsequence, we may assume without loss of generality that $\widehat{f_n}(\pi)\mathcal{F}(f^*)(\pi) \to \mathcal{F}(f * f^*)(\pi)$ in $L^2(\mathcal{H}_\pi)$ for μ- almost every π. Again by Plancherel's theorem,

$$
\int_{\widehat{G}} ||\widehat{f_n}(\pi) - \mathcal{F}(f)(\pi)||^2 \, d\mu(\pi) \to 0.
$$

Therefore, we can pass on to a subsequence again, and assume like above that $\widehat{f_n}(\pi) \to \mathcal{F}(f)(\pi)$ in $L^2(\mathcal{H}_\pi)$ for μ- almost every π. Since multiplication by any bounded operator preserves Hilbert-Schmidt convergence, we arrive at

$$
\mathcal{F}(f * f^*)(\pi) = \mathcal{F}(f)(\pi)\mathcal{F}(f)(\pi)^*
$$

for almost every π. Finally, using (6) we get

$$
\mathcal{F}(f)(\pi)\mathcal{F}(f)(\pi)^* = I \tag{3.39}
$$

for almost every π. Note that a similar argument, alongwith (7), yields

$$\mathcal{F}(\phi)(\pi)\mathcal{F}(f)(\pi)^* = O \tag{3.40}$$

almost everywhere.

Since $\mathcal{F}(f)(\pi)$ is Hilbert-Schmidt, it is compact. This implies

$$\dim\,(\mathcal{H}_\pi) < \infty$$

and $\mathcal{F}(f)(\pi)$ is unitary for almost every π. This, together with (11), gives

$$\mathcal{F}(\phi)(\pi) = O$$

for $\mu-$ almost every π, contradicting Plancherel's theorem since $||\phi|| = 1$. $\square$

Corresponding Cuntz-Pimsner Construction

Recalling the construction in [Kwa17], the C^*-correspondence associated with the system $(C^*(G), E_H)$ is given by

$$X_G := \overline{C^*(G) \odot C^*(G) \,/\, \mathcal{N}},$$

where the completion is taken with respect to the seminorm induced by the $C^*(G)$-valued inner product

$$\langle a \odot b,\, c \odot d\rangle := b^* E_H(a^*c)d, \qquad a, b, c, d \in C^*(G). \tag{3.41}$$

Here,

$$\mathcal{N} := \{\, x \odot y \in C^*(G) \odot C^*(G) \mid \langle x \odot y, x \odot y\rangle = 0 \,\}$$

denotes the null space of the seminorm.

The right module action is defined by

$$(b \otimes c) \cdot a := b \otimes (ca), \qquad a, b, c \in C^*(G),$$

and the left action is given by the $*$-homomorphism

$$\pi_G : C^*(G) \to \mathcal{L}(X_G), \qquad \pi_G(a)([b \otimes c]) := [ab \otimes c].$$

As we discussed earlier this construction coincides with the KSGNS construction [Lan95]

$$C^*(G) \otimes_{E_H} C^*(G),$$

where $C^*(G)$ is viewed as a C^*-correspondence over itself.
We now perform a minimal KSGNS construction for the system $(C^*(G), E_H)$. Let

$$X := \ell^2(G/H) \otimes C^*(G), \qquad \Omega := \xi_H \otimes 1 \in X,$$

and define a representation

$$\tilde{\pi}_G : C^*(G) \longrightarrow \mathcal{L}(X)$$

by setting

$$\tilde{\pi}_G(u_g) := \lambda_{G/H}(g) \otimes u_g, \qquad g \in G,$$

where $\lambda_{G/H}$ denotes the quasi-regular representation of G on $\ell^2(G/H)$, given by

$$\lambda_{G/H}(g)(\xi_{xH}) = \xi_{gxH}, \qquad x \in G.$$

The family $\{\xi_{gH} : g \in G\}$ forms an orthonormal basis of $\ell^2(G/H)$.

A direct computation shows that

$$
\begin{aligned}
\langle \Omega, \tilde{\pi}_G(u_g)\Omega\rangle &= \langle \xi_H \otimes 1, \xi_{gH} \otimes u_g\rangle \\
&= 1_H(g)u_g \\
&= E_H(u_g)
\end{aligned}
\tag{3.42}
$$

and hence the triple $(X, \tilde{\pi}_G, \Omega)$ realizes the conditional expectation E_H. Moreover, the representation is minimal, since

$$\overline{\tilde{\pi}_G(C^*(G))\,\Omega\,C^*(G)} = \ell^2(G/H) \otimes C^*(G).$$

Although $\tilde{\pi}_G$ does not coincide with the left action arising from the correspondence $(C^*(G), X_G)$, the uniqueness of the KSGNS construction implies that there exists a unitary isomorphism

$$u \in \mathcal{U}(X, X_G)$$

such that

$$\tilde{\pi}_G(a) = u^*\,\pi_G(a)\,u, \qquad a \in C^*(G). \tag{3.43}$$

Analysis of Redundancy:

If we recall Definition 2.6.5 and Proposition 2.9.9, we can conclude that for a representation (π, S) for $(C^*(G), E_H)$ and the corresponding representation (π, π_{X_G}) for $(C^*(G), X_G)$ we say that if $(\pi(a), k)$ is a redundancy for the representation (π, S) iff $a \in J_{X_G}$ and $(\pi, \pi_{X_G})^{(1)}(\pi_G(a))$ where $J_{X_G} = J(X_G)\bigcap(\ker(\pi_G))^{\perp}$ and $\pi_G : A \to L(X_G)$ is the $*$-homomorphism giving the left action for $(C^*(G), X_G)$.

Proposition 3.3.10. *Assuming G is amenable and $[G : H] = \infty$ we claim that there exist no redundancy for any representation (π, S) for $(C^*(G), E_H)$ or in other words any representation (π, π_{X_G}) is a covariant representation for $(C^*(G), X_G)$.*

Proof. Suppose (π, π_{X_G}) is a representation for $(C^*(G), X_G)$, to prove it is a covariant representation we need to prove that for any $a \in J_{X_G}$ we have $(\pi, \pi_{X_G})^{(1)}(\pi_G(a)) = \pi(a)$.

$$\begin{aligned} \text{Now, } a \in J_{X_G} \Rightarrow \pi_G(a) \in \mathcal{K}(X_G) \Leftrightarrow \tilde{\pi}_G(a) \in \mathcal{K}(X) &= \mathcal{K}(\ell^2(G/H) \otimes C^*(G)) \\ &= \mathcal{K}(\ell^2(G/H)) \otimes C^*(G) \end{aligned} \tag{3.44}$$

We can recall that,

$$\tilde{\pi}_G(\sum_{s \in G} a_s u_s) = \sum_{s \in G} a_s \lambda_{G/H}(s) \otimes u_s \tag{3.45}$$

Now if $\tilde{\pi}_G(\sum_{s \in G} a_s u_s) \in \mathcal{K}(\ell^2(G/H)) \otimes C^*(G)$ then for $h \in G$ such that $a_h \neq 0$ in $\sum_{s \in G} a_s u_s$ and $\tau_h(\sum_s a_s u_s) = \tau((\sum_s a_s u_s) h^{-1})$ where τ_h is the canonical trace we can define on $C^*(G)$ due to G being amenable, we get

$$\begin{aligned} &\mathrm{id} \otimes \tau_h(\tilde{\pi}_G(\sum_{s \in G} a_s u_s)) \in \mathcal{K}(\ell^2(G/H)) \\ \Rightarrow &a_h \lambda_{G/H}(h) \in \mathcal{K}(\ell^2(G/H)) \end{aligned} \tag{3.46}$$

Now we arrive at a contradiction as $\lambda_{G/H}(h)$ is a unitary and since $[G : H] = \infty$ implies that $\lambda_{G/H}(h) \notin \mathcal{K}(\ell^2(G/H))$. Hence we have that $\tilde{\pi}_G(\sum_s a_s u_s) \notin \mathcal{K}(\ell^2(G/H) \otimes C^*(G))$. For a particular $h \in G$ we have $(id \otimes \tau_h)\pi_G(\sum_s a_s u_s) = a_h \lambda_{G/H}(u_h) = \tau(\sum_s a_s u_s)\lambda_{G/H}(u_h)$ for all $\sum_s a_s u_s \in \mathbb{C}[G]$. Thus, we have

$$(id \otimes \tau_h)\pi_G(a) = \tau_h(a)\lambda_{G/H}(u_h) \quad \forall a \in C^*(G) \tag{3.47}$$

now if $\tau_h(a) \neq 0$ we can safely conclude that $(id \otimes \tau_h)\pi_G(a) \notin \mathcal{K}(\ell^2(G/H))$ and arrive at our contradiction. Hence if for some $a \in C^*(G)$ we get $(id \otimes \tau_h)\pi_G(a) \in \mathcal{K}(\ell^2(G/H)) \Rightarrow \tau_h(a) = 0 \ \forall \ h \in G$ let $\pi_\lambda : C^*(G) \to B(\ell^2(G))$ be the extension of the left regular representation $\lambda : G \to \mathcal{U}(\ell^2(G))$. Now,

$$\begin{aligned}\tau_h(\sum_s a_s u_s) = a_h &= \langle (\sum_s a_s u_s)\xi_g, \xi_{hg}\rangle \\ &= \langle (\pi_\lambda(\sum_s a_s u_s)\xi_g, \xi_{hg}\rangle\end{aligned} \tag{3.48}$$

Thus we have, $\tau_h(a) = \langle(\pi_\lambda(a)\xi_g, \xi_{hg}\rangle$

$$\begin{aligned}&\tau_h(a) = 0 \\ \Rightarrow &\langle(\pi_\lambda(a)\xi_g, \xi_{hg}\rangle = 0 \\ \Rightarrow &\ \pi_\lambda(a)\xi_g = 0 \text{ for } g \in G \\ \Rightarrow &\ \pi_\lambda(a) = 0\end{aligned} \tag{3.49}$$

Now, G is amenable $\Rightarrow$ π_λ is faithful, hence $a = 0$. Hence, we have that $\tilde{\pi}_G(a) \notin \mathcal{K}(\ell^2(G/H)) \otimes C^*(G) \Rightarrow \pi_G(a) \notin \mathcal{K}(X_G)$ which implies that any representation (π, π_{X_G}) of $(C^*(G), X_G)$ is a covariant representation. □

Spatial Representation for $(C^*(G), E_H)$

In this section we essentially construct a spatial representation for $(C^*(G), E_H)$, we have already shown that there does not exist an isometry $S \in \ell^2(G)$ such that (γ_λ, S) is a representation for $(C^*(G), E_H)$. Thus we have the following

Proposition 3.3.11. *For the Hilbert space,*

$$\mathcal{H} = \ell^2(G) \oplus \bigoplus_{n\geq 1} (\ell^2(G/H))^{\otimes n} \otimes \ell^2(G) \tag{3.50}$$

and $T \in B(\mathcal{H})$ defined by $T(\xi) = \delta_H \otimes \xi$ we can define a representation $\pi : C^*(G) \to B(\mathcal{H})$ is given by,

$$\pi(u_g) = \lambda(g) \oplus \bigoplus_{n\geq 1} \lambda_{G/H}^{\otimes n}(g) \otimes \lambda_(g) \tag{3.51}$$

such that (π, T) is a representation.

Proof. To prove that (π, T) is a representation of $(C^*(G), E_H)$ we have to show that,

$$T^*\pi(a)T = \pi(E_H(a)),\ a \in C^*(G) \tag{3.52}$$

for $\epsilon \in \ell^2(G)$ we have, $\pi(u_g)T\epsilon = \xi_{gH} \otimes \lambda(g)\epsilon$ and for $\eta \in \ell^2(G/H)^{\otimes n}$ we have,

$$\pi(u_g)T\eta \otimes \epsilon = \xi_{gH} \otimes \lambda^{\otimes n}_{G/H}(g)\eta \otimes \lambda(g)\epsilon \tag{3.53}$$

Now, for any $\zeta \in \mathcal{H}$

$$\begin{aligned}
\langle T^*\pi(u_g)T\eta \otimes \epsilon, \zeta\rangle &= \langle \pi(u_g)T\eta \otimes \epsilon, T\zeta\rangle \\
&= \langle \xi_{gH} \otimes \lambda^{\otimes n}_{G/H}(g)\eta \otimes \lambda(g)\epsilon, \xi_H \otimes \zeta\rangle \\
&= \langle 1_H(g)\lambda^{\otimes n}_{G/H}(g)\eta \otimes \lambda(g)\epsilon, \zeta\rangle \\
&= \langle \pi(E_H(g))\eta \otimes \epsilon, \zeta\rangle
\end{aligned} \tag{3.54}$$

From (3.49) we can say that $T^*\pi(g)T(h) = \pi(E_H(g))(h)$ for all $h \in \mathcal{H}$, thus proving that (π, T) is a representation for $(C^*(G), E_H)$. □

Let the C^*-algebra associated with (π, T) be $C^*(\pi, T)$ which actually gives us our spatial C^*-algebra construction. Now we try to construct a gauge action on the C^*-algebra $C^*(\pi, T)$. Let $\beta : \mathbb{T} \to Aut(C^*(\pi, T))$ be given by $\beta(z)(x) = U_z x U_{\bar{z}}$ for $x \in C^*(\pi, T)$ where $U_z|_{\ell^2(G)} = \text{id}$ and $U_z(\eta \otimes \epsilon) = z^n \eta \otimes \epsilon$ where $\eta \in \ell^2(G/H)^{\otimes n}$ and $\epsilon \in \ell^2(G)$. We first try to show that β gives us a gauge action of $\mathbb{T}$ on $C^*(\pi, T)$.

$$\begin{aligned}
/U_z\pi(u_g)U_{\bar{z}}(\eta \otimes \epsilon) &= U_z\pi(u_g)(\bar{z}^n\eta \otimes \epsilon) \\
&= \bar{z}^n U_z \lambda^{\otimes n}_{G/H}(g)\eta \otimes \lambda(g)\epsilon \\
&= |z|^n \pi(g)(\eta \otimes \epsilon) = \pi(g)(\eta \otimes \epsilon)
\end{aligned} \tag{3.55}$$

and

$$
\begin{aligned}
U_z T U_{\bar{z}}(\eta \otimes \epsilon) &= (\bar{z})^n U_z(\xi_H \otimes \eta \otimes \epsilon) \\
&= (\bar{z})^n z^{n+1} T(\eta \otimes \epsilon) \\
&= z T(\eta \otimes \epsilon)
\end{aligned}
\tag{3.56}
$$

From (3.56) and (3.57) it follows that β actually gives us a gauge action of $\mathbb{T}$ on $C^*(\pi, T)$.

Proposition 3.3.12. *If we assume that* $[G : H] = \infty$ *we have that,*

$$
C^*(C^*(G), E_H) \cong C^*(\pi, T) \tag{3.57}
$$

Proof. Since $[G : H] = \infty$ from Proposition 3.3.10 we can conclude that (π, S) gives us a covariant representation and then Using (3.56), (3.57) and Proposition 2.9.12 we can conclude that,

$$
C^*(C^*(G), E_H) \cong C^*(\pi, T) \tag{3.58}
$$

□

The next remark is not quite important for our discussion but still its quite interesting

Remark 3.3.13. *Let* G *be a connected LCA group with Haar measure* μ*. For every* $g \in G$ *let* $U_g \in B(L^2(G))$ *be given by*

$$
U_g(f)(x) = f(x - g)
$$

for all $x \in G$*. These are the shift operators. Note that the map* $g \to U_g$ *is a strongly continuous unitary representation of* G*. Let* H *be a proper subgroup of* G *which is not dense. Then :*

There does not exist $V \in B(L^2(G))$ such that $V^* U_g V = U_g$ for all $g \in H$ and $V^* U_g V = O$ for all $g \in G \setminus H$.

By strong continuity, we can assume WLOG that H *is closed. If* $\mu(H) > 0$*, there exists an open neighborhood* U *of* 0 *such that* $U \subset H - H \subset H$*, by Steinhaus' theorem. Hence, for every* $g \in H$*,*

$g + U \subset H$, implying that H is open. Since H is closed as well, connectedness implies $H = G$, a contradiction.

Hence, we must have $\mu(H) = 0$. Since any open set has positive Haar measure, this implies $G \setminus H$ is dense in G. By strong continuity again,

$$V^* U_g V = O$$

for all $g \in G$. This contradicts the fact that $V^* V = I$ since 0 is always in H.

Chapter 4

Operator Space Structure associated to the generators of $\mathcal{U}_n^{nc}$ and related C^*-algebras

This chapter is mostly based on the work [BMP25]. We make a detailed study of operator spaces generated by the generators of Brown's noncommutative unitary C^*-algebra $\mathcal{U}_n^{nc}$ and related C^*-algebras. In particular, we identify the universal C^*-algebra $C^*\langle M_n(\mathbb{C})^*\rangle$ associated with the operator space $M_n(\mathbb{C})^*$ with the non-commutative C^*-algebra $\mathcal{C}_n^{nc}$. This algebra $\mathcal{C}_n^{nc}$ is the universal unital C^*-algebra generated by elements u_{ij}, $1 \leq i, j \leq n$, subject to the condition that the matrix $[u_{ij}]$ is contractive. We further describe several operator algebraic features of $C^*\langle M_n(\mathbb{C})^*\rangle$: specifically, we analyze the Lifting Property (LP), residual finite dimensionality, and primitivity. In addition, we investigate the maximal and minimal operator space structures of the standard generators of $\mathcal{U}_n^{nc}$ as well as $\mathcal{U}_{n,red}^{nc}$. Finally, we explore a natural compact quantum semigroup structure on $C^*\langle M_n(\mathbb{C})^*\rangle$, demonstrating separation by finite-dimensional (co)-representations and characterizing invertible elements in its state space with respect to convolution.

4.1 Universal C*-algebras and Operator Spaces

Starting from an operator space, we can obtain a universal C^*-algebra as in the following

Remark 4.1.1. *Let E be an operator space. Then, $C^*\langle E\rangle$ is the universal C^* algebra generated by E such that there exists a complete isometric embedding $j : E \to C^*\langle E\rangle$ of E in $C^*\langle E\rangle$. This C^*-algebra has the following universal property.*

- *The algebra generated by $j(E)$ is dense in $C^*\langle E\rangle$.*

- *For any C^*-algebra $\mathbf{B}$ and any completely contractive map $u : E \to \mathbf{B}$, there exists a $*$-representation $\pi : C^*\langle E\rangle \to \mathbf{B}$ such that $\pi \circ j = u$.*

The density of $j(E)$ inside $C^*\langle E\rangle$ ensures the uniqueness of the representation π.

Existence of such an object is proved in [Pes94a].

The following is well known and can be found, for instance in [Pis20]. For the sake of completeness, we provide a proof here. Let us denote $\xi_{ij} \in M_n(\mathbb{C})^*$ as the biorthogonal basis of $M_n(\mathbb{C})^*$ with respect to the usual basis of (e_{ij}) of $M_n(\mathbb{C})$ i.e., $\xi_{ij}(e_{mk}) = \delta_{im}\delta_{jk},\ i, j, m, k \in \{1, 2, \cdots, n\}$.

Let $\mathcal{E}_n^{nc}$ be the operator space generated by the span of $\{u_{ij}\}$ where u_{ij}'s are the standard generating elements of $\mathcal{U}_n^{nc}$.

Theorem 4.1.2. *The operator space $\mathcal{E}_n^{nc}$ is completely isometric to the dual operator space of $M_n(\mathbb{C})$.*

Proof. For any C^*-algebra $\mathcal{A}$, we show $\|\sum \xi_{ij} \otimes a_{ij}\|_{C^*\langle M_n(\mathbb{C})^*\rangle \otimes_{\max} \mathcal{A}} = \|\sum u_{ij} \otimes a_{ij}\|_{\mathcal{U}_n^{nc} \otimes_{\max} \mathcal{A}}$, where $\sum_{i,j=1}^n \xi_{ij} \otimes a_{ij} \in C^*\langle M_n(\mathbb{C})^*\rangle \otimes \mathcal{A}$ and $a_{ij} \in \mathcal{A}$. Let S be the set of all $*$-homomorphisms $\pi : \mathcal{U}_n^{nc} \to \mathcal{D}$, where $\mathcal{D}$ is a C^*-algebra. Then using the universal property of $\mathcal{U}_n^{nc}$ along with the Theorem 2.3.3, there is a one-to-one correspondence between S and $\mathcal{U}(M_n(\mathcal{D}))$, the space of all unitary matrices in $M_n(\mathcal{D})$. Therefore, for any $x = (x_{ij}) \in M_n(\mathbf{B}(\mathcal{H}))$ and a direct application of Russo-Dye Theorem, one has

$$\begin{aligned} & \sup\left\{\left\|\sum \pi(u_{ij})x_{ij}\right\| : \pi : \mathcal{U}_n^{nc} \to \mathcal{D}\right\} \\ = \ & \sup\left\{\left\|\sum u_{ij}x_{ij}\right\| : u = (u_{ij}) \in \mathcal{U}(M_n(\mathcal{D}))\right\} \\ = \ & \sup\left\{\left\|\sum z_{ij}x_{ij}\right\| : z \in \mathbf{B}_{M_n(\mathcal{D})}\right\}. \end{aligned}$$

Let $\sigma : \mathcal{A} \to \mathbf{B}(\mathcal{H})$ be a $*$-homomorphism. Consider $\mathcal{D} = \sigma(\mathcal{A})^c$ and $x_{ij} = \sigma(a_{ij})$, we have

$$\sup\left\{\left\|\sum \pi(u_{ij})\sigma(a_{ij})\right\| : \pi : \mathcal{U}_n^{nc} \to \sigma(\mathcal{A})^c\right\} = \sup\left\{\left\|\sum z_{ij}\sigma(a_{ij})\right\| : z \in \mathbf{B}_{M_n(\sigma(\mathcal{A})^c)}\right\}.$$

Claim: $\|\sum \xi_{ij} \otimes a_{ij}\|_{C^*\langle M_n(\mathbb{C})^*\rangle \otimes_{\max} \mathcal{A}} = \sup\left\{\|\sum z_{ij}\sigma(a_{ij})\| : z \in \mathbf{B}_{M_n(\sigma(\mathcal{A})^c)},\ \sigma : \mathcal{A} \to \mathbf{B}(\mathcal{H})\right\}$. To see this, consider $z \in \mathbf{B}_{M_n(\sigma(\mathcal{A})^c)} \subset M_n(\mathbb{C}) \otimes \sigma(\mathcal{A})^c \cong \mathrm{CB}(M_n(\mathbb{C})^*, \sigma(\mathcal{A})^c)$. Let $\tilde{z} \in \mathrm{CB}(M_n(\mathbb{C})^*, \sigma(\mathcal{A})^c)$ be the map corresponding to an element $z \in \mathbf{B}_{M_n(\sigma(\mathcal{A})^c)}$. Note that $\|\tilde{z}\|_{\mathrm{cb}} = \|z\| \leq 1$. In particular, for $z = \sum e_{ij} \otimes z_{ij}$ we have $\tilde{z}(\xi) = \sum \xi(e_{ij})z_{ij}$, $\forall \xi \in M_n(\mathbb{C})^*$. Let $j : M_n(\mathbb{C})^* \to C^*\langle M_n(\mathbb{C})^*\rangle$ be the completely isometry of $M_n(\mathbb{C})^*$ inside $C^*\langle M_n(\mathbb{C})^*\rangle$. Since $\|\tilde{z}\|_{\mathrm{cb}} \leq 1$, using the universal property of $C^*\langle M_n(\mathbb{C})^*\rangle$, there exists a $*$- homomorphism $\pi_z : C^*\langle M_n(\mathbb{C})^*\rangle \to \sigma(\mathcal{A})^c$ such that $\pi_z(j(\xi)) = \tilde{z}(\xi)$. Since $\tilde{z}(\xi_{ij}) = z_{ij}$, $\pi_z(j(\xi_{ij})) = z_{ij}$. Therefore,

$$\begin{aligned} \left\|\sum \xi_{ij} \otimes a_{ij}\right\|_{C^*\langle\mathcal{E}\rangle \otimes_{\max} \mathcal{A}} \ & \geq \ \|\pi_z(\xi_{ij})\sigma(a_{ij})\| \\ & = \ \left\|\sum z_{ij}\sigma(a_{ij})\right\|. \end{aligned}$$

Thus, $\sup\left\{\|\sum z_{ij}\sigma(a_{ij})\| :\ z \in \mathbf{B}_{M_n(\sigma(\mathcal{A})^c)},\ \sigma : \mathcal{A} \to \mathbf{B}(\mathcal{H})\right\} \leq \|\sum \xi_{ij} \otimes a_{ij}\|_{C^*\langle M_n(\mathbb{C})^*\rangle \otimes_{\max} \mathcal{A}}$. Conversely, for all $\epsilon > 0$, there exists $*$-homomorphisms $\sigma_\circ : \mathcal{A} \to \mathbf{B}(\mathcal{K})$ and $\alpha : C^*\langle M_n(\mathbb{C})^*\rangle \to \mathbf{B}(\mathcal{K})$ s.t. $\alpha(C^*\langle M_n(\mathbb{C})^*\rangle) \subseteq \sigma_\circ(\mathcal{A})^c$ and we have the following inequality $\|\sum \alpha(\xi_{ij})\sigma_\circ(a_{ij})\| > \|\sum \xi_{ij} \otimes a_{ij}\|_{C^*\langle M_n(\mathbb{C})^*\rangle \otimes_{\max} \mathcal{A}} - \epsilon$. Clearly, $\alpha \circ j : \mathcal{E} \to \alpha(C^*\langle M_n(\mathbb{C})^*\rangle) \subseteq \sigma_\circ(\mathcal{A})^c$ and $\alpha \circ j$ is completely contractive with $\alpha \circ j \in \mathrm{CB}(M_n(\mathbb{C})^*, \sigma_\circ(\mathcal{A})^c) = M_n(\mathbb{C}) \otimes \sigma_\circ(\mathcal{A})^c$. Therefore, there exists an element $t \in M_n(\mathbb{C}) \otimes \sigma_\circ(\mathcal{A})^c$ such that $\tilde{t} = \alpha \circ j \in$

$\mathrm{CB}(M_n(\mathbb{C})^*, \sigma_\circ(\mathcal{A})^c)$. Hence, $t = \sum e_{ij} \otimes \alpha(\xi_{ij}) \in M_n(\mathbb{C}) \otimes \sigma_\circ(\mathcal{A})^c$ and $t_{ij} = \alpha(\xi_{ij})$ with

$$\left\|\sum t_{ij}\sigma_\circ(a_{ij})\right\| \;=\; \left\|\sum \alpha(\xi_{ij})\sigma_\circ(a_{ij})\right\| > \left\|\sum \xi_{ij} \otimes_{\max} a_{ij}\right\| - \epsilon.$$

Therefore, we have our equality. Now, consider the map $\varphi : M_n(\mathbb{C})^* \to \mathcal{E}_n^{nc}$ by $\xi_{ij} \mapsto u_{ij}$ and use the above inequality considering the C^*-algebra $\mathcal{A}$ to be $M_m(\mathbb{C})$, we can conclude that the map is a complete isometry. Hence, we are done with the proof. $\square$

Now we propose the next lemma which will be very important to our discussion.

Lemma 4.1.3. *Let $\mathcal{A}$ be a C^*-algebra, generated by $\{x_{ij} : 1 \leq i,j \leq n,\ \|[x_{ij}]\| \leq 1\}$. Then there always exist a surjective $*$-homomorphism*

$$\varphi : C^*\langle M_n(\mathbb{C})^*\rangle \to \mathcal{A},\ \xi_{ij} \mapsto x_{ij}$$

and a unital completely positive map

$$\Psi : \mathcal{U}_n^{nc} \to \mathcal{A},\ u_{ij} \mapsto x_{ij}.$$

Proof. Since $A = [x_{ij}]$ is a contractive matrix in $M_n(\mathcal{A})$, consider the unitary dilation of $[x_{ij}]$

$$\hat{A} = \begin{bmatrix} A & (I - AA^*)^{1/2} \\ (I - A^*A)^{1/2} & -A^* \end{bmatrix} \in M_{2n}(\mathcal{A}).$$

Let us denote $(1 - AA^*)^{1/2} = [a_{ij}]$, $(1 - A^*A)^{1/2} = [b_{ij}]$. Now rewrite the unitary matrix

$$\hat{A} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} & a_{11} & a_{12} & \dots & a_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} & a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} & a_{n1} & a_{n2} & \dots & a_{nn} \\ b_{11} & b_{12} & \dots & b_{1n} & -x_{11}^* & -x_{21}^* & \dots & -x_{n1}^* \\ b_{21} & b_{22} & \dots & b_{2n} & -x_{12}^* & -x_{22}^* & \dots & -x_{n2}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} & -x_{1n}^* & -x_{2n}^* & \dots & -x_{nn}^* \end{bmatrix}.$$

Now consider the matrix

$$
U = \left[\begin{array}{ccccc|ccccc|c|ccccccc}
x_{11} & 0 & 0 & \dots & 0 & x_{12} & 0 & 0 & \dots & 0 & \dots & x_{1n} & 0 & \dots & 0 & a_{11} & \dots & a_{1n} \\
0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\hline
x_{21} & 0 & 0 & \dots & 0 & x_{22} & 0 & 0 & \dots & 0 & \dots & x_{2n} & 0 & \dots & 0 & a_{21} & \dots & a_{2n} \\
0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
x_{n1} & 0 & 0 & \dots & 0 & x_{n2} & 0 & 0 & \dots & 0 & \dots & x_{nn} & 0 & \dots & 0 & a_{n1} & \dots & a_{nn} \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\
b_{11} & 0 & 0 & \dots & 0 & b_{12} & 0 & 0 & \dots & 0 & \dots & b_{1n} & 0 & \dots & 0 & -x_{11}^* & \dots & -x_{n1}^* \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
b_{n1} & 0 & 0 & \dots & 0 & b_{n2} & 0 & 0 & \dots & 0 & \dots & b_{nn} & 0 & \dots & 0 & -x_{1n}^* & \dots & -x_{nn}^*
\end{array}\right].
$$

Here every block of the matrix U has dimension $2n \times 2n$. It is straightforward to see that U is a unitary matrix. Therefore, by the universal property of $\mathcal{U}_n^{nc}$, there exists a $*$-homomorphism

$$\rho : \mathcal{U}_n^{nc} \longrightarrow M_{2n}(\mathcal{A})$$

such that

$$\rho(u_{ij}) = \begin{bmatrix} x_{ij} & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{bmatrix}_{2n\times 2n}.$$

Consider the UCP. map

$$\Psi : \mathcal{U}_n^{nc} \longrightarrow \mathcal{A}, \text{ defined by } \Psi(x) = (\rho(x))_{11},$$

where $(\rho(x))_{11}$ denotes the $(1,1)$ corner of the matrix and $x \in \mathcal{U}_n^{nc}$. Let

$$\pi : M_n(\mathbb{C})^* \longrightarrow \mathcal{U}_n^{nc}$$

be the natural completely contractive map. Then $\varphi = \Psi \circ \pi : M_n(\mathbb{C})^* \to \mathcal{A}$ is a complete contraction. Hence using the universal property of $C^*\langle M_n(\mathbb{C})^*\rangle$, we have a surjective $*$-homomorphism

$$\varphi : C^*\langle M_n(\mathbb{C})^*\rangle \to \mathcal{A},\ \xi_{ij} \mapsto x_{ij}.$$

□

4.1.1 Comparison between $\mathcal{U}_n^{nc}$ and $C^*\langle M_n(\mathbb{C})^*\rangle$

A natural question now is whether $\mathcal{U}_n^{nc}$ is in fact also the universal C^*-algebra associated to the dual of $M_n(\mathbb{C})$. To answer this in the negative, we will now study the abelianization of $C^*\langle M_n(\mathbb{C})^*\rangle$ and $\mathcal{U}_n^{nc}$. The prove for the following theorems can be found in [Man23a] and [BMP25]

Theorem 4.1.4. *Let $\mathbf{B}_{M_n(\mathbb{C})}$ be the closed unit ball in $M_n(\mathbb{C})$. Then, the abelianization of $C^*\langle M_n(\mathbb{C})^*\rangle$ is the C^*-algebra $C(\mathbf{B}_{M_n(\mathbb{C})})$, the continuous functions on $\mathbf{B}_{M_n(\mathbb{C})}$.*

The following is easy and was noted by Brown [Bro81].

Theorem 4.1.5. *The abelianization of $\mathcal{U}_n^{nc}$ is the C^*-algebra $C(\mathcal{U}(n))$, the space of all continuous functions on $\mathcal{U}(n)$, where $\mathcal{U}(n)$ is the space of all complex $n \times n$ unitary matrices.*

Remark 4.1.6. *Indeed, this shows that the universal C^*-algebra $\mathcal{U}_n^{nc}$ is not $*$-isomorphic to the universal C^*-algebra $C^*\langle M_n(\mathbb{C})^*\rangle$. So, $\mathcal{U}_n^{nc}$ is only a quotient C^*-algebra of $C^*\langle M_n(\mathbb{C})^*\rangle$.*

4.1.2 $\mathcal{C}_n^{nc}$ and its properties

We introduce the following C*-algebra, which will a central role for us.

Definition 4.1.7. *Let $\mathcal{C}_n^{nc}$ be the universal unital C^*-algebra generated by n^2 elements u_{ij}, $1 \leq i, j \leq n$ such that the matrix $[u_{ij}]$ is a contraction in $M_n(\mathcal{C}_n^{nc})$. This C^*-algebra has the following universal property:*

- *For any C^*-algebra $\mathcal{A}$ with generators v_{ij}, $1 \leq i, j \leq n$ such that the matrix $[v_{ij}]$ is a contractive matrix in $M_n(\mathcal{A})$. Then there exists a unique surjective $*$-homomorphism $\pi : \mathcal{C}_n^{nc} \to \mathcal{A}$ such that $\pi(u_{ij}) = v_{ij}$ for all $1 \leq i, j \leq n$.*

Let $\mathcal{F}$ be the free $*$-algebra generated by $\{u_{ij}\}_{1 \leq i,j \leq n}$. Consider the class $\mathcal{R}$ of all $*$-representations $\pi : \mathcal{F} \to B(H_\pi)$ such that $\left\|[\pi(u_{ij})]\right\| \leq 1$ in $M_n(B(H_\pi))$.
If $u = [u_{ij}]$, then for every $\pi \in \mathcal{R}$, $\|\pi(u_{ij})\| \leq \|[\pi(u_{ij})]\| \leq 1$. Hence each generator is uniformly bounded by 1 in every admissible representation. It follows that for any monomial w in the generators and their adjoints, $\|\pi(w)\| \leq 1$. Consequently, for any element

$$a = \sum_{\alpha=1}^{N} c_\alpha w_\alpha \in \mathcal{F}, \text{ where } w_\alpha \text{ are monomials}$$

we have

$$\|\pi(a)\| \leq \sum_{\alpha=1}^{N} |c_\alpha|.$$

Thus

$$\|a\|_{\max} := \sup_{\pi \in \mathcal{R}} \|\pi(a)\| < \infty.$$

The function $\|\cdot\|_{\max}$ is therefore a C^*-seminorm on $\mathcal{F}$. Let

$$\mathcal{N} = \{a \in \mathcal{F} : \|a\|_{\max} = 0\}.$$

Then the completion of $\mathcal{F}/\mathcal{N}$ with respect to $\|\cdot\|_{\max}$ is a C^*-algebra satisfying the required universal property.

Proposition 4.1.8. *The universal C^*-algebra $C^*\langle M_n(\mathbb{C})^*\rangle$ is $*$-isomorphic to the universal C^*-algebra $\mathcal{C}_n^{nc}$.*

Proof. Let us recall that $\xi_{ij} \in M_n(\mathbb{C})^*$ denotes the biorthogonal basis of $M_n(\mathbb{C})^*$ with respect to the standard basis (e_{ij}) of $M_n(\mathbb{C})$, that is, $\xi_{ij}(e_{mk}) = \delta_{im}\delta_{jk},\ i,j,m,k \in \{1,2,\ldots,n\}$. Note that $[\xi_{ij}]$ is a contractive matrix as a consequence of Theorem 4.1.2, taking $A = \mathbb{C}$. Hence, by the universal property of $\mathcal{C}_n^{\mathrm{nc}}$, there exists a surjective $*$-homomorphism.

$$\pi : \mathcal{C}_n^{\mathrm{nc}} \to C^*\langle M_n(\mathbb{C})^*\rangle$$

such that

$$\pi(u_{ij}) = \xi_{ij}, \quad 1 \le i,j \le n.$$

Now we claim that $\pi : \mathcal{C}_n^{nc} \to C^*\langle M_n(\mathbb{C})^*\rangle$ is an isomorphism. Let us consider a faithful $*$-homomorphism $\Psi : \mathcal{C}_n^{nc} \to \mathbf{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. Using the Lemma 4.1.3, we have the following diagram commutes

$$\begin{tikzcd} \mathcal{C}_n^{nc} \arrow[r, "\pi"] \arrow[dr, "\Psi"'] & C^*\langle M_n(\mathbb{C})^*\rangle \arrow[d, "\varphi"] \\ & \mathbf{B}(\mathcal{H}). \end{tikzcd}$$

Therefore, $C^*\langle M_n(\mathbb{C})^*\rangle$ is $*$-isomorphic to the universal C^*-algebra $\mathcal{C}_n^{nc}$. $\square$

Next, we demonstrate several structural properties of $\mathcal{C}_n^{nc}$. First, we show that $\mathcal{C}_n^{nc}$ has the Lifting property. Before proceeding further, we recall the following definitions. More details can be found in [BO08].

Definition 4.1.9. *Let $\mathcal{E}$ be an operator space. For $\lambda > 0$, we say $\mathcal{E}$ has λ-OLP if given any ideal $\mathcal{J}$ of any C^*-algebra $\mathbf{B}$ and complete contraction $\varphi : \mathcal{E} \to \frac{\mathbf{B}}{\mathcal{J}}$, there exists a map $\psi : \mathcal{E} \to \mathbf{B}$ with $\|\psi\|_{cb} \leq \lambda$ such that $\pi \circ \psi = \varphi$, where $\pi : \mathbf{B} \to \frac{\mathbf{B}}{\mathcal{J}}$ is the canonical quotient homomorphism.*

Definition 4.1.10. *Suppose $\mathcal{A}$ and $\mathbf{B}$ are C^*-algebras, and let $\mathcal{J}$ be a closed two-sided ideal in $\mathbf{B}$. A completely contractive positive (CCP.) map $\varphi : \mathcal{A} \to \mathbf{B}/\mathcal{J}$ is said to be* liftable *provided there exists a CCP. map $\psi : \mathcal{A} \to \mathbf{B}$ with $\pi \circ \psi = \varphi$. We say that a unital C^*-algebra $\mathcal{A}$ has the* lifting property *if, for every C^*-algebra $\mathbf{B}$ and every closed ideal $\mathcal{J} \subseteq \mathbf{B}$, any CCP. map $\varphi : \mathcal{A} \to \mathbf{B}/\mathcal{J}$ admits such a lift.*

Theorem 4.1.11. *$\mathcal{C}_n^{nc}$ has the Lifting property.*

Proof. Observe that $C^*\langle M_n(\mathbb{C})^*\rangle$ is a separable C^*-algebra. Consequently, by combining Proposition 4.1.8 with [Oza01, Proposition 2.2], it suffices to verify that $M_n(\mathbb{C})^*$ possesses the 1-OLP. Let $\theta : M_n(\mathbb{C})^* \to \mathbf{B}/\mathcal{J}$ be a complete contraction, where $\mathbf{B}$ is a C^*-algebra and $\mathcal{J}$ is a closed two-sided ideal in $\mathbf{B}$. Denote by

$$\pi : \mathbf{B} \to \mathbf{B}/\mathcal{J}$$

the canonical quotient map. The elements $(\theta(\xi_{ij}))$ then form a contractive matrix in $M_n(\mathbf{B}/\mathcal{J})$. Choose a contractive matrix $A = [x_{ij}]$ in $M_n(\mathbf{B})$ such that $\pi(x_{ij}) = \theta(u_{ij})$. By Lemma 4.1.3, one obtains a unital completely positive map

$$\Psi : \mathcal{U}_n^{nc} \to \mathbf{B}, \qquad u_{ij} \mapsto x_{ij}.$$

Furthermore, there is a natural complete contraction $\iota : M_n(\mathbb{C})^* \to \mathcal{U}_n^{nc}$, defined by $\xi_{ij} \mapsto \xi_{ij}$. The composition of ι with Ψ provides the required lifting. Therefore, $\mathcal{C}_n^{nc}$ satisfies the Lifting Property. $\square$

Its important to note that in lifting property has been "localized" to define the local lifting property for C^*-algebras. The local lifting property (LLP) was introduced by Kirchberg in [Kir93] in the study of tensor products and extension theory of C^*-algebras. In this work, Kirchberg formulated lifting conditions for completely positive maps from quotient C^*-algebras, requiring liftings only on finite-dimensional operator subsystems. This notion plays a fundamental role in the theory of tensor products and is closely related to the Connes–Kirchberg conjecture.

Definition 4.1.12. *[BO08]For each finite-dimensional operator system $\mathcal{E} \subseteq \mathcal{A}$, consider a CCP. map $\varphi : \mathcal{A} \to \mathbf{B}/\mathcal{J}$, where $\mathcal{A}$ and $\mathbf{B}$ are C^*-algebras and $\mathcal{J}$ is a closed two-sided ideal of $\mathbf{B}$. We say that φ is* locally liftable *if there exists a CCP. map $\psi : \mathcal{A} \to \mathbf{B}$ such that $\pi \circ \psi = \varphi|_{\mathcal{E}}$, with $\pi : \mathbf{B} \to \mathbf{B}/\mathcal{J}$ the canonical quotient map. A unital C^*-algebra $\mathcal{A}$ is said to possess the* local lifting property *whenever every such φ is locally liftable, regardless of the choice of $\mathbf{B}$ and $\mathcal{J}$.*

It follows directly from the definition that the lifting property for a C^*-algebra implies the local lifting property. In view of the preceding theorem, we therefore deduce that $\mathcal{C}_n^{nc}$ possesses the local lifting property. As a consequence, for any Hilbert space $\mathcal{H}$, one obtains the equality

$$\mathcal{C}_n^{nc} \otimes_{\max} \mathbf{B}(\mathcal{H}) \;=\; \mathcal{C}_n^{nc} \otimes \mathbf{B}(\mathcal{H}).$$

Now we study the residually finite-dimensional (RFD) property of $\mathcal{C}_n^{nc}$. Recall that a unital C^*-algebra $\mathcal{A}$ is called a residually finite-dimensional (RFD) C^*-algebra if $\mathcal{A}$ admits a separating family of finite-dimensional representations. Note that all abelian C^*-algebras are RFD C^*-algebra as characters of an abelian C^*-algebra separate the points of $\mathcal{A}$.

In the following theorem, we study the RFD property of the C*-algebras $\mathcal{C}_n^{nc}$, adpating a technique of [Cho80] to establish the result.

Theorem 4.1.13. *$\mathcal{C}_n^{nc}$ is a* RFD *C^*-algebra.*

Proof. The algebra $\mathcal{C}_n^{nc}$ is separable, and therefore we may realize it as a $*$-subalgebra of $\mathbf{B}(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$. Consider an increasing sequence of projections $\{P_k\} \subseteq \mathbf{B}(\mathcal{H})$ with $\operatorname{rank}(P_k) = k$ and $P_k \to I$ strongly, where I is the identity operator.

With this setup, the matrix $u = (u_{ij})$ is realised as a contractive element of $M_n(\mathbf{B}(\mathcal{H}))$. Now, consider the associated contractive matrix.

$$A_k = \begin{bmatrix} P_k & & \\ & \ddots & \\ & & P_k \end{bmatrix} u \begin{bmatrix} P_k & & \\ & \ddots & \\ & & P_k \end{bmatrix} = [d_{ij}^k].$$

By the universality of $\mathcal{C}_n^{nc}$, we have a $*$-homomorphism from $\pi_k : \mathcal{C}_n^{nc} \to M_k(\mathbb{C})$ by $\pi_k(u_{ij}) = d_{ij}^k$. We now claim that the collection of representations $\{\pi_k\}_{k=1}^{\infty}$ forms a separating family. In other words, the corresponding $*$-homomorphism

$$\pi : \mathcal{C}_n^{nc} \longrightarrow \bigoplus_{k=1}^{\infty} M_k(\mathbb{C})$$

is injective.

Note that

$$\begin{bmatrix} P_k u_{11} P_k & P_k u_{12} P_k & \dots & P_k u_{1n} P_k \\ P_k u_{21} P_k & P_k u_{22} P_k & \dots & P_k u_{2n} P_k \\ \vdots & \vdots & \ddots & \vdots \\ P_k u_{n1} P_k & P_k u_{n2} P_k & \dots & P_k u_{nn} P_k \end{bmatrix} \to \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{bmatrix} \text{ in S.O.T.}$$

Hence, d_{ij}^k converges to u_{ij} in S.O.T. Thus, $F(d_{11}^k, d_{12}^k, \cdots, d_{nn}^k) \to F(u_{11}, u_{12}, \cdots, u_{nn})$ in S.O.T., where $F(\cdot,\cdot)$ is a $*$-polynomial in n^2 non-commutating variables. Therefore, for any $\epsilon > 0$ and given $\|F(u_{11}, u_{12}, \cdots, u_{nn})\| = 1$ we have $\|F(d_{11}^k, d_{12}^k, \cdots, d_{nn}^k)\| \geq 1 - \epsilon$ for large enough k. Now we have

$$\begin{aligned} \|\pi(F(u_{11}, u_{12}, \dots, u_{nn}))\| &\geq \|\pi_k(F(u_{11}, u_{12}, \cdots, u_{nn}))\| \\ &= \|F(d_{11}^k, d_{12}^k, \cdots, d_{nn}^k)\| \\ &\geq 1 - \epsilon. \end{aligned}$$

Hence the result is proved. $\square$

Remark 4.1.14. *Note that this can also be proved using [Pes94b, Theorem 4.1].*

The following corollaries are direct applications of the above theorem.

Corollary 4.1.15. *There exists a faithful tracial state for the universal C^*-algebra $\mathcal{C}_n^{nc}$.*

Proof. The embedding of $\mathcal{C}_n^{nc}$ into $\bigoplus_k M_k(\mathbb{C})$ as a C^*-subalgebra gives us our desired result. Denote by τ_k the normalized trace on $M_k(\mathbb{C})$. We then define a tracial functional

$$\tau : \bigoplus_{k=1}^{\infty} M_k(\mathbb{C}) \longrightarrow \mathbb{C}, \qquad \tau\left(\bigoplus_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \frac{\tau_k(A_k)}{2^k}.$$

It follows immediately from construction that τ is a faithful tracial state on $\bigoplus_{k=1}^{\infty} M_k(\mathbb{C})$, as required.

$\square$

Definition 4.1.16. *An operator A in a C^*-algebra satisfying $A^*A \geq AA^*$ is called hyponormal.*

Note that, normal operators are hyponormal by definition but we will see that for $\mathcal{C}_n^{nc}$ we have the following rigidity property.

Corollary 4.1.17. *In $\mathcal{C}_n^{nc}$ every hyponormal operator is normal.*

Proof. As $\mathcal{C}_n^{nc}$ is RFD C^*-algebra, we have an embedding $\mathcal{C}_n^{nc}$ into $\bigoplus_k M_k(\mathbb{C})$ as a C^*-subalgebra. Let $A = \bigoplus A_k \in \bigoplus_k M_k(\mathbb{C})$ be a hyponormal operator on $\mathcal{C}_n^{nc}$. Therefore, for each k, A_k is hyponormal matrix. It is well-known that every hyponormal matrix is normal matrix. Therefore, A_k is normal matrix for each k. Hence, A is normal operator. $\square$

To study the primitivity of $\mathcal{C}_n^{nc}$, we will need the following results. First, we show that the C^*-algebra $\mathcal{C}_n^{nc}$ has no non-trivial projections, generalizing Choi [Cho80] for the free group case.

Theorem 4.1.18. *The universal C^*-algebra $\mathcal{C}_n^{nc}$ has no non-trivial projection.*

Proof. We may assume $\mathcal{C}_n^{nc} \subset \mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Consider the C^*-algebra $\mathfrak{A} = \{\varphi : [0,1] \to \mathbf{B}(\mathcal{H}) : \ \varphi \text{ is norm continuous and } \varphi(0) \in \mathbb{C}1\}$. Note that there is an isometric embedding of $\mathcal{C}_n^{nc}$ into $\mathfrak{A}$. Indeed, consider the $*$-homomorphism $\Psi : \mathcal{C}_n^{nc} \to \mathfrak{A}$ defined by

$$\Psi(u_{ij})(t) = t.u_{ij},\ t \in [0,1],\ 1 \le i,j \le n.$$

Clearly, $[\Psi(u_{ij})]$ is contractive matrix in $M_n(\mathfrak{A})$. Also, it is easy to check that $\Psi : \mathcal{C}_n^{nc} \to \mathfrak{A}$ is injective. Now we claim that $\mathfrak{A}$ is projection-less. Suppose that there exists a projection φ in $\mathfrak{A}$. Then $\varphi(0)$ is a projection in $\mathbb{C}1$. Therefore, either $\varphi(0) = 0$ or $\varphi(0) = 1$. First consider $\varphi(0) = 0$. Then let $h : [0,1] \to \{0,1\}$ be defined by $h(t) = \|\varphi(t)\|$, $t \in [0,1]$. Since h is continuous, $h([0,1])$ is connected subset in $\{0,1\}$. Therefore, $\varphi(t) = 0$ for all $t \in [0,1]$. Thus, $\mathfrak{A}$ is projection-less. Therefore, $\mathcal{C}_n^{nc}$ has no non-trivial projection. □

Now, we establish that the C^*-algebra $\mathcal{C}_n^{nc}$ is primitive; which means that, it has a faithful, irreducible representation in $\mathbf{B}(\mathcal{H})$. To prove this, we first need the following lemmas.

Remark 4.1.19. *Since $\mathcal{C}_n^{\mathrm{nc}}$ admits a surjective $*$-homomorphism onto $\mathcal{U}_n^{\mathrm{nc}}$, and $\mathcal{U}_n^{\mathrm{nc}}$ in turn admits a surjective $*$-homomorphism onto $C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$, which is infinite-dimensional (since $\mathbb{T}^n$ is an infinite compact space), it follows that $\mathcal{C}_n^{\mathrm{nc}}$ is infinite-dimensional.*

Since, we have already shown that $\mathcal{C}_n^{\mathrm{nc}}$ is projectionless*, and a projectionless C^*-algebra cannot be finite-dimensional, this provides another way to see that $\mathcal{C}_n^{\mathrm{nc}}$ is infinite-dimensional.*

Lemma 4.1.20. *Suppose $\pi : \mathcal{C}_n^{nc} \to \mathbf{B}(\mathcal{H})$ is a faithful representation for some Hilbert space $\mathcal{H}$. Then, the image $\pi(\mathcal{C}_n^{nc})$ contains no non-zero compact operator.*

Proof. Assume, for contradiction, that there exists a non-zero compact operator $T \in \pi(\mathcal{C}_n^{nc})$. Then $T^*T \in \pi(\mathcal{C}_n^{nc})$ is a non-zero positive compact operator. By the spectral theorem for compact self-adjoint operators, there exists a non-zero eigenvalue $\lambda > 0$, and hence a corresponding non-zero finite-rank spectral projection P onto $\ker(T^*T - \lambda I)$ and since P is a finite rank projection and $\mathcal{C}_n^{nc}$ is infinite-dimensional (see Remark 4.1.19), we can further conclude $\pi(1) \neq P$.

Since $T^*T \in \pi(\mathcal{C}_n^{nc})$ and $\pi(\mathcal{C}_n^{nc})$ is a C^*-algebra, the projection P belongs to the C^*-algebra generated by T^*T, and hence $P \in \pi(\mathcal{C}_n^{nc})$. Because π is faithful, $\pi(\mathcal{C}_n^{nc})$ is $*$-isomorphic to $\mathcal{C}_n^{nc}$. Thus, the existence of a non-trivial projection $P \in \pi(\mathcal{C}_n^{nc})$ implies that $\mathcal{C}_n^{nc}$ contains a non-zero projection, contradicting the fact that $\mathcal{C}_n^{nc}$ is projectionless. Therefore, $\pi(\mathcal{C}_n^{nc})$ contains no non-zero compact operators. □

The next lemma shows that compact perturbations of the generators of $\mathcal{C}_n^{\mathrm{nc}}$, provided that the resulting matrix remains a contraction, yield the same universal C^*-algebra.

Lemma 4.1.21. *Let $\mathcal{H}$ be a Hilbert space such that $\mathcal{C}_n^{nc}$ is realized as a C^*-subalgebra of $B(\mathcal{H})$, and let $k_{ij} \in \mathcal{K}(\mathcal{H})$ for $1 \leq i, j \leq n$. We can define*

$$\mathcal{A} := C^*\Big(\{\, u_{ij} + k_{ij} : 1 \leq i, j \leq n,\ \|(u_{ij} + k_{ij})\| \leq 1\}\Big) \subset B(\mathcal{H}),$$

. Then $\mathcal{A}$ is isomorphic to $\mathcal{C}_n^{\mathrm{nc}}$ induced by the map $u_{ij} \mapsto u_{ij} + k_{ij}$.

Proof. Using universal property of $\mathcal{C}_n^{nc}$, there exists a surjective $*$-homomorphism

$$\Psi : \mathcal{C}_n^{nc} \to \mathcal{A},\ u_{ij} \mapsto u_{ij} + k_{ij}.$$

Consider the quotient $*$-homomorphism $\pi : \mathbf{B}(\mathcal{H}) \to \frac{\mathbf{B}(\mathcal{H})}{\mathcal{K}(\mathcal{H})}$. Let $\pi_{\mathcal{C}_n^{nc}}$ and $\pi_{\mathcal{A}}$ denote the restrictions of π to $\mathcal{C}_n^{nc}$ and $\mathcal{A}$ respectively. Note that $\pi_{\mathcal{C}_n^{nc}} = \pi_{\mathcal{A}} \circ \Psi$. Moreover, $\ker(\pi_{\mathcal{C}_n^{nc}}) = \mathcal{C}_n^{nc} \cap \mathcal{K}(\mathcal{H})$. Thus, using previous Lemma 4.1.20, $\ker(\pi_{\mathcal{C}_n^{nc}}) = \{0\}$ i.e., $\pi_{\mathcal{C}_n^{nc}} : \mathcal{C}_n^{nc} \to \frac{\mathbf{B}(\mathcal{H})}{\mathcal{K}(\mathcal{H})}$ is injective. Hence, $\Psi : \mathcal{C}_n^{nc} \to \mathcal{A}$ is $*$-isomorphism. □

In the following theorem, we establish the primitivity of $\mathcal{C}_n^{nc}$.

Theorem 4.1.22. *The universal C^*-algebra $\mathcal{C}_n^{nc}$ is primitive.*

Proof. To prove primitivity of $\mathcal{C}_n^{nc}$, it suffices to construct a faithful irreducible representation.

Since $\mathcal{C}_n^{nc}$ is separable and residually finite-dimensional, there exists a sequence of finite-dimensional representations

$$\pi_k : \mathcal{C}_n^{nc} \to M_{d_k}(\mathbb{C})$$

such that

$$\bigcap_{k=1}^{\infty} \ker(\pi_k) = \{0\}.$$

Equivalently, the direct sum representation

$$\pi := \bigoplus_{k=1}^{\infty} \pi_k$$

is faithful.

For each $k \geq 1$, define

$$V_k := [\pi_k(u_{ij})] \in M_n(M_{d_k}) \cong M_{nd_k}.$$

Since $[u_{ij}]$ is a contraction in $M_n(\mathcal{C}_n^{nc})$, each V_k is a contraction, $\|V_k\| \leq 1$.

Choose a sequence (r_k) such that

$$0 < r_k < 1, \qquad r_k \uparrow 1.$$

Define

$$T_k := r_k V_k$$

and

$$T := \bigoplus_{k=1}^{\infty} T_k.$$

Then

$$\|T\| = \sup_k \|T_k\| = \sup_k r_k \|V_k\| \leq \sup_k r_k \leq 1.$$

Hence T is a contraction but at the same time every T_k are strict contraction.

Now observe that

$$\pi([u_{ij}]) - T = \bigoplus_{k=1}^{\infty} (1 - r_k) V_k.$$

Since

$$\|(1 - r_k) V_k\| \leq 1 - r_k \to 0,$$

the operator

$$\pi([u_{ij}]) - T$$

is compact. Indeed, a block diagonal operator is compact whenever the norms of its blocks converge to zero and the blocks are finite dimensional.

Therefore each entry of T differs from the corresponding entry of $\pi([u_{ij}])$ by a compact operator. Hence

$$T_{ij} = \pi(u_{ij}) + k_{ij}, \qquad k_{ij} \in \mathcal{K}(H).$$

By Lemma 4.1.21,

$$C^*(T_{ij}, 1 \leq i, j \leq n) \cong \mathcal{C}_n^{nc}.$$

We now modify T in order to obtain an irreducible representation.

Since each block T_k acts on a finite-dimensional Hilbert space, we may unitarily conjugate each block so that $\mathrm{Re}(T_k)$ is diagonal. Since unitary conjugation preserves norms and block structure, the resulting operator remains a strict contraction and is still block diagonal.

For each $k \geq 1$, we have

$$\|T_k\| < 1.$$

Since T_k acts on a finite-dimensional Hilbert space,

$$\mathrm{Re}(T_k) = \frac{1}{2}(T_k + T_k^*)$$

is self-adjoint. Hence there exists a unitary U_k such that $U_k \,\mathrm{Re}(T_k) U_k^*$ is diagonal.

Replacing T_k by $U_k T_k U_k^*$, we may therefore assume that $\mathrm{Re}(T_k)$ is diagonal for every k. Since unitary conjugation preserves norms,

$$\|U_k T_k U_k^*\| = \|T_k\| < 1,$$

so each block remains a strict contraction.

Now perturb the diagonal entries of $\mathrm{Re}(T_k)$ by arbitrarily small real numbers so that:

1. the diagonal entries within each block are pairwise distinct;

2. the diagonal entries belonging to different blocks are also distinct.

Since $\|T_k\| < 1$, there exists $\varepsilon_k > 0$ such that every perturbation of T_k having norm less than ε_k is still a strict contraction. Choose the diagonal perturbation in the k-th block to have norm smaller than $\varepsilon_k/2$. Denote the resulting perturbed block by T_k'. Then

$$\|T_k'\| < 1$$

for every k.

Moreover, we choose the perturbation sizes tending to zero as $k \to \infty$. Therefore

$$T' - T = \bigoplus_{k=1}^{\infty}(T_k' - T_k)$$

is compact, since the norms of the blocks converge to zero.

Define

$$T' = \bigoplus_{k=1}^{\infty} T_k'.$$

Then

$$\|T'\| = \sup_k \|T_k'\| \leq 1,$$

so T' is a contraction.

By construction, $\mathrm{Re}(T')$ is diagonal with pairwise distinct diagonal entries.

We now construct a compact perturbation of T' which connects the different blocks.

Recall that

$$H = \bigoplus_{k=1}^{\infty} H_k, \qquad H_k \cong \mathbb{C}^{nd_k},$$

and

$$T' = \bigoplus_{k=1}^{\infty} T_k'$$

is block diagonal with respect to this decomposition.

Choose the standard basis in each block H_k, and denote it by

$$e_{k,1}, e_{k,2}, \ldots, e_{k,nd_k}.$$

In particular, $e_{1,1}$ denotes the first basis vector of the first block H_1.

Since T' is block diagonal, all off-diagonal blocks are zero. In particular,

$$\langle T' e_{1,1}, e_{k,j} \rangle = \langle T' e_{k,j}, e_{1,1} \rangle = 0$$

for every $k \geq 2$ and every $1 \leq j \leq nd_k$.

Choose nonzero real numbers (λ_k) such that $\lambda_k \to 0$. Define

$$R = \sum_{k=2}^{\infty} \sum_{j=1}^{nd_k} i\lambda_k \left(e_{k,j} \otimes e_{1,1}^* + e_{1,1} \otimes e_{k,j}^* \right).$$

Recall that for vectors $x, y \in H$, the rank-one operator $x \otimes y^*$ is defined by

$$(x \otimes y^*)(v) = \langle v, y \rangle x.$$

Thus $e_{k,j} \otimes e_{1,1}^*$ maps the vector $e_{1,1}$ to $e_{k,j}$, while $e_{1,1} \otimes e_{k,j}^*$ maps $e_{k,j}$ to $e_{1,1}$. Therefore R creates two-way interaction between the first block and every other block.

Define

$$T'' := T' + R.$$

Since

$$H = \bigoplus_{k=1}^{\infty} \mathbb{C}^{nd_k},$$

the operator T' has block matrix form

$$T' = \begin{pmatrix} T_1' & 0 & 0 & 0 & \cdots \\ 0 & T_2' & 0 & 0 & \cdots \\ 0 & 0 & T_3' & 0 & \cdots \\ 0 & 0 & 0 & T_4' & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each $T_k' \in M_{nd_k}$.

The perturbation R changes only the first row and first column of blocks. More precisely,

$$T'' = \begin{pmatrix} T_1' & R_{12} & R_{13} & R_{14} & \cdots \\ R_{21} & T_2' & 0 & 0 & \cdots \\ R_{31} & 0 & T_3' & 0 & \cdots \\ R_{41} & 0 & 0 & T_4' & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where for each $k \geq 2$,

$$R_{1k} = \begin{pmatrix} i\lambda_k & i\lambda_k & \cdots & i\lambda_k \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{nd_1 \times nd_k},$$

and $R_{k1} = R_{1k}^*$.

Hence vectors from different blocks now interact under T'', and the original block decomposition is no longer reducing.

For each $N \geq 2$, define

$$R_N = \sum_{k=2}^{N} \sum_{j=1}^{nd_k} i\lambda_k \left(e_{k,j} \otimes e_{1,1}^* + e_{1,1} \otimes e_{k,j}^* \right).$$

Each R_N has finite rank. Moreover,

$$\|R - R_N\| \leq 2 \sup_{k>N} |\lambda_k|.$$

Since $\lambda_k \to 0$, we obtain

$$\|R - R_N\| \to 0.$$

Hence R is compact as a norm limit of finite-rank operators.

Since $\|T_k'\| < 1$ for every k, we may choose the sequence (λ_k) sufficiently small so that each block of T'' remains strictly contractive. Consequently,

$$\|T''\| \leq 1,$$

and therefore T'' is a contraction.

Finally, since R is compact and $T' - T$ is compact, it follows that

$$T'' - T$$

is compact.

Since R is compact and $T' - \pi([u_{ij}])$ is compact, it follows that

$$T'' - \pi([u_{ij}])$$

is compact. Therefore, by Lemma 4.1.21,

$$C^*(T_{ij}'' : 1 \leq i, j \leq n) \cong \mathcal{C}_n^{nc}.$$

We now show that the operators

$$\{T_{ij}'' : 1 \leq i, j \leq n\}$$

have no nontrivial common reducing subspace.

Suppose, $W \subseteq H$ is a common reducing subspace for all the operators T''_{ij}, $1 \le i, j \le n$. Let $P_W : H \to H$ denote the orthogonal projection onto W. Since W reduces each T''_{ij}, we have $T''_{ij} P_W = P_W T''_{ij}$ for all $1 \le i, j \le n$.

Therefore the block diagonal operator

$$P := \begin{bmatrix} P_W & & \\ & \ddots & \\ & & P_W \end{bmatrix}$$

commutes with the matrix T''.

Equivalently,

$$PT'' = T''P.$$

Hence P commutes with T'', and therefore also with T''^*. It follows that P commutes with the self-adjoint operators

$$\mathrm{Re}(T'') = \frac{T'' + T''^*}{2} \qquad \text{and} \qquad \mathrm{Im}(T'') = \frac{T'' - T''^*}{2i}.$$

To complete the argument, we use a result of Radjavi and Rosenthal [RR73]. They showed that if $A, B \in B(\mathcal{H})$ are represented with respect to a fixed orthonormal basis so that A is diagonal with distinct diagonal entries and every entry in the first column of B is nonzero, then A and B admit no common nontrivial reducing subspace. Since this fact is central to our argument, we include a detailed proof for the reader's convenience.

Since R is purely imaginary, $\mathrm{Re}(T'') = \mathrm{Re}(T')$. By construction, $\mathrm{Re}(T')$ is diagonal with distinct diagonal entries. Therefore any operator commuting with it must itself be diagonal in the same basis. Hence

$$P = \mathrm{diag}(\alpha_1, \alpha_2, \dots), \qquad \alpha_m \in \{0, 1\}.$$

Now P also commutes with $\mathrm{Im}(T'')$. By construction, the $(1, m)$-entry of $\mathrm{Im}(T'')$ is nonzero whenever the coordinate indexed by m corresponds to one of the vectors

$$e_{k,j}, \qquad k \geq 2.$$

Let, $A := \mathrm{Im}(T'')$. Since, $PT'' = T''P$, we also have $PA = AP$.

We already know that

$$P = \mathrm{diag}(\alpha_1, \alpha_2, \dots).$$

Now compare the $(1, m)$-entries of both sides.

Since P is diagonal,

$$(PA)_{1m} = \sum_k P_{1k} A_{km}.$$

The only nonzero term occurs when $k = 1$, so

$$(PA)_{1m} = P_{11} A_{1m} = \alpha_1 A_{1m}.$$

Similarly,

$$(AP)_{1m} = \sum_k A_{1k} P_{km}.$$

Again, since P is diagonal, the only nonzero term occurs when $k = m$. Hence

$$(AP)_{1m} = A_{1m} P_{mm} = A_{1m} \alpha_m.$$

Since

$$PA = AP,$$

we obtain

$$\alpha_1 A_{1m} = A_{1m} \alpha_m.$$

By construction,

$$A_{1m} \neq 0$$

whenever the coordinate indexed by m corresponds to one of the vectors

$$e_{k,j}, \qquad k \geq 2.$$

Therefore,

$$\alpha_1 = \alpha_m.$$

Since this holds for every such m, all diagonal entries of P are equal. Hence

$$P = 0 \qquad \text{or} \qquad P = I.$$

Looking at the $(1, m)$-entry of the relation

$$P \operatorname{Im}(T'') = \operatorname{Im}(T'')P,$$

we obtain

$$\alpha_1 = \alpha_m$$

for every such index m.

Hence all diagonal entries of P are equal. Therefore

$$P = 0 \qquad \text{or} \qquad P = I.$$

Thus there are no nontrivial reducing subspaces, so the representation generated by T'' is irreducible.

Since the representation is faithful, we conclude that $\mathcal{C}_n^{nc}$ is primitive. □

Corollary 4.1.23. *The universal C^*-algebra $\mathcal{C}_n^{nc}$ has trivial center.*

Proof. By primitiveness of $\mathcal{C}_n^{nc}$, it admits a faithful irreducible representation $\pi : \mathcal{C}_n^{nc} \to \mathcal{B}(H)$. It is then easy to see that by irreducibility and Schur's lemma, we have $\pi(\mathcal{C}_n^{nc})' = \mathbb{C}I$. Since $\pi(\mathcal{Z}(\mathcal{C}_n^{nc})) \subseteq \pi(\mathcal{C}_n^{nc})'$, it follows that $\pi(z) = \lambda I$ for all $z \in \mathcal{Z}(\mathcal{C}_n^{nc})$. The faithfulness of π then implies $z = \lambda 1$, and hence $\mathcal{Z}(\mathcal{C}_n^{nc}) = \mathbb{C}1$. $\square$

4.2 MAX Operator Space structure associated to the generators of $\mathcal{U}_n^{nc}$

Maximal operator space structure (Definition 2.1.10) of the generators of $C^*(\mathbb{F}_n)$ was studied in [Pau96] (see also [Zha95]), where it was proved that the maximal operator space MAX(ℓ_n^1) is isometrically embedded inside $C^*(\mathbb{F}_n)$. In this context, we observe the following.

Theorem 4.2.1. *Any contractive map $\varphi : \ell_n^1\{= (\ell_n^\infty)^*\} \to \mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ is completely contractive, here $(\ell_n^\infty)^*$ is the dual operator space of ℓ_n^∞.*

Proof. Let us denote $\{e_i : 1 \leq i \leq n\}$ the standard basis of ℓ_n^1. Now, consider $\varphi : \ell_n^1 \to \mathbf{B}(\mathcal{H})$ is a contractive map. Therefore, $\|\varphi(e_i)\| \leq 1$. Hence, using dilation theory and universal property of $C^*(\mathbb{F}_n)$, we have a UCP. map $\phi : C^*(\mathbb{F}_n) \to \mathbf{B}(\mathcal{H})$ such that $\phi(u_i) = \varphi(e_i)$ for all i. Also, note that ℓ_n^1 is completely isometrically embedded into $C^*(\mathbb{F}_n)$ with $e_i \mapsto u_i$. Therefore, $\varphi = \phi \circ \iota$, where $\iota : \ell_n^1 \to C^*(\mathbb{F}_n)$ is the completely isometric embedding. Thus, $\varphi : \ell_n^1 \to \mathbf{B}(\mathcal{H})$ is completely contractive. $\square$

In the following theorems, we study the maximal operator space structure of the standard generators of $\mathcal{U}_n^{nc}$.

Theorem 4.2.2. *Let $M_n(\mathbb{C})$ be the space of all $n \times n$ complex matrices. Then the following are equivalent.*

1. *The maximal operator space MAX$(M_n(\mathbb{C})^*)$ is completely isometrically embedded into $\mathcal{U}_n^{nc}$ with $\xi_{ij} \mapsto u_{ij}$.*

2. *Every contractive map $\varphi : M_n(\mathbb{C})^* \to \mathbf{B}(\mathcal{H})$ is completely contractive, where $\mathcal{H}$ is a Hilbert space and M_n^* is the operator space dual of M_n.*

Proof. (2) $\implies$ (1) Let $\rho : \text{MAX}(M_n(\mathbb{C})^*) \to \mathcal{U}_n^{nc}$ be the map defined by $\rho(\xi_{ij}) = u_{ij}$. Our claim is that ρ is an isometry. From the Theorem 4.1.2, it follows that $\|\sum \xi_{ij} \otimes a_{ij}\| = \|\sum u_{ij} \otimes a_{ij}\|$, where a_{ij} are elements of a C^*-algebra $\mathcal{A}$. If we take $\mathcal{A} = M_n(\mathbb{C})$, then the norm of the matrix $[\xi_{ij}]$ is 1. Now consider an element $\sum c_{ij}\xi_{ij}$ in $\text{MAX}(M_n(\mathbb{C})^*)$. Therefore,

$$\begin{aligned}\left\|\rho(\sum c_{ij}\xi_{ij})\right\| &= \left\|\sum c_{ij}u_{ij}\right\| \\ &= \left\|\sum c_{ij}\xi_{ij}\right\|.\end{aligned}$$

The last equality is due to Theorem 4.1.2 considering $\mathcal{A} = \mathbb{C}$. Hence $\|\rho(\sum c_{ij}\xi_{ij})\| \leq \|\sum c_{ij}\xi_{ij}\|_{\text{MAX}(M_n(\mathbb{C})^*)}$. For the other direction, we use the unitary dilation theory. Let $\varphi : M_n(\mathbb{C})^* \to \mathbf{B}(\mathcal{H})$ be a contractive map. Then $[\varphi(\xi_{ij})]$ is a contractive matrix. Hence there exists a UCP map $\pi : \mathcal{U}_n^{nc} \to C^*(\varphi(M_n(\mathbb{C})^*))$ such that $\pi(u_{ij}) = \varphi(\xi_{ij})$ i.e., the following diagram commutes

$$\begin{tikzcd} M_n(\mathbb{C})^* \arrow[r, "\rho"] \arrow[dr, "\varphi"'] & \mathcal{U}_n^{nc} \arrow[d, "\pi"] \\ & C^*(\varphi(M_n(\mathbb{C})^*)). \end{tikzcd}$$

Thus, $\|\varphi(\sum c_{ij}\xi_{ij})\| \leq \|\sum c_{ij}u_{ij}\|$ i.e., $\|\sum c_{ij}\xi_{ij}\|_{\text{MAX}(M_n(\mathbb{C})^*)} \leq \|\sum c_{ij}u_{ij}\|$. Hence we obtain $\rho : \text{MAX}(M_n(\mathbb{C})^*) \to \mathcal{U}_n^{nc}$ is an isometry. Now we show that $\rho : \text{MAX}(M_n(\mathbb{C})^*) \to \mathcal{U}_n^{nc}$ is completely isometry. Let $[x_{ij}]$ be any element in $M_m(\text{MAX}(M_n(\mathbb{C})^*))$. Then x_{ij} can be written in the form $x_{ij} = \sum_{k,l=1}^{n} c_{kl}^{ij}\xi_{kl}$, where $c_{kl}^{ij} \in \mathbb{C}$, for all i, j, k, l. Hence, we have

$$
\begin{aligned}
\|[\rho(x_{ij})]\|_{M_m(\mathcal{U}_n^{nc})} &= \left\| \left[\rho\left(\sum_{k,l=1}^{n} c_{kl}^{ij} \xi_{kl} \right) \right] \right\|_{M_m(\mathcal{U}_n^{nc})} \\
&= \left\| \left[\sum_{k,l=1}^{n} c_{kl}^{ij} \rho(\xi_{kl}) \right] \right\|_{M_m(\mathcal{U}_n^{nc})} \\
&= \left\| \left[\sum_{k,l=1}^{n} c_{kl}^{ij} u_{kl} \right] \right\|_{M_m(\mathcal{U}_n^{nc})} \\
&= \left\| \sum_{i,j=1}^{m} e_{ij} \otimes \left(\sum_{k,l=1}^{n} c_{kl}^{ij} u_{kl} \right) \right\|_{M_m(\mathcal{U}_n^{nc})} \\
&= \left\| \sum_{i,j=1}^{m} e_{ij} \otimes \left(\sum_{k,l=1}^{n} c_{kl}^{ij} \xi_{kl} \right) \right\|_{M_m(M_n(\mathbb{C})^*)} \\
&= \left\| \left[\sum_{k,l=1}^{n} c_{kl}^{ij} \xi_{kl} \right] \right\|_{M_m(M_n(\mathbb{C})^*)} \\
&\leq \|[x_{ij}]\|_{M_m(\mathrm{MAX}(M_n(\mathbb{C})^*))}
\end{aligned}
$$

For the other inequality, consider a contractive map $\varphi : M_n(\mathbb{C})^* \to \mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then the matrix $[\varphi(\xi_{ij})]$ is contractive. Again using the above argument, we have a unital completely positive map $\pi : \mathcal{U}_n^{nc} \to C^*(\varphi(M_n(\mathbb{C})^*))$ such that $\pi(u_{ij}) = \varphi(\xi_{ij})$. Therefore, $\|[\sum c_{ij}\varphi(\xi_{ij})]\| \leq \|[\sum c_{ij}u_{ij}]\|$. This holds for all contractive map $\varphi : M_n(\mathbb{C})^* \to \mathbf{B}(\mathcal{H})$. Hence $\|[\sum c_{ij}\xi_{ij}]\|_{M_m(\mathrm{MAX}(M_n(\mathbb{C})^*))} \leq \|[\sum c_{ij}u_{ij}]\|$. Thus, the isometry $\rho : \mathrm{MAX}(M_n(\mathbb{C})^*) \to \mathcal{U}_n^{nc}$ is a completely isometry.

(1) $\implies$ (2) Since the maximal operator space $\mathrm{MAX}(M_n(\mathbb{C})^*)$ is isometrically embedded into $\mathcal{U}_n^{nc}$ with $\xi_{ij} \mapsto u_{ij}$, $\|[\xi_{ij}]\|_{\mathrm{MAX}(M_n(\mathbb{C})^*)} \leq 1$. Let $\varphi : M_n(\mathbb{C})^* \to \mathbf{B}(\mathcal{H})$ be a contractive map. Then, $\|[\varphi(\xi_{ij})]\| \leq 1$. Using the Lemma 4.1.3, there exists a UCP map $\Psi : \mathcal{U}_n^{nc} \to C^*(\varphi(M_n(\mathbb{C})^*))$ such that $\Psi(u_{ij}) = \varphi(\xi_{ij})$, $1 \leq i,j \leq n$. Let $\pi : M_n(\mathbb{C})^* \to \mathcal{U}_n^{nc}$ be the natural completely contractive map. Note that $\Psi \circ \pi = \varphi$. Therefore, $\varphi : M_n(\mathbb{C})^* \to \mathbf{B}(\mathcal{H})$ is completely contractive. $\square$

Remark 4.2.3. *Consider the map $\varphi : M_n(\mathbb{C})^* \to M_n(\mathbb{C})$ defined by $\xi_{ij} \mapsto e_{ij}$, for $1 \leq i,j \leq n$.*

We show that this map is contractive but fails to be completely contractive. We know that $M_n(\mathbb{C})^*$ is isometrically isometric to $(M_n(\mathbb{C}), \|.\|_1)$, where $\|.\|_1$ is the trace norm.

Since $\|(a_{ij})\|_{op} \leq \|(a_{ij})\|_1$ for $(a_{ij}) \in M_n(\mathbb{C})$, the map $\varphi : M_n(\mathbb{C})^* \to M_n(\mathbb{C})$ is contractive. To prove that φ is not completely contractive, consider the map $\varphi_n := \mathrm{id} \otimes \varphi : M_n(\mathbb{C}) \otimes M_n(\mathbb{C})^* \to M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. Now, using Theorem 4.1.2, we have $\|\sum_{i,j} e_{ij} \otimes \xi_{ij}\| = 1$, and note that $\|\sum e_{ij} \otimes e_{ij}\| = \sqrt{n}$. Hence, the map φ_n cannot be contractive, which implies that φ is not completely contractive. Using Theorem 4.2.2, we conclude that MAX$(M_n(\mathbb{C})^*)$ is not isometrically embedded into $\mathcal{U}_n^{nc}$ with $\xi_{ij} \mapsto u_{ij}$.

In Theorem 4.2.2, we observed a correlation between contractive maps being completely contractive for an operator space and the operator space structure being maximal. This result can be generalised as follows.

Theorem 4.2.4. *Let $\mathcal{E}$ be an operator space. Then, the following are equivalent.*

1. *$\mathcal{E}$ is completely isometric to MAX$(\mathcal{E})$ by the identity map.*
2. *Every contractive map $\varphi : \mathcal{E} \to \mathbf{B}(\mathcal{H})$ is completely contractive, where $\mathcal{H}$ is a Hilbert space.*

Proof. (2) $\implies$ (1). Let $\varphi : \mathcal{E} \to \mathbf{B}(\mathcal{H})$ be a contractive map, where $\mathcal{H}$ is a Hilbert space. Then, $\varphi : \mathcal{E} \to \mathbf{B}(\mathcal{H})$ is completely contractive. Therefore, $\|(\varphi(x_{ij}))\| \leq \|(x_{ij})\|$. This shows that $\|(x_{ij})\|_{\mathrm{MAX}} \leq \|(x_{ij})\|$. Further, using maximality for MAX norm we conclude that $\|(x_{ij})\|_{\mathrm{MAX}} = \|(x_{ij})\|$. Hence, MAX$(\mathcal{E})$ is completely isometric with $\mathcal{E}$.

(1) $\implies$ (2). Let $\varphi : \mathcal{E} \to \mathbf{B}(\mathcal{H})$ be a contractive map, where $\mathcal{H}$ is a Hilbert space. Since MAX$(\mathcal{E})$ is completely isometric to $\mathcal{E}$, $\|(\varphi(x_{ij}))\| \leq \|(x_{ij})\|_{\mathrm{MAX}} = \|(x_{ij})\|$ for all $(x_{ij}) \in M_n(\mathcal{E})$ and $n \in \mathbb{N}$. Hence $\varphi : \mathcal{E} \to \mathbf{B}(\mathcal{H})$ is completely contractive. □

Remark 4.2.5. *Using Theorems Theorem 4.2.1 and Theorem 4.2.4, we can conclude that MAX(ℓ_n^1) is completely isometric to the operator space structure of $\ell_n^1 = (\ell_n^\infty)^*$. However, for $M_n(\mathbb{C})^*$, using Theorem 4.2.4 and Remark 4.2.3, it follows that MAX$(M_n(\mathbb{C})^*)$ is not completely isometric to the usual operator space structure of $M_n(\mathbb{C})^*$.*

4.2.1 Minimal Operator Space Structure

The following structural theorem for $\mathrm{MIN}(M_n(\mathbb{C})^*)$ was proven in [Man23a] and is also present in our paper [BMP25] but we include the proof for continuity.

Theorem 4.2.6. *The minimal operator space MIN$(M_n(\mathbb{C})^*)$ is completely isometric into the C^*-algebra $C(\mathcal{U}(n))$, the space of all continuous functions on the compact set of $n \times n$ unitary matrix.*

Proof. Let us first consider $v_{ij} : \mathcal{U}(n) \to \mathbb{C}$ be the continuous functions defined by

$$v_{ij}\left([a_{ij}]\right) = a_{ij},\ [a_{ij}] \in \mathcal{U}(n).$$

Clearly, $[v_{ij}]$ is a unitary matrix in $M_n(C(\mathcal{U}(n)))$. Then there exists a $*$-homomorphism $\pi : \mathcal{U}_n^{nc} \to C(\mathcal{U}(n))$ such that $\pi(u_{ij}) = v_{ij}$. Let us define a linear map $\rho : \mathrm{MIN}(M_n(\mathbb{C})^*) \to C(\mathcal{U}(n))$ by $\rho(\xi_{ij}) = v_{ij},\ \forall i, j$. Now we show that $\|\sum c_{ij}\xi_{ij}\|_{\mathrm{MIN}} = \|\sum c_{ij}v_{ij}\|_\infty$. Note that

$$\begin{aligned}
\left\|\sum c_{ij}\xi_{ij}\right\|_{\mathrm{MIN}} &= \left\|\sum c_{ij}\xi_{ij}\right\| \\
&= \left\|\sum c_{ij}u_{ij}\right\| \\
&\geq \left\|\sum c_{ij}v_{ij}\right\|_\infty .
\end{aligned}$$

For the other direction, consider the MIN norm as follows

$$\left\|\sum c_{ij}\xi_{ij}\right\|_{\mathrm{MIN}} = \sup\left\{\left|f\left(\sum c_{ij}\xi_{ij}\right)\right| :\ f : M_n(\mathbb{C})^* \to \mathbb{C},\ \|f\| \leq 1\right\}.$$

For a fixed matrix $[c_{ij}] \in M_n(\mathbb{C})$. Define a map $\varphi : \mathbb{B}_{M_n(\mathbb{C})} \to \mathbb{R}$ by $\varphi([b_{ij}]) = \left|\sum_{i,j} c_{ij}b_{ij}\right|$, where $\mathbb{B}_{M_n(\mathbb{C})}$ denotes the closed unit ball of $M_n(\mathbb{C})$. Since the convex hull of the unitary group is weak*-dense in $\mathrm{B}_{M_n(\mathbb{C})}$, the supremum is attained on unitary matrices. Moreover, since $\|[f(\xi_{ij})]\| \leq 1$, it follows that

$$\left\|\sum c_{ij}\xi_{ij}\right\|_{\mathrm{MIN}} \leq \sup_{U\in\mathcal{U}(n)} \left\|\sum c_{ij}u_{ij}\right\| = \left\|\sum c_{ij}v_{ij}\right\|_{\infty}.$$

Next, we show that $\rho : \mathrm{MIN}(M_n(\mathbb{C})^*) \to C(\mathcal{U}(n))$ is a completely isometry. Let $[x_{ij}]$ be any element in $M_m(\mathrm{MIN}(M_n(\mathbb{C})^*))$. Then x_{ij} can be written in the form $x_{ij} = \sum_{k,l=1}^{n} c_{kl}^{ij}\xi_{kl}$, where $c_{kl}^{ij} \in \mathbb{C}$, for all i, j, k, l. Hence, we have

$$\begin{aligned}
\|[\rho(x_{ij})]\|_{\infty} &= \left\|\left[\rho\left(\sum_{k,l=1}^{n} c_{kl}^{ij}\xi_{kl}\right)\right]\right\|_{\infty} \\
&= \left\|\left[\sum_{k,l=1}^{n} c_{kl}^{ij}v_{kl}\right]\right\|_{\infty} \\
&= \sup\left\{\left\|\left[\sum_{k,l=1}^{n} c_{kl}^{ij}a_{kl}\right]\right\|_{M_m(\mathbb{C})} : [a_{ij}] \in \mathcal{U}(n)\right\}.
\end{aligned}$$

Also, note that

$$\begin{aligned}
\left\|\left[\sum_{k,l=1}^{n} c_{kl}^{ij}\xi_{kl}\right]\right\|_{\mathrm{MIN}(M_n(\mathbb{C})^*)} &= \sup\left\{\left\|\left[\sum_{k,l=1}^{n} c_{kl}^{ij}f(\xi_{kl})\right]\right\| : f : M_n(\mathbb{C})^* \to \mathbb{C},\ \|f\| \leq 1\right\} \\
&= \sup\left\{\left\|\left[\sum_{k,l=1}^{n} c_{kl}^{ij}a_{kl}\right]\right\|_{M_m(\mathbb{C})} : [a_{ij}] \in \mathbf{B}_{M_n(\mathbb{C})}\right\}.
\end{aligned}$$

Therefore, using Russo-Dye theorem, we get

$$\sup\left\{\left\|\left[\sum_{k,l=1}^{n} c_{kl}^{ij}a_{kl}\right]\right\|_{M_m(\mathbb{C})} : [a_{ij}] \in \mathcal{U}(n)\right\} = \sup\left\{\left\|\left[\sum_{k,l=1}^{n} c_{kl}^{ij}a_{kl}\right]\right\|_{M_m(\mathbb{C})} : [a_{ij}] \in \mathbf{B}_{M_n(\mathbb{C})}\right\}.$$

Hence, the minimal operator space $\mathrm{MIN}(M_n(\mathbb{C})^*)$ is embedded completely isometrically into the C^*-algebra $C(\mathcal{U}(n))$. $\square$

4.3 Operator Space structure associated to the generators of $\mathcal{U}^{nc}_{n,red}$

We have defined $\mathcal{U}^{nc}_{n,red}$ in Section 2.3.3. In this section, we study the operator space structure of the reduced C^*-algebra $\mathcal{U}^{nc}_{n,\mathrm{red}}$, building upon the work of [HP93] in the case of the free group. Our first step is to establish the following result, which plays a crucial role in the subsequent discussion.

Theorem 4.3.1. *For any C^*-algebra $\mathcal{A}$ and for any finite collection $\{a_{ij} \mid a_{ij} \in \mathcal{A}\}$ we have,*

$$\begin{aligned}\frac{1}{\sqrt{n}} \max\left\{ \left\|\sum a_{ij}^* a_{ij}\right\|^{1/2}, \left\|\sum a_{ij} a_{ij}^*\right\|^{1/2} \right\} &\leq \left\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij}) \otimes a_{ij}\right\| \\ &\leq \sqrt{n}\ \max\left\{ \left\|\sum a_{ij}^* a_{ij}\right\|^{1/2}, \left\|\sum a_{ij} a_{ij}^*\right\|^{1/2} \right\}.\end{aligned}$$

Proof. We may assume $A \subseteq B(K)$ for some Hilbert space K. Let ξ be a unit vector in K.

$$\begin{aligned}\left\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij}) \otimes a_{ij}\right\| &\geq \left\|\left(\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij}) \otimes a_{ij}\right)\left(\xi_{\mathrm{Tr}*h_{\mathbb{T}}} \otimes \xi\right)\right\| \\ &= \left\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}} \otimes a_{ij}\xi\right\|\end{aligned}$$

We identify $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})$ with u_{ij} as well as $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(x)$ with x for any $x \in M_n *^{\mathrm{alg}}_{C} C(\mathbb{T})$. We now show that $\{\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}} : i, j\}$ is an orthogonal collection. First, we compute the following.

$$\begin{aligned}\langle \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}}, \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{i'j'})\xi_{\mathrm{Tr}*h_{\mathbb{T}}}\rangle &= \langle \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u^*_{i'j'})\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}}, \xi_{\mathrm{Tr}*h_{\mathbb{T}}}\rangle \\ &= \mathrm{Tr}*h_{\mathbb{T}}(u^*_{i'j'}u_{ij}) \\ &= \mathrm{Tr}*h_{\mathbb{T}}((\sum_{k'} e_{k'i'}ue_{j'k'})^*(\sum_k e_{ki}ue_{jk})) \\ &= \mathrm{Tr}*h_{\mathbb{T}}((\sum_{k'} e_{k'j'}u^*e_{i'k'})(\sum_k e_{ki}ue_{jk})) \\ &= \sum_k \mathrm{Tr}*h_{\mathbb{T}}(e_{kj'}u^*e_{i'i}ue_{jk})\end{aligned}$$

Since $\mathrm{Tr}*h_{\mathbb{T}}$ is a trace, we have

$$\begin{aligned}\langle \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}}, \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{i'j'})\xi_{\mathrm{Tr}*h_{\mathbb{T}}}\rangle &= \sum_k \mathrm{Tr}*h_{\mathbb{T}}(e_{kj'}u^*e_{i'i}ue_{jk}) \\ &= \sum_k \mathrm{Tr}*h_{\mathbb{T}}(e_{jk}e_{kj'}u^*e_{i'i}u) \\ &= n\mathrm{Tr}*h_{\mathbb{T}}(e_{jj'}u^*e_{i'i}u)\end{aligned}$$

If $i \neq i'$ and $j \neq j'$, then$\mathrm{Tr}(e_{jj'}) = \mathrm{Tr}(e_{i'i}) = 0$ and $h_{\mathbb{T}}(u) = h_{\mathbb{T}}(u^*) = 0$. Consequently,

$$(\mathrm{Tr}*h_{\mathbb{T}})(e_{jj'}u^*e_{i'i}u) = 0.$$

This follows from the fact that M_n and $C(\mathbb{T})$ form a free family of unital subalgebras in the free product $M_n * C(\mathbb{T})$.

Now if $i = i'$ and $j \neq j'$ we can write $e_{ii} = \frac{1}{n}I_n + (e_{ii} - \frac{1}{n}I_n) = \frac{1}{n}I_n + E_{ii}$, where $E_{ii} = e_{ii} - \frac{1}{n}I_n$, also we observe that, $\mathrm{Tr}(E_{ii}) = 0$

$$\begin{aligned}\mathrm{Tr}*h_{\mathbb{T}}(e_{jj'}u^*e_{ii}u) &= \mathrm{Tr}*h_{\mathbb{T}}(e_{jj'}u^*(\frac{1}{n}I_n + E_{ii})u) \\ &= \frac{1}{n}\mathrm{Tr}*h_{\mathbb{T}}(e_{jj'}u^*u) + \mathrm{Tr}*h_{\mathbb{T}}(e_{jj'}u^*E_{ii}u) \\ &= 0\end{aligned}$$

The case for $i \neq i'$ and $j = j'$ follows same above logic. Hence we can conclude that $\{\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}} : i, j\}$ is an orthogonal set.
So, $\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}} \otimes a_{ij}\xi\| = (\sum \|\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}}\|^2 \|a_{ij}\xi\|^2)^{1/2}$

Now, $\|\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}}\|^2 = n\mathrm{Tr} * h_{\mathbb{T}}(e_{jj}u^*e_{ii}u)$ Hence,

$$\begin{aligned}
\|\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}}\|^2 &= n\mathrm{Tr} * h_{\mathbb{T}}(e_{jj}u^*e_{ii}u) \\
&= n\mathrm{Tr} * h_{\mathbb{T}}(((1/n)I_n + E_{jj})u^*((1/n)I_n + E_{ii})u) \\
&= n[\frac{1}{n^2}\mathrm{Tr} * h_{\mathbb{T}}(u^*u)] \\
&= \frac{1}{n}
\end{aligned}$$

So,

$$\begin{aligned}
\left\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}} \otimes a_{ij}\xi\right\| &= (\sum \|\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\xi_{\mathrm{Tr}*h_{\mathbb{T}}}\|^2 \|a_{ij}\xi\|^2)^{1/2} \\
&= \frac{1}{\sqrt{n}}(\sum \|a_{ij}\xi\|^2)^{1/2} \\
&= \frac{1}{\sqrt{n}}\langle(\sum a_{ij}^*a_{ij})\xi, \xi\rangle^{1/2}
\end{aligned}$$

Taking the supremum over all unit vectors $\xi \in K$, we get,

$$\left\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij}) \otimes a_{ij}\right\| \geq (1/\sqrt{n})\left\|\sum a_{ij}^*a_{ij}\right\|^{1/2}$$

Since $\xi \in K$ is an arbitrary unit vector, $\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij}) \otimes a_{ij}\| \geq \frac{1}{\sqrt{n}}\left\|\sum a_{ij}^*a_{ij}\right\|^{1/2}$. Taking adjoint on both sides, we have $\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij}) \otimes a_{ij}\| \geq \frac{1}{\sqrt{n}}\left\|\sum a_{ij}a_{ij}^*\right\|^{1/2}$.
Hence, $\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij}) \otimes a_{ij}\| \geq \frac{1}{\sqrt{n}}\max\left\{\left\|\sum a_{ij}^*a_{ij}\right\|^{1/2}, \left\|\sum a_{ij}a_{ij}^*\right\|^{1/2}\right\}$.

Next, we compute the following:

$$\begin{aligned}\left\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\otimes a_{ij}\right\| &= \left\|\sum(\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\otimes 1)(1\otimes a_{ij})\right\| \\ &\leq \left\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij}^*)\right\|^{\frac{1}{2}}\left\|\sum a_{ij}^* a_{ij}\right\|^{\frac{1}{2}} \\ &= \left\|\sum_{i,j=1}^{n}\sum_{k=1}^{n} \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ik})\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{kj}^*)\right\|^{\frac{1}{2}}\left\|\sum a_{ij}^* a_{ij}\right\|^{\frac{1}{2}}\end{aligned}$$

Since $[u_{ij}]$ is a unitary matrix, we have $\sum_{k=1}^{n} u_{ik}u_{kj}^* = \delta_{ij}1$. Applying the representation $\pi_{\mathrm{Tr}*h_{\mathbb{T}}}$, it follows that$\sum_{k=1}^{n} \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ik})\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{kj}^*) = \delta_{ij}1$. Summing over i, j, we obtain

$$\sum_{i,j=1}^{n}\sum_{k=1}^{n} \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ik})\pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{kj}^*) = n\,1.$$

Thus we have,

$$\begin{aligned}\left\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\otimes a_{ij}\right\| &= \sqrt{n}\left\|\sum a_{ij}^* a_{ij}\right\|^{\frac{1}{2}} \\ &\leq \sqrt{n}\ \max\left\{\left\|\sum a_{ij}^* a_{ij}\right\|^{1/2}, \left\|\sum a_{ij} a_{ij}^*\right\|^{1/2}\right\}.\end{aligned} \tag{4.1}$$

Therefore, combining last two equations, one has

$$\begin{aligned}\frac{1}{\sqrt{n}}\max\left\{\left\|\sum a_{ij}^* a_{ij}\right\|^{1/2}, \left\|\sum a_{ij} a_{ij}^*\right\|^{1/2}\right\} &\leq \left\|\sum \pi_{\mathrm{Tr}*h_{\mathbb{T}}}(u_{ij})\otimes a_{ij}\right\| \\ &\leq \sqrt{n}\ \max\left\{\left\|\sum a_{ij}^* a_{ij}\right\|^{1/2}, \left\|\sum a_{ij} a_{ij}^*\right\|^{1/2}\right\}\end{aligned}$$

□

To proceed further, we recall some definitions that will play a crucial role in our discussion.

Definition 4.3.2. *In $\mathbf{B}(\ell_2)$ we can consider the column Hilbert space $\mathcal{C} = \overline{span}\{e_{i1} \mid i \in \mathbb{N}\}$ and the row Hilbert space $\mathcal{R} = \overline{span}\{e_{1i} \mid i \in \mathbb{N}\}$ and their corresponding finite dimensional versions $\mathcal{C}_n = \overline{span}\{e_{i1} \mid 1 \leq i \leq n\}$ and $\mathcal{R}_n = \overline{span}\{e_{1i} \mid 1 \leq i \leq n\}$. Consider the space $\mathbf{B}(\ell_2) \oplus \mathbf{B}(\ell_2)$, equipped with the norm $\|x \oplus y\| = \max\{\|x\|, \|y\|\}$. Consider the vectors*

$\delta_i = e_{1i} \oplus e_{i1} \in \mathcal{R} \oplus \mathcal{C} \subset \mathbf{B}(\ell^2) \oplus \mathbf{B}(\ell^2)$. We define $\mathcal{R} \cap \mathcal{C}$ by the closed subspace spanned in $\mathcal{R} \oplus \mathcal{C}$ spanned by vectors δ_i. We define $\mathcal{R}_n \cap \mathcal{C}_n$ by the subspace of $\mathcal{R} \cap \mathcal{C}$ spanned by $\{\delta_i \mid 1 \le i \le n\}$.

Definition 4.3.3. *$\mathcal{R}_{n^2} \cap \mathcal{C}_{n^2} \subset M_{n^2}(\mathbb{C}) \oplus M_{n^2}(\mathbb{C})$ be the operator space. Let $\{\delta_{ij}\}$ be the basis of $\mathcal{R}_{n^2} \cap \mathcal{C}_{n^2}$, where $\delta_{ij} = e_{1,(i-1)n+j} \oplus e_{(i-1)n+j,1}$*

$$\mathcal{R}_{n^2} \cap \mathcal{C}_{n^2} = \left\{ \begin{bmatrix} c_{11} & \\ c_{12} & \\ \vdots & \\ c_{1n} & \\ \vdots & 0 \\ c_{n1} & \\ c_{n2} & \\ \vdots & \\ c_{nn} & \end{bmatrix} \bigoplus \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} & \dots & c_{n1} & \dots c_{nn} \\ & & & & & & \\ & & & 0 & & & \\ & & & & & & \end{bmatrix} \middle| \, c_{ij} \in \mathbb{C}, 1 \le i, j \le n \right\}.$$

In the following theorem, we obtain the structure of the operator space spanned by $\pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij})$.

Theorem 4.3.4. *Let $\mathcal{E}^{nc}_{n,red}$ be the operator space spanned by $\pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij})$. Then, there are completely bounded maps $w : \mathcal{R}_{n^2} \cap \mathcal{C}_{n^2} \to \mathcal{E}^{nc}_{n,red}$ and $v : \mathcal{E}^{nc}_{n,red} \to \mathcal{R}_{n^2} \cap \mathcal{C}_{n^2}$ such that $vw = \frac{1}{n} I_{\mathcal{R}_{n^2} \cap \mathcal{C}_{n^2}}$ and $\|v\|_{cb} \|w\|_{cb} \le 1$. In particular; the operator space $\mathcal{E}^{nc}_{n,red}$ is completely isomorphic to $\mathcal{R}_{n^2} \cap \mathcal{C}_{n^2}$.*

Proof. Let us first consider a linear map $w : \mathcal{R}_{n^2} \cap \mathcal{C}_{n^2} \to \mathcal{E}^{nc}_{n,red}$ defined by

$$w\left(\sum_{i,j=1}^{n} c_{ij}\delta_{ij}\right) = \sum_{i,j=1}^{n} c_{ij}\pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij}).$$

For each $k \in \mathbb{N}$ and any set $\{a_{ij} : 1 \le i, j \le n\}$ of n^2 elements in $M_k(\mathbb{C})$, the following holds:

$$(w \otimes \mathrm{id}) \left(\sum_{i,j=1}^{n} \delta_{ij} \otimes a_{ij} \right) = \sum_{i,j=1}^{n} \pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij}) \otimes a_{ij}. \tag{4.2}$$

Note that

$$\begin{aligned} \left\| \sum_{i=1}^{n} \delta_{ij} \otimes a_{ij} \right\| &= \left\| \begin{bmatrix} a_{11} & & \\ a_{12} & & \\ \vdots & & \\ a_{1n} & & \\ \vdots & 0 & \\ a_{n1} & & \\ a_{n2} & & \\ \vdots & & \\ a_{nn} & & \end{bmatrix} \oplus \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & \dots & a_{n1} & \dots a_{nn} \\ & & & & & & \\ & & & 0 & & & \\ & & & & & & \end{bmatrix} \right\| \\ &= \max \left\{ \left\| \sum_{i,j=1}^{n} a_{ij}^* a_{ij} \right\|^{1/2}, \left\| \sum_{i,j=1}^{n} a_{ij} a_{ij}^* \right\|^{1/2} \right\}. \end{aligned} \tag{4.3}$$

From the Theorem 4.3.1, one has, $\|w \otimes \mathrm{id}\| \leq \sqrt{n}$ for all k i.e., $\|w\|_{\mathrm{cb}} \leq \sqrt{n}$. Next, we define the following linear map $v : \mathcal{E}_{n,red}^{nc} \to \mathcal{R}_{n^2} \cap \mathcal{C}_{n^2}$:

$$v(x) = \sum_{i,j=1}^{n} \mathrm{Tr} * h_{\mathbb{T}}(\pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij})^* x) \delta_{ij}, \ x \in \mathcal{E}_{n,red}^{nc}.$$

Note that

$$\mathrm{Tr} * h_{\mathbb{T}}(\pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij}^* u_{pq})) = \begin{cases} \frac{1}{n} & i = p, \ j = q; \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Thus, we have $v\left(\sum_{i,j} c_{ij} \pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij})\right) = \frac{1}{n} \sum_{i,j} c_{ij} \delta_{ij}$. Now, we obtain the following computations:

$$
\begin{aligned}
& \left\| (v \otimes \mathrm{id}) \left(\sum_{i,j} \pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij}) \otimes a_{ij} \right) \right\| \\
= \quad & \frac{1}{n} \left\| \sum_{i,j} \delta_{ij} \otimes a_{ij} \right\| \\
= \quad & \frac{1}{n} \max \left\{ \left\| \sum_{i,j=1}^{n} a_{ij}^* a_{ij} \right\|^{1/2}, \left\| \sum_{i,j=1}^{n} a_{ij} a_{ij}^* \right\|^{1/2} \right\} \\
\leq \quad & \frac{1}{\sqrt{n}} \left\| \sum_{i,j} \pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij}) \otimes a_{ij} \right\|.
\end{aligned}
$$

Thus, for all k, $\|v \otimes \mathrm{id}\| \leq \frac{1}{\sqrt{n}}$ i.e., $\|v\|_{\mathrm{cb}} \leq \frac{1}{\sqrt{n}}$. Therefore, we have $\|v\|_{\mathrm{cb}} \|w\|_{\mathrm{cb}} \leq 1$. Clearly, we have the following computations:

$$
vw \left(\sum_{i,j} c_{ij} \delta_{ij} \right) = v \left(\sum_{i,j} c_{ij} \pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij}) \right) = \frac{1}{n} \sum_{i,j} c_{ij} \delta_{ij}.
$$

$$
wv \left(\sum_{i,j} c_{ij} \pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij}) \right) = \frac{1}{n} w \left(\sum_{i,j} c_{ij} \delta_{ij} \right) = \frac{1}{n} \sum_{i,j} c_{ij} \pi_{\mathrm{Tr} * h_{\mathbb{T}}}(u_{ij}).
$$

Therefore, the operator space $\mathcal{E}_{n,red}^{nc}$ is completely isomorphic to $\mathcal{R}_{n^2} \cap \mathcal{C}_{n^2}$. $\square$

Chapter 5

Quantum semigroup structure

In this section, we study the compact quantum semigroup structure on $\mathcal{C}_n^{nc}$. We begin by recalling the definition of a compact quantum group, referring to [Wor87] for more details.

Definition 5.0.1. *A* compact quantum group *is a pair* $\mathbb{G} = (\mathcal{A}, \Delta)$*, where* $\mathcal{A}$ *is a unital* C^**-algebra and* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ *is a unital* $*$*-homomorphism, called the* comultiplication*, such that:*

1. Δ *is coassociative, i.e.*

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta \quad \text{as maps from } \mathcal{A} \text{ to } \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A};$$

2. *the linear spans of* $(\mathcal{A} \otimes 1)\Delta(\mathcal{A})$ *and* $(1 \otimes \mathcal{A})\Delta(\mathcal{A})$ *are both dense in* $\mathcal{A} \otimes \mathcal{A}$ *(this is referred to as the* cancellation law*).*

If only condition (1) holds, then the pair $(\mathcal{A}, \Delta)$ *is called a* compact quantum semigroup.

Definition 5.0.2. *A* quantum subsemigroup *of* $(\mathcal{A}, \Delta)$ *is a pair* $(\mathcal{B}, \Delta_{\mathcal{B}})$ *such that:*

1. $\mathcal{B}$ *is a unital* C^**-algebra,*

2. *there exists a surjective unital* $*$*-homomorphism*

$$\pi : \mathcal{A} \to \mathcal{B},$$

3. *the comultiplication descends to $\mathcal{B}$, i.e.*

$$\Delta_{\mathcal{B}} \circ \pi = (\pi \otimes \pi) \circ \Delta,$$

equivalently, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow \pi & & \downarrow \pi \otimes \pi \\ B & \xrightarrow{\Delta_{\mathcal{B}}} & \mathcal{B} \otimes \mathcal{B} \end{array}$$

Note that $\mathcal{U}_n^{nc}$ has a natural quantum semigroup structure with co-multiplication $\Delta_{\mathcal{U}_n^{nc}}$: $\mathcal{U}_n^{nc} \to \mathcal{U}_n^{nc} \otimes \mathcal{U}_n^{nc}$, studied in detail in [Man23b].

Theorem 5.0.3. *The following holds.*

1. *There exists a unique unital $*$-homomorphism $\Delta : \mathcal{C}_n^{nc} \to \mathcal{C}_n^{nc} \otimes \mathcal{C}_n^{nc}$ such that $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$ and $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$.*

2. *There exists a unital $*$-homomorphism $\epsilon : \mathcal{C}_n^{nc} \to \mathbb{C}$ such that $(\epsilon \otimes \mathrm{id})\Delta(a) = a = (\mathrm{id} \otimes \epsilon)\Delta(a),\ a \in \mathcal{C}_n^{nc}$.*

Proof.

1. Using the universal property of the C^*-algebra $\mathcal{C}_n^{\mathrm{nc}}$, it suffices to show that the matrix $[\sum_{k=1}^n u_{ik} \otimes u_{kj}]_{i,j=1}^n$ is contractive in $M_n(\mathcal{C}_n^{\mathrm{nc}} \otimes \mathcal{C}_n^{\mathrm{nc}})$ in order to construct a $*$-homomorphism

$$\Delta : \mathcal{C}_n^{\mathrm{nc}} \to \mathcal{C}_n^{\mathrm{nc}} \otimes \mathcal{C}_n^{\mathrm{nc}}, \qquad u_{ij} \mapsto \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

Observe that, by the universal property of $\mathcal{C}_n^{\mathrm{nc}}$, there exists a $*$-homomorphism

$$\varphi : \mathcal{C}_n^{\mathrm{nc}} \to \mathcal{U}_n^{\mathrm{nc}}, \qquad u_{ij} \mapsto v_{ij}.$$

By Lemma 4.1.3, there exists a unital completely positive map

$$\Psi : \mathcal{U}_n^{\mathrm{nc}} \to \mathcal{C}_n^{\mathrm{nc}}, \quad v_{ij} \mapsto u_{ij}.$$

Hence, we have the UCP map

$$\Psi \otimes \Psi : \mathcal{U}_n^{\mathrm{nc}} \otimes \mathcal{U}_n^{\mathrm{nc}} \to \mathcal{C}_n^{\mathrm{nc}} \otimes \mathcal{C}_n^{\mathrm{nc}},$$

for which

$$(\Psi \otimes \Psi)\left(\sum_{k=1}^{n} v_{ik} \otimes v_{kj}\right) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

Then note that $(\Psi\otimes\Psi)\circ\Delta_{\mathcal{U}_n^{nc}}\circ\varphi : \mathcal{C}_n^{nc} \to \mathcal{C}_n^{nc}\otimes\mathcal{C}_n^{nc}$ is a UCP mapping $u_{ij} \mapsto \sum_{k=1}^{n} u_{ik}\otimes u_{kj}$ and therefore, we have that $[\sum_{k=1}^{n} u_{ik} \otimes u_{kj}]_n$ is contraction in $M_n(\mathcal{C}_n^{nc}\otimes\mathcal{C}_n^{nc})$. Thus by universal property of $\mathcal{C}_n^{nc}$ we obtain a $*$-homomorphism $\Delta : \mathcal{C}_n^{nc} \to \mathcal{C}_n^{nc} \otimes \mathcal{C}_n^{nc}$ such that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ and the uniqueness follows from the defining properties on the generators. Now, we show that Δ satisfies the co-associative property.

$$\begin{aligned}
(\Delta \otimes \mathrm{id})\Delta(u_{ij}) =&(\Delta \otimes \mathrm{id})\Big(\sum_k u_{ik} \otimes u_{kj}\Big) \\
=&\sum_k \Big(\sum_l u_{il} \otimes u_{lk}\Big) \otimes u_{kj} \\
=&\sum_{k,l} (u_{il} \otimes u_{lk}) \otimes u_{kj} \\
=&\sum_{k,l} u_{il} \otimes (u_{lk} \otimes u_{kj}) \\
=&\sum_{l} u_{il} \otimes \Delta(u_{lj}) \\
=&(\mathrm{id} \otimes \Delta)\Big(\sum_l u_{il} \otimes u_{lj}\Big) \\
=&(\mathrm{id} \otimes \Delta)\Delta(u_{ij}).
\end{aligned}$$

Thus our result is proved.

2. Since the identity matrix is contractive, there exists a unital $*$-homomorphism $\epsilon : \mathcal{C}_n^{nc} \to \mathbb{C}$ such that $\epsilon(u_{ij}) = \delta_{ij}$. Clearly, $\epsilon : \mathcal{C}_n^{nc} \to \mathbb{C}$ satisfies the following relation

$$(\epsilon \otimes \mathrm{id})\Delta(a) = a = (\mathrm{id} \otimes \epsilon)\Delta(a), \;\; a \in \mathcal{C}_n^{nc}.$$

$\square$

Next, we show that $\mathcal{C}_n^{nc}$ is not a compact quantum group with respect to the above co-multiplication $\Delta : \mathcal{C}_n^{nc} \to \mathcal{C}_n^{nc} \otimes \mathcal{C}_n^{nc}$. The following is well known and can be found, for instance in [Pat13]. We sketch the proof here to ensure self-containment.

Lemma 5.0.4. *Suppose that $(\mathcal{A}, \Delta_{\mathcal{A}})$ is a compact quantum subsemigroup of a compact quantum group $(\mathbf{B}, \Delta_{\mathbf{B}})$. Then $(\mathcal{A}, \Delta_{\mathcal{A}})$ is a compact quantum group.*

Proof. Let $\pi : \mathbf{B} \to \mathcal{A}$ be a surjective $*$-homomorphism such that $(\pi \otimes \pi)\Delta_{\mathbf{B}} = \Delta_{\mathcal{A}}\pi$. Since $(\mathbf{B}\otimes 1)\Delta_{\mathbf{B}}(\mathbf{B})$ is dense in $\mathbf{B}\otimes\mathbf{B}$, $(\pi\otimes\pi)((\mathbf{B}\otimes 1)\Delta_{\mathbf{B}}(\mathbf{B}))$ is dense in $(\pi\otimes\pi)(\mathbf{B}\otimes\mathbf{B}) = \mathcal{A}\otimes\mathcal{A}$. Therefore, $(\pi(\mathbf{B}) \otimes 1)(\pi \otimes \pi)\Delta_{\mathbf{B}}(\mathbf{B})$ is dense in $\mathcal{A} \otimes \mathcal{A}$. Hence, $(\mathcal{A} \otimes 1)\Delta_{\mathcal{A}}(\mathcal{A})$ is dense in $\mathcal{A} \otimes \mathcal{A}$. Similarly, $(1 \otimes \mathcal{A})\Delta_{\mathcal{A}}(\mathcal{A})$ is dense in $\mathcal{A} \otimes \mathcal{A}$. $\square$

Theorem 5.0.5. *$\mathcal{C}_n^{nc}$ is not a compact quantum group.*

Proof. First, we show that the universal C^*-algebra $\mathcal{U}_n^{nc}$ is not a compact quantum group. This was exhibited by Wang in [Wan95]. For the sake of completeness, we provide the proof here. Assume for contradiction that $\mathcal{U}_n^{nc}$ is a compact quantum group. In that case, the matrix $\overline{u} = (u_{ij}^*)$ must be invertible in $M_n(\mathcal{U}_n^{nc})$, since $u = (u_{ij})$ is a unitary representation of $\mathcal{U}_n^{nc}$. This implies that u^t is also an invertible matrix in $M_n(\mathcal{U}_n^{nc})$. Next, we consider the unitary matrix $u' = (u'_{ij})$ belonging to $M_n(M_2(\mathbb{C}))$, defined as follows.

$$u' = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & I_{2n-4} \end{bmatrix},$$

$$\text{where } a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, d = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

By universal property, we have a $*$-homomorphism $\pi : \mathcal{U}_n^{nc} \to M_2(\mathbb{C})$ such that $\pi(u_{ij}) = u'_{ij}$. Since u^t is invertible, $(u')^t$ is invertible in $M_n(M_2(\mathbb{C}))$. This is a contradiction. Hence, $\mathcal{U}_n^{nc}$ is not a compact quantum group. Also, we have a natural surjective $*$-homomorphism $\pi : \mathcal{C}_n^{nc} \to \mathcal{U}_n^{nc}$ with the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}_n^{nc} \arrow[r, "\Delta"] \arrow[d, "\pi"'] & \mathcal{C}_n^{nc} \otimes \mathcal{C}_n^{nc} \arrow[d, "\pi\otimes\pi"] \\
\mathcal{U}_n^{nc} \arrow[r, "\Delta_{\mathcal{U}_n^{nc}}"] & \mathcal{U}_n^{nc} \otimes \mathcal{U}_n^{nc}
\end{tikzcd}$$

Thus, $(\mathcal{U}_n^{nc},\ \Delta_{\mathcal{U}_n^{nc}})$ is a compact quantum sub-semigroup of $(\mathcal{C}_n^{nc},\ \Delta)$. Since $(\mathcal{U}_n^{nc},\ \Delta_{\mathcal{U}_n^{nc}})$ is not a compact quantum group, Lemma 5.0.4 implies that $\mathcal{C}_n^{nc}$ is not a compact quantum group either. □

5.1 Finite Dimensional Representations of $\mathcal{C}_n^{nc}$.

For compact quantum semigroup, we also define the finite-dimensional representation.

Definition 5.1.1. *For a compact quantum semigroup $(\mathcal{A}, \Delta)$ we define a finite dimensional representation on a finite-dimensional Hilbert space $\mathcal{H}$ to be an element $u \in \mathbf{B}(\mathcal{H}) \otimes \mathcal{A}$ satisfying*

$$(\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13}.$$

In general, the element u is not assumed to be invertible. The dimension of the Hilbert space $\mathcal{H}$ specifies the dimension of the representation. For the algebra $\mathcal{C}_n^{nc}$, consider the generators $\{u_{ij} : 1 \leq i, j \leq n\}$. The matrix $u = (u_{ij})$ defines a representation of the compact quantum semigroup $(\mathcal{C}_n^{nc}, \Delta)$, since

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

Moreover, the conjugate matrix $\bar{u} = (u_{ij}^*)$ also gives a representation of $\mathcal{C}_n^{nc}$, although it is not invertible.

Finally, we observe that the notion of tensor product of representations extends naturally to compact quantum semigroups. Let v and w be two representations of $(\mathcal{A}, \Delta)$. The tensor product representation is defined by

$$v \otimes w = v_{13} w_{23}.$$

It is straightforward to verify that

$$(\mathrm{id} \otimes \Delta)(v \otimes w) = (v \otimes w)_{12}(v \otimes w)_{13},$$

which shows that $v \otimes w$ is again a representation of $\mathcal{A}$. Now we prove the following result for $\mathcal{C}_n^{nc}$.

Proposition 5.1.2. *The linear span of matrix coefficients corresponding to finite dimensional representations of $\mathcal{C}_n^{nc}$ is dense inside $\mathcal{C}_n^{nc}$.*

Proof. To show this, it suffices to prove that for any elements of $\mathcal{C}_n^{nc}$ of the form $u_{i_1 j_1}^{r_1} u_{i_2 j_2}^{r_2} ... u_{i_p j_p}^{r_p}$, $r_k \in \{1, *\}$ and $1 \leq i_1, j_1, \cdots, i_k, j_k \leq n,\ k = 1, 2, \cdots, p$ becomes matrix coefficients corresponding to some finite-dimensional representations. Now observe, $u = (u_{ij})$ and $\bar{u} = (u_{ij}^*)$ gives us finite dimensional representations of $\mathcal{C}_n^{nc}$. Let, $u^{r_k} = u$ when $r_k = 1$ and $u^{r_k} = \bar{u}$ when $r_k = *$. Therefore, we obtain the tensor product representation $v = u^{r_1} \otimes \cdots \otimes u^{r_p}$ of $\mathcal{C}_n^{nc}$, This yields a finite-dimensional representation. Corresponding to such a representation v, we obtain the associated matrix coefficient $u_{i_1 j_1}^{r_1} u_{i_2 j_2}^{r_2} ... u_{i_p j_p}^{r_p}$. $\square$

Now we focus on examining the state space structure of $\mathcal{C}_n^{nc}$. Let $S(\mathcal{C}_n^{nc})$ be the space of all states on $\mathcal{C}_n^{nc}$. Now for φ, ψ in $S(\mathcal{C}_n^{nc})$, define the convolution product as follows:

$$\varphi \star \psi : \mathcal{C}_n^{nc} \to \mathbb{C},\ \varphi \star \psi := (\varphi \otimes \psi) \circ \Delta.$$

$S(\mathcal{C}_n^{nc})$ with this convolution product forms a topological semigroup. Let us recall the following.

Definition 5.1.3. *For any C^*-algebra $\mathcal{A}$, a non-zero $*$-homomorphism $\varphi : \mathcal{A} \to \mathbb{C}$ from $\mathcal{A}$ onto $\mathbb{C}$ is called a character.*

Considering that we have a comultiplication defined on $\mathcal{C}_n^{nc}$, its possible to define invertible elements for the semigroup $S(\mathcal{C}_n^{nc})$ as follows.

Definition 5.1.4. *$\varphi \in S(\mathcal{C}_n^{nc})$ is called invertible if there exists $\psi \in S(\mathcal{C}_n^{nc})$ such that*

$$\varphi \star \psi = \epsilon = \psi \star \varphi$$

.

Now, we determine all characters on $\mathcal{C}_n^{nc}$. Before proceeding to the next theorem, we note that for any matrix $(\alpha_{ij}) \in \mathbf{B}_{M_n(\mathbb{C})}$, one can associate a character $\chi_{(\alpha_{ij})}$ defined by

$$\chi_{(\alpha_{ij})}(u_{ij}) = \alpha_{ij}.$$

Theorem 5.1.5. *We denote the set of all characters on $\mathcal{C}_n^{nc}$ by $\mathcal{X}(\mathcal{C}_n^{nc})$. Then $\mathcal{X}(\mathcal{C}_n^{nc})$ is isomorphic to $\mathbf{B}_{M_n(\mathbb{C})}$ as topological semigroup.*

Proof. First, observe that $\mathcal{X}(\mathcal{C}_n^{nc})$ with the convolution product is a semigroup. We intend to prove that $\mathcal{X}(\mathcal{C}_n^{nc}) \cong \mathbf{B}_{M_n(\mathbb{C})}$ as topological semigroup. Consider the map $\varphi_1 : \mathcal{X}(\mathcal{C}_n^{nc}) \to \mathbf{B}_{M_n(\mathbb{C})}$ is defined by

$$\varphi_1(\chi) = (\chi(u_{ij})), \ \chi \in \mathcal{X}(\mathcal{C}_n^{nc}).$$

for $\eta_1, \eta_2 \in \mathcal{X}(\mathcal{C}_n^{nc})$. We have

$$\begin{aligned} \varphi_1(\eta_1 \star \eta_2) &= ((\eta_1 \star \eta_2)(u_{ij}))_{i,j} \\ &= \left(\sum_{k=1}^{n} \eta_1(u_{ik}) \eta_2(u_{kj}) \right)_{i,j} \\ &= \varphi_1(\eta_1)\varphi_1(\eta_2). \end{aligned} \tag{5.1}$$

Hence $\varphi_1 : \mathcal{X}(\mathcal{C}_n^{nc}) \to \mathbf{B}_{M_n(\mathbb{C})}$ is a semigroup homomorphism. Now conversely consider a map $\varphi_2 : \mathbf{B}_{M_n(\mathbb{C})} \to \mathcal{X}(\mathcal{C}_n^{nc})$ is defined by

$$\varphi_2((\alpha_{ij})) = \chi_{(\alpha_{ij})},\ (\alpha_{ij}) \in \mathbf{B}_{M_n(\mathbb{C})}.$$

Similarly, one can prove that $\varphi_2 : \mathbf{B}_{M_n(\mathbb{C})} \to \mathcal{X}(\mathcal{C}_n^{nc})$ is a semigroup homomorphism. Now, let (α_{ij}) be an element in $\mathbf{B}_{M_n(\mathbb{C})}$. Therefore, $\varphi_1 \circ \varphi_2((\alpha_{ij})) = \varphi_1(\chi_{(\alpha_{ij})}) = (\chi_{(\alpha_{ij})}(u_{ij}))_{i,j} = (\alpha_{ij})$. Hence we have $\varphi_1 \circ \varphi_2 = \text{id}$. Following the same line of proof, we can get $\varphi_2 \circ \varphi_1 = \text{id}$. Thus proving that, $\mathcal{X}(\mathcal{C}_n^{nc})$ is isomorphic to $\mathbf{B}_{M_n(\mathbb{C})}$ as semigroups. Now we prove that the group homomorphism $\varphi_1 : \mathcal{X}(\mathcal{C}_n^{nc}) \to \mathbf{B}_{M_n(\mathbb{C})}$ is a continuous map concerning their respective topologies.Let (χ_α) be a net in $\mathcal{X}(\mathcal{C}_n^{nc})$ such that χ_α converges to some $\chi \in \mathcal{X}(\mathcal{C}_n^{nc})$ with respect to the weak*-topology. Thus $\chi_\alpha(u_{ij})$ converges to $\chi(u_{ij})$ for all i, j. Hence, $(\chi_\alpha(u_{ij}))$ converges to $(\chi(u_{ij}))$ in $\mathbf{B}_{M_n(\mathbb{C})}$. Therefore, $\varphi_1(\chi_\alpha)$ converges to $\varphi_1(\chi)$ in $\mathbf{B}_{M_n(\mathbb{C})}$. Hence $\varphi_1 : \mathcal{X}(\mathcal{C}_n^{nc}) \to \mathbf{B}_{M_n(\mathbb{C})}$ is continuous homomorphism. Since $\mathcal{X}(\mathcal{C}_n^{nc})$ is compact semigroup and $\mathbf{B}_{M_n(\mathbb{C})}$ is Hausdorff space, $\varphi_1 : \mathcal{X}(\mathcal{C}_n^{nc}) \to \mathbf{B}_{M_n(\mathbb{C})}$ is a homeomorphism. We therefore obtain that $\mathcal{X}(\mathcal{C}_n^{nc}) \cong \mathbf{B}_{M_n(\mathbb{C})}$ as topological semigroups. □

Its natural to wonder about invertible elements inside $S(\mathcal{C}_n^{nc})$. Hence, we try to characterize them.

Theorem 5.1.6. *Let us denote the set $\mathcal{I}(\mathcal{C}_n^{nc})$ as all invertible elements in $S(\mathcal{C}_n^{nc})$. Then, $\mathcal{I}(\mathcal{C}_n^{nc})$ is group isomorphic with $\mathcal{U}(n)$ as topological group.*

Proof. Using the previous Proposition, it is enough to prove that $[\varphi(u_{ij})]$ is a unitary matrix in $M_n(\mathbb{C})$. Consider an invertible state on $\mathcal{C}_n^{nc}$, say $\alpha : \mathcal{C}_n^{nc} \to \mathbb{C}$ and φ being invertible there exists a state $\beta : \mathcal{C}_n^{nc} \to \mathbb{C}$ such that $\alpha \star \beta = \mathcal{E} = \beta \star \alpha$, where the co-unit of $\mathcal{C}_n^{nc}$ is given by

$\mathcal{E} : \mathcal{C}_n^{nc} \to \mathbb{C}$. Thus, we get

$$\begin{aligned}
\alpha \star \beta(u_{ij}) &= (\alpha \otimes \beta) \circ \Delta(u_{ij}) \\
&= (\alpha \otimes \beta) \left(\sum_{k=1}^{n} u_{ik} \otimes u_{kj} \right) \\
&= \sum_{k=1}^{n} \alpha(u_{ik}) \beta(u_{kj}) \\
&= \delta_{ij},
\end{aligned}$$

for all $1 \leq i, j \leq n$. Thus, we obtain $[\alpha(u_{ij})][\beta(u_{ij})] = I_n$. Similarly, $[\beta(u_{ij})][\alpha(u_{ij})] = I_n$. Which proves that $[\alpha(u_{ij})]$ is an invertible matrix, inverse $[\beta(u_{ij})]$. Since α and β are states on $\mathcal{C}_n^{nc}$, we obtain $\|[\alpha(u_{ij})]\| \leq 1$ and $\|[\beta(u_{ij})]\| \leq 1$. Thus $[\alpha(u_{ij})]$ is an invertible contraction matrix whose inverse is also a contraction. Hence $[\alpha(u_{ij})]$ is a unitary matrix.

$\square$

Finally, we have the following.

Corollary 5.1.7. *Let $\varphi : \mathcal{C}_n^{nc} \to \mathbb{C}$ be an invertible state on $\mathcal{C}_n^{nc}$. Then, $\varphi : \mathcal{C}_n^{nc} \to \mathbb{C}$ is a character. The converse is not always true; a character $\chi : \mathcal{C}_n^{nc} \to \mathbb{C}$ need not be invertible concerning the above convolution product.*

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Preprints

The following preprints are based on the research presented in this thesis:

1. Sayan K. Banik, *Generalised Crossed Products of group C^*-algebras*, Manuscript submitted for publication, preprint available upon request, 2025.

2. Sayan K. Banik, Malay Mandal, and Issan Patri, *Brown's Universal Noncommutative C^*-algebra and Universal C^*-algebras Associated to Matrix Operator Spaces*, Manuscript submitted for publication, preprint available upon request, Indian Statistical Institute, 2025.