INDIAN STATISTICAL INSTITUTE

Semestral Examination

Second semester 2024–2025

M. Stat (First year)

Metric Topology and Complex Analysis

Date: May 2, 2025 Maximum Marks: 60

Duration: 3 hours 30 minutes

Answer all questions and give a detailed answer.

If you use a theorem then it must be clearly stated.

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

(1) (a) If f and g are entire functions such that $g \circ f$ is constant then show that either f or g is constant.

(b) Suppose f is an entire function which maps the real line into itself. Then show that f has a power series representation whose coefficients are all real.

(c) Prove that the zeros of the polynomial $P(z) = \sum_{k=0}^{n} 2^{k-n} z^k$ lie in $\mathbb{D}$.

(d) Show that $\sec z$ has a simple pole at $z = \pi/2$. Find the residue of the function at $\pi/2$. 3+4+4+4

(2) Let $R > 0$. Given that f is an entire function satisfying the condition $|f(z)| \leq |z|^2$ for $|z| > R$. Prove that f is a quadratic function. 8

(3) Suppose $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function which extends to a continuous function on $\bar{\mathbb{D}}$. Assume that

(a) $|f(z)| = 1$ for all $z \in \partial\mathbb{D}$, that is if $|z| = 1$;

(b) f does not vanish on $\mathbb{D}$.

Prove that f is a constant function. 7

(4) Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$ using contour integration when $a > 1$. Hint: Write cos in terms of exp function. **10**

(5) Suppose that w_1, w_2 are two complex numbers such that w_2/w_1 is not a real number. Let f be a (non-constant) meromorphic function on $\mathbb{C}$ such that

$$f(z+w_1) = f(z) \quad \text{and} \quad f(z+w_2) = f(z)$$

whenever $f(z)$ is defined.
Consider the parallelegram P defined by the vertices 0, w_1, w_2, $w_1 + w_2$:

$$P = \{t_1 w_1 + t_2 w_2 \,|\, 0 \leq t_1, t_2 \leq 1\}$$

Assume that f has no pole or zero on the boundary of P.

(a) Show that f has equal number of zeros and poles within P, where a zero or a pole is counted as many times as its order.
(b) Prove that f must have a zero in P.
(c) Conclude that f has infinitely many zeros and poles. 6+6+2

(6) Suppose $0, a, b, c$ are 4 distinct points in $\mathbb{D}$. Find a biholomorphic map $f : \mathbb{D} \to \mathbb{D}$ such that $f(0) = a$ and $f(b) = c$. 7

(7) Find a conformal map from the upper half space $\mathbb{H}$ onto the annulus $1 < |z| < 2$. 7