Indian Statistical Institute
Semester 1 (2025-2026)
M. Stat 1st Year
Semestral Examination
Stochastic Processes

Friday, 21.11.2025, Time: 10:30 AM to 1:30 PM — Total Points: $5 \times (7+7) = 70$

Answers must be justified with clear and precise arguments. More than one uncrossed answers to the same question or part of a question will not be entertained and only the first uncrossed answer will be graded. If you use any theorem or result proved in class state it explicitly.

1. Suppose that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rates λ_1 and λ_2 respectively. (a) Show that $\{N_1(t) + N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$. (b) Show that the probability that the first event of the combined process comes from $\{N_1(t), t \geq 0\}$ is $\lambda_1/(\lambda_1 + \lambda_2)$.

2. (a) Let $X(t)$ be a pure birth continuous time MC. Assume that

$$P(\text{ an event happens in } (t, t+h] | X(t) = \text{ odd}) = \lambda_1 h + o(h),$$

$$P(\text{ an event happens in } (t, t+h] | X(t) = \text{ even}) = \lambda_2 h + o(h),$$

(plus usual assumptions on the probability of more than one event in a small time interval, etc.). Find differential equations for $P_1(t) = P(X(t) = \text{ odd }), P_2(t) = P(X(t) = \text{ even })$ and show that

$$P_1(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2}(1 - \exp(-(\lambda_1 + \lambda_2)t)),$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\exp(-(\lambda_1 + \lambda_2)t).$$

(b) In the midterm you considered a continuous time Markov chain on $S = \{0, 1, 2\}$ for which the infinitesimal matrix is of the form

$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 \\ 0 & \mu_2 & -\mu_2 \end{pmatrix},$$

and found $E_0 T_0$, where T_0 is the first return time to 0 after at least one journey through states other than 0, by deriving linear equations for $E_0 T_0, E_1 T_0, E_2 T_0$. This solution assumes that $E_0 T_0$ is finite, but the expectation of a nonnegative integer valued random variable can be infinite also. **Prove** by a separate argument that $E_0 T_0 < \infty$.

3. Patients suffering from a certain disease come to a health center in the manner of a Poisson (λ) process. Each patient undergoing treatment has a recovery time X with distribution function F. Assuming the center has unlimited capacity to serve (a) show that the probability generating function $\phi(t) = Ez^{N(t)}$, where $N(t) =$ the number of patients still recovering at t, satisfies the renewal-type equation $\phi(t) = e^{-\lambda t} + \int_0^t \{(1 - F(t-u))z + F(t-u)\}\phi(t-u)\lambda e^{-\lambda u} du$, and (b) by a change of variable $t - u = \omega$ solve the above equation (convert it to a differential equation) to explicitly find $\phi(t)$, and identify the distribution of $N(t)$.

4. (a) At time 0 there is an organism that lives a random length of time $\sim F$ at the end of which it (dies itself and) gives rise to a mean number m of organisms which follow a similar life cycle. Find a renewal equation for the mean number $M(t)$ of organisms at time t. (Hint: if at time x the organism dies giving birth to a mean of m organisms then in the remaining $(t - x)$ time one adds in the mean $mM(t-x)$ many organisms.)

   (b) Write down the statements of Blackwell's renewal theorem and the key renewal theorem and for the case $m = 1$ discuss if both of these are applicable to (a) or not and the final result of the application under suitable conditions.

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5. (a) Consider a linear birth and death plus immigration process $X(t)$ with parameters λ, μ, ν respectively. Assuming $X(0) = 0$ the pgf of $X(t)$ has been derived to be

$$\left(\frac{\mu - \lambda \mathcal{P}(z)}{\mu - \lambda z}\right)^{\nu/\lambda}.$$

Discuss the limits of the above and the limiting distribution, given (the pgf of a birth and death process starting from 1 with absorption at zero) $\mathcal{P}(z) = \{\mu(z-1) - (\lambda z - \mu)e^{(\mu-\lambda)t}\}/\{\lambda(z-1) - (\lambda z - \mu)e^{(\mu-\lambda)t}\}$ in the cases (i) $\lambda < \mu$, (ii) $\lambda > \mu$.

(b) A bacterium enters the system and gives birth or dies from an antibiotic in the manner of a linear birth and death process (with absorption at 0) with rates λ and μ respectively modelled as a Markov branching process. If $\lambda < \mu$ then is the expected time to extinction finite, i.e. is $E_1 T_0 < \infty$?